(p,q)-REGULAR OPERATORS BETWEEN BANACH LATTICES

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Abstract. We study the class of (p,q)-regular operators between quasi-Banach lattices. In particular, a representation of this class as the dual of a certain tensor norm for Banach lattices is given. We also provide some factorization results for (p,q)-regular operators yielding new Marcinkiewicz-Zygmund type inequalities for Banach function spaces. An extension theorem for (q,∞)-regular operators defined on a subspace of L_q is also given.

1. Introduction

This paper is devoted to study operators between Banach or quasi-Banach lattices satisfying estimates of the form

$$\Big\| \Big(\sum_{i=1}^n |Tx_i|^p \Big)^{\frac{1}{p}} \Big\| \leq K \Big\| \Big(\sum_{i=1}^n |x_i|^q \Big)^{\frac{1}{q}} \Big\|,$$

for every choice of vectors $\{x_i\}_{i=1}^n$. The operators for which this inequality holds are called (p,q)-regular, and as far as we know were introduced by A. Bukhvalov in [2] in connection with the interpolation of Banach lattices (see also [3]). The aim of this note is to make a systematic study of the class of (p,q)-regular operators.

It should be noted that the notion of regular operator has different usages in the literature: these can refer to operators that can be written as a difference of positive operators (cf. [1]), or for the terminology used in [20], these refer to operators $T : X \to Y$ for which there is $K > 0$ such that

$$\Big\| \bigvee_{i=1}^n |Tx_i| \Big\| \leq K \Big\| \bigvee_{i=1}^n |x_i| \Big\|$$

for every $\{x_i\}_{i=1}^n \subset X$. The latter correspond in the current terminology to (∞,∞)-regular operators.

There has been a considerable interest in the literature to determine conditions under which every operator between two Banach lattices is (r,r)-regular (or for brevity r-regular). In particular, an application of Grothendieck's inequality due to J. L. Krivine [14] (see also [17, Theorem 1.f.14]) yields that for any Banach lattices X, Y, every bounded linear operator $T : X \to Y$ is 2-regular. This fact has been

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later extended by N. J. Kalton to a quite large family of quasi-Banach lattices [13], and is also related to the complexification constants of an operator (cf. [10]). On the other hand, it is also known that positive operators between Banach lattices are always (p,q)-regular for $q \leq p$. The case of $(\infty,1)$-regular operators, which contain all (p,q)-regular operators, have been recently shown to have good interpolation properties with respect to the Calderón-Lozanovskii construction [23].

In the particular case of operators between L_p spaces, the question whether every operator is r-regular can be traced back to classical works of R. Paley, J. Marcinkiewicz, A. Zygmund and S. Kwapien, and has been completely settled by A. Defant and M. Junge in [9] (see also the references therein). In particular, in that paper, the authors characterize the triples (p,q,r) for which every operator $T: L_p \to L_q$ is r-regular, even providing quantitative versions and asymptotic estimates of the constants involved in some cases. In this paper, we will analyze the analogous situation concerning (p,q)-regular operators.

It goes without saying that there is also a natural interplay between (p,q)-regularity and summability properties. This connection stems from the fact that for a Banach lattice X and $\{x_i\}_{i=1}^n \subset X$ it holds that

$$\sup_{x^*\in B_{X^*}} \Big(\sum |\langle x_i, x^*\rangle|^p\Big)^{1/p} \leq \Big\|\Big(\sum |x_i|^p\Big)^{1/p}\Big\|_X.$$

In particular, the above inequality yields that every lattice (p,q)-summing operator is (p,q)-regular (see [5, 18] concerning lattice summing operators). The natural connection with convexity and concavity will also be explored.

The paper is organized as follows: after Section 2, where a preliminary discussion about the basics on (p,q)-regular operators is given, in Section 3 we present several facts for this class of operators in the framework of Lapresté tensor norms (see [7]). In particular, this allows to represent the class of (p,q)-regular operators as the dual of a certain tensor product in a standard way. It must be noted though that this point of view has been historically considered for Banach spaces, or locally convex vector spaces, but it is not so common for the case of Banach lattices. This approach is not to be confused with that of Banach lattice tensor norms, which studies the conditions under which a particular topology in a tensor product of Banach lattices becomes itself a Banach lattice (see the founding paper of D. Fremlin [12] and related recent work of A. Schep in [24]).

Next part of the paper, Section 4, is devoted to the peculiarities of (p,q)-regular operators between L_r spaces. We first study the factorization properties of these operators in terms of the Maurey-Rosenthal theory (see Theorems 4.1 and 4.4) in order to provide a characterization of a specific class of operators factoring through L_r-spaces. As a main result of this section, we give a new class of Marcinkiewicz-Zygmund inequalities involving norms of general Banach function lattices.

The paper is finished by showing the extension properties of the (∞,q)-regular operators defined on a subspace of a Banach lattice (Theorem 5.2), which provide a version of a result of G. Pisier on extension of ∞-regular operators (see [20]).

We use standard terminology from Banach spaces, Banach lattices and operator theory. For any unexplained notion the reader is referred to the monographs [1, 7, 17].

2. Definitions and preliminaries

Suppose X is a quasi-Banach lattice of measurable functions on a measure space (Ω,Σ,μ). Given $\{x_i\}_{i=1}^n \subset X$ and $p\in(0,\infty)$, expressions of the form $(\sum_{i=1}^n |x_i|^p)^{1/p}$ can be defined pointwise (μ-almost everywhere). Although Krivine's functional calculus ([17, Theorem 1.d.1], see also [22] for the non-locally convex setting) gives a meaning to this kind of expressions for abstract quasi-Banach lattices, for most applications we will only be concerned with the case of measurable functions.

Definition 2.1. Given quasi-Banach lattices X, Y, and $0<p,q<\infty$ a linear map $T: X\to Y$ is (p,q)*-regular* if there is a constant $K>0$ such that for every $\{x_i\}_{i=1}^n\subset X$,

$$\Big\|\Big(\sum_{i=1}^n |Tx_i|^p\Big)^{\frac1p}\Big\| \le K\Big\|\Big(\sum_{i=1}^n |x_i|^q\Big)^{\frac1q}\Big\|.$$

Similarly, T is (p,∞)-regular (respectively, (∞,q)-regular) when

$$\Big\|\Big(\sum_{i=1}^n |Tx_i|^p\Big)^{\frac1p}\Big\| \le K\Big\|\bigvee_{i=1}^n |x_i|\Big\|. \qquad \Big(\text{resp. } \Big\|\bigvee_{i=1}^n |Tx_i|\Big\| \le K\Big\|\Big(\sum_{i=1}^n |x_i|^q\Big)^{\frac1q}\Big\|.\Big)$$

For simplicity, when $p=q$ we will say that T is p-regular. With this notation, the well-known notion of regular operator (cf. [20]) corresponds to the case of ∞-regular operator.

We will write $R_{p,q}(X,Y)$ for the space of (p,q)-regular operators between X and Y. We will denote by $\rho_{p,q}(T)$ the smallest $K>0$ for which the inequalities appearing in Definition 2.1 hold for arbitrary elements in X. The following facts are straightforward:

Proposition 2.2.

(1) *Every (p,q)-regular linear map T is bounded with $\|T\|\le\rho_{p,q}(T)$.*
(2) *Let $p_1\ge p$ and $q_1\le q$. If T is (p,q)-regular, then T is (p_1,q_1)-regular with $\rho_{p_1,q_1}(T)\le\rho_{p,q}(T)$.*

Given $\{x_i\}_{i=1}^n\subset X$, for $p\ge1$, taking $1/p+1/p'=1$ we can write

$$\Big(\sum_{i=1}^n |x_i|^p\Big)^{\frac1p} = \sup\Big\{\sum_{i=1}^n a_ix_i : \sum_{i=1}^n |a_i|^{p'}\le 1\Big\},$$

(the supremum being taken in the sense of order in the lattice X). In particular when T is a positive operator, we have the inequalities

$$\Big(\sum_{i=1}^n |Tx_i|^p\Big)^{\frac1p} = \sup_{\sum_{i=1}^n |a_i|^{p'}\le1} T\Big(\sum_{i=1}^n a_ix_i\Big) \le T\Big(\sup_{\sum_{i=1}^n |a_i|^{p'}\le1}\sum_{i=1}^n a_ix_i\Big) = T\Big[\Big(\sum_{i=1}^n |x_i|^p\Big)^{\frac1p}\Big]$$

More generally, if the modulus of an operator $T: X\to Y$ exists (cf. [1, Chapter 1]), then it is p-regular for every $1\le p\le\infty$, and $\rho_{p,p}(T)\le\||T|\|$. In particular, this happens when Y is a Dedekind complete Banach lattice and T can be written as a difference of two positive operators. Conversely, suppose Y is complemented by a positive projection in its topological bidual Y^{**}, then every 1-regular operator $T: X\to Y$ can be written as a difference of two positive operators [15, p. 307].

The definition of a (p,q)-regular operator suggests a connection with convexity and concavity. Indeed, recall that an operator $T:X\to Y$ is (p,q)-concave (cf. [11, p. 330]) whenever there is a constant $C>0$ such that for every $\{x_i\}_{i=1}^n\subset X$

$$\Big(\sum_{i=1}^n \|Tx_i\|^p\Big)^{\frac{1}{p}}\leq C\Big\|\Big(\sum_{i=1}^n |x_i|^q\Big)^{\frac{1}{q}}\Big\|.$$

Similarly, $T:X\to Y$ is (p,q)-convex whenever there is a constant $C>0$ such that for every $\{x_i\}_{i=1}^n\subset X$

$$\Big\|\Big(\sum_{i=1}^n |Tx_i|^p\Big)^{\frac{1}{p}}\Big\|\leq C\Big(\sum_{i=1}^n \|x_i\|^q\Big)^{\frac{1}{q}}.$$

In particular, a Banach lattice is called p-concave, respectively p-convex, when the identity is (p,p)-concave, respectively (p,p)-convex. It is straightforward to check that, for $p\geq q$, if $T:X\to Y$ is (p,q)-concave and $S:Y\to Z$ is p-convex, then ST is (p,q)-regular with $\rho_{p,q}(ST)\leq M^{(p)}(S)K_{p,q}(T)$ (where, $M^{(p)}(S)$ and $K_{p,q}(T)$ denote, respectively, the p-convexity constant of S and the (p,q)-concavity constant of T, cf. [17, 1.d.3]).

We will also make use of the concavification of a Banach lattice (cf. [17, 1.d]). Namely, if X is a p-convex Banach lattice of measurable functions over some measure space (Ω,Σ,μ), we define its p-concavification $X_{[p]}$ which consists of those measurable functions f such that $|f|^{\frac{1}{p}}\in X$, endowed with the norm

$$\|f\|_{X_{[p]}}=\||f|^{\frac{1}{p}}\|_X^p.$$

We mentioned above that for a large class of quasi-Banach lattices every bounded linear operator is 2-regular. In fact, an application of Grothendieck's inequality due to J. L. Krivine [14] (see also [17, Theorem 1.f.14]) yields that for Banach lattices X,Y, every bounded linear operator $T:X\to Y$ is 2-regular with $\rho_{2,2}(T)\leq K_G\|T\|$, where K_G denotes Grothendieck's constant.

This fact was extended by N. J. Kalton to L-convex quasi-Banach lattices [13]. Recall that a quasi-Banach lattice X is L-convex whenever its order intervals are uniformly locally convex, that is, whenever there exists $0<\varepsilon<1$ so that if $u\in X_+$ with $\|u\|=1$ and $0\leq x_i\leq u$ (for $i=1,\dots,n$) satisfy

$$\frac{1}{n}(x_1+\ldots+x_n)\geq(1-\varepsilon)u,$$

then

$$\max_{1\leq i\leq n}\|x_i\|\geq\varepsilon.$$

This class includes every quasi-Banach lattice which is the p-concavification of a Banach lattice (for instance, L_p, $\Lambda(W,p)$ and $L_{p,\infty}$ for $0<p<\infty$). Kalton's result states that if Y is an L-convex quasi-Banach lattice, then for every quasi-Banach lattice X, every operator $T:X\to Y$ is 2-regular [13, Theorem 3.3].

Proposition 2.3. *Given quasi-Banach lattices X,Y and $0<p,q\leq\infty$, the space $(R_{p,q}(X,Y),\rho_{p,q}(\cdot))$ is a quasi-Banach space.*

Proof. This is straightforward. For completeness, just note that $\|T\|\leq\rho_{p,q}(T)$. □

Proposition 2.4. *Let X,Y be quasi-Banach lattices.*

(i) *Let $0<p<q\leq\infty$. Then $R_{p,q}(X,Y)=\{0\}$.*

(ii) *Suppose Y is L-convex, then for every $p \geq 2 \geq q$ we have $R_{p,q}(X,Y) = L(X,Y)$.*
(iii) *For $0 < q \leq p$, if X is q-concave and Y is p-convex, then $R_{p,q}(X,Y) = L(X,Y)$.*

Proof. (i) Let $T \in R_{p,q}(X,Y)$. Suppose there is $x \in X$ such that $Tx \neq 0$. Then for each $n \in \mathbb{N}$,

$$n^{\frac{1}{p}} \|Tx\| = \|(\sum_{i=1}^n |Tx|^p)^{\frac{1}{p}}\| \leq \rho_{p,q}(T) \|(\sum_{i=1}^n |x|^q)^{\frac{1}{q}}\| = \rho_{p,q}(T)\, n^{\frac{1}{q}} \|x\|.$$

Since this is impossible for large n, we have that $T = 0$.

(ii) This follows from [13, Theorem 3.3] and Proposition 2.2.

(iii) Let $T : X \to Y$ be an operator. For $(x_i)_{i=1}^n \subset X$ we have

$$\begin{aligned} \|(\sum_{i=1}^n |Tx_i|^p)^{\frac{1}{p}}\| &\leq M^{(p)}(Y) (\sum_{i=1}^n \|Tx_i\|^p)^{\frac{1}{p}} \leq M^{(p)}(Y) (\sum_{i=1}^n \|Tx_i\|^q)^{\frac{1}{q}} \\ &\leq M^{(p)}(Y) \|T\| \, (\sum_{i=1}^n \|x_i\|^q)^{\frac{1}{q}} \leq M^{(p)}(Y) \|T\| M_{(q)}(X) \, \|(\sum_{i=1}^n |x_i|^q)^{\frac{1}{q}}\|. \end{aligned}$$

□

In the rest of the section we characterize (p,q)-regularity of operators in terms of the bilinear maps defined by them. Some of the results presented here are well-known; but we include them here in a unified way and with complete proofs for the aim of completeness. We first analyze a special type of duality in Köthe-Bochner spaces, which will be done in what follows.

Recall that for a Banach lattice X and $1 \leq p < \infty$, $X(\ell_p)$ is the closed subspace of sequences $x = (x_n)_{n \in \mathbb{N}} \subset X$ for which

$$\|x\|_{X(\ell_p)} = \sup_k \Big\| \Big(\sum_{n=1}^k |x_n|^p \Big)^{\frac{1}{p}} \Big\| < \infty,$$

and which is spanned by the eventually null sequences. Similarly, $X(\ell_\infty)$ corresponds to the space of those sequences with

$$\|x\|_{X(\ell_\infty)} = \sup_k \Big\| \bigvee_{n=1}^k |x_n| \Big\| < \infty.$$

These are the natural generalization of Bochner (or Köthe-Bochner) spaces for abstract Banach lattices. Indeed, let X be an order continuous quasi-Banach function space over the measure space (Ω, Σ, μ) and let $0 < r \leq \infty$. The Köthe-Bochner space $X(\Omega, \Sigma, \mu; \ell^r)$ is defined to be the space of strongly Σ-measurable functions $\phi : \Omega \to \ell^r$ with the quasi-norm given by

$$\|\phi\|_{X(\Omega,\Sigma,\mu;\ell^r)} := \big\| \phi_{\|\cdot\|_r} \big\|_X,$$

where $\phi_{\|\cdot\|_r} : \Omega \to \mathbb{R}$ is given by $\omega \in \Omega \mapsto \|\phi(\omega)\|_{\ell^r}$.

Lemma 2.5. *Let X be an order continuous quasi-Banach function space over (Ω, Σ, μ), $1 \leq r \leq \infty$ and let $(e_i)_{i \in \mathbb{N}}$ denote the unit basis of ℓ_r. For any $(x_i)_{i=1}^n \subset X$ we have*

(i) *the function* $\phi : \Omega \to \ell^r$ *given by* $\phi(\omega) = \sum_{i=1}^n x_i(\omega) e_i$ *belongs to* $X(\Omega, \Sigma, \mu; \ell^r)$ *with*

$$\|\phi\|_{X(\Omega,\Sigma,\mu;\ell^r)} := \big\| \big(\sum_{i=1}^n |x_i|^r \big)^{\frac{1}{r}} \big\|_X,$$

(ii) *the set* $X(\Omega, \Sigma, \mu; \ell^r)_0$ *of all the functions defined in this way is dense in* $X(\Omega, \Sigma, \mu; \ell^r)$.

Proof. (i) Since X is order continuous, each function x_i can be approximated by simple functions. Therefore, as the sequence is finite, a direct calculation shows that there is a sequence of ℓ^r-valued simple functions converging in the quasi-norm of $X(\Omega, \Sigma, \mu; \ell^r)$ to ϕ, and also that there is a subsequence of it that converges μ-almost everywhere. Therefore, ϕ is strongly measurable. The formula for the quasi-norm is just the definition of the quasi-norm in the Köthe-Bochner space.

(ii) By the order continuity of X, it can be easily seen using also that the functions are strongly measurable that vector valued simple functions are dense in $X(\Omega, \Sigma, \mu; \ell^r)$.

Indeed, take a sequence of simple functions (x_i) converging μ-a.e. to a function $x \in X(\Omega, \Sigma, \mu; \ell^r)$. We can choose a sequence (y_i) such that for every i, $\|x(w) - y_{i+1}(w)\|_X \leq \|x(w) - y_i(w)\|_X$ holds μ-a.e. In order to see this, just take $x_1 = y_1$ and consider the measurable set $A_2 := \{w | \|x(w) - x_1(w)\| \leq \|x(w) - x_2(w)\|\}$, and define the simple function $y_2 = y_1 \chi_{A_2} + x_2 \chi_{A_2^c}$. Now, define y_3 in the same way using y_2 and x_3, and so on. The resulting sequence satisfies the requirement. Now, we have that the real valued functions $\tau_i(w) = \|x(w) - y_i(w)\|$ are non-negative, decreasing and converge to 0 μ-a.e. The order continuity of $X(\mu)$ gives then that $\lim_i \tau_i = 0$ in the norm of $X(\mu)$, that is, $\lim_i y_i = x$ in $X(\Omega, \Sigma, \mu; \ell^r)$.

So it is enough to prove that there is a sequence of functions as in (i) converging to every simple function like $\psi = \sum_{i=1}^m u_i \chi_{A_i}$ in $X(\Omega, \Sigma, \mu; \ell^r)$, with $u_i = \sum_j \lambda_{i,j} e_j \in \ell_r$. In order to see that, note that for every k we can write ψ as

$$\psi = \sum_{i=1}^m \Big(\sum_{j=1}^k \lambda_{i,j} e_j \Big) \cdot \chi_{A_i} + \sum_{i=1}^m \Big(\sum_{j>k} \lambda_{i,j} e_j \Big) \cdot \chi_{A_i}.$$

Clearly, the first member in this sum, say $\psi_k = \sum_{i=1}^m (\sum_{j=1}^k \lambda_{i,j} e_j) \cdot \chi_{A_i}$, belongs to the above family. Thus, it is enough to check that the second member in the sum converges to 0 in the quasi-norm, and so $\psi_k \to_k \psi$. Indeed,

$$\Big\| \sum_{i=1}^m \Big(\sum_{j>k} \lambda_{i,j} e_j \Big) \cdot \chi_{A_i} \Big\|_{X(\Omega,\Sigma,\mu;\ell^r)} \leq \sum_{i=1}^m \Big(\sum_{j=k+1}^\infty |\lambda_{i,j}|^r \Big)^{\frac{1}{r}} \|\chi_{A_i}\|_X \to_k 0.$$

□

Let $1 \leq r, p, s \leq \infty$ such that $1/r = 1/p + 1/s$, and X a Banach lattice with dual X^*. A similar argument as that of [17, Proposition 1.d.2] yields that for any $\{x_i\}_{i=1}^n \subset X$, $\{x_i^*\}_{i=1}^n \subset X^*$ it holds that

$$(1) \qquad \Big(\sum_{i=1}^n |\langle x_i^*, x_i \rangle|^r \Big)^{\frac{1}{r}} \leq \Big\langle \Big(\sum_{i=1}^n |x_i^*|^s \Big)^{\frac{1}{s}}, \Big(\sum_{i=1}^n |x_i|^p \Big)^{\frac{1}{p}} \Big\rangle.$$

For $r = 1$ this inequality is sharp, in the sense of [15, 7.2.2 (2)]. For $r > 1$ this need not be the case, but in the Köthe-Bochner setting there is an improvement which will be the key for our purposes. This can be understood as the consequence of some "duality in the range" between norms of vector valued function spaces.

We will consider the (μ-almost everywhere) pointwise product of vector valued functions as follows: for $\phi \in X(\Omega,\Sigma,\mu;\ell^p)$ and $\psi \in X^*(\Omega,\Sigma,\mu;\ell^s)$,

$$(\phi \cdot \psi)(\omega) := \sum_{i=1}^{\infty} \langle \phi(\omega), e_i\rangle \langle \psi(\omega), e_i\rangle\, e_i. \tag{2}$$

Hölder's inequality in the range of the function yields that that $(\phi \cdot \psi)(w) \in \ell^r$ μ-almost everywhere. In fact we have the following:

Lemma 2.6. *(Hölder's inequality for ℓ^r-valued Köthe-Bochner functions) Let X be an order continuous Banach function space over (Ω,Σ,μ), and $1 \le r \le p, s \le \infty$ such that $1/r = 1/p + 1/s$. If $\phi \in X(\Omega,\Sigma,\mu;\ell^p)$ and $\psi \in X^*(\Omega,\Sigma,\mu;\ell^s)$, then*

$$\big\|\phi \cdot \psi\big\|_{L^1(\Omega,\Sigma,\mu;\ell^r)} \le \big\|\phi\big\|_{X(\Omega,\Sigma,\mu;\ell^p)} \cdot \big\|\psi\big\|_{X^*(\Omega,\Sigma,\mu;\ell^s)}.$$

Proof. The order continuity of X gives the representation of the duality by means of the integral with respect to μ, which can be taken as a probability measure (cf. [17, Theorem 1.b.14]. Let ϕ and ψ as in the statement. Then

$$\begin{aligned}
\|\phi \cdot \psi\|_{L^1(\Omega,\Sigma,\mu;\ell^r)} &= \int \big(\sum_{i=1}^{\infty} |\langle \phi(w), e_i\rangle \langle \psi(w), e_i\rangle|^r\big)^{\frac{1}{r}} d\mu \\
&\le \int \big(\sum_{i=1}^{\infty} |\langle \phi(w), e_i\rangle|^p\big)^{\frac{1}{p}} \cdot \big(\sum_{i=1}^{\infty} |\langle \psi(w), e_i\rangle|^s\big)^{\frac{1}{s}} d\mu \\
&= \int \big\|\phi(w)\big\|_{\ell^p} \cdot \big\|\psi(w)\big\|_{\ell^s} d\mu \\
&\le \big\|\phi\big\|_{X(\Omega,\Sigma,\mu;\ell^p)} \cdot \big\|\psi\big\|_{X^*(\Omega,\Sigma,\mu;\ell^s)}.
\end{aligned}$$

□

Lemma 2.7. *Let X be an order continuous Banach function space over (Ω,Σ,μ) and consider $1 \le r \le p, s \le \infty$ satisfying that $1/r = 1/p + 1/s$. Given $\{x_i\}_{i=1}^n \subset X$ let $\phi = \sum_{i=1}^n x_i e_i \in X(\Omega,\Sigma,\mu;\ell^p)$. Then*

$$\|\phi\|_{X(\Omega,\Sigma,\mu;\ell^p)} = \sup_{\psi \in B_{X^*(\Omega,\Sigma,\mu;\ell^s)}} \big\|\phi \cdot \psi\big\|_{L^1(\Omega,\Sigma,\mu;\ell^r)}.$$

Actually, the functions in $B_{X^*(\mu,\ell^s)}$ for the computation of the norm can be taken of the form $\sum_{i=1}^n x_i^* e_i \in X^*(\Omega,\Sigma,\mu;\ell^s)$, that is

$$\big\|\big(\sum_{i=1}^n |x_i|^p\big)^{\frac{1}{p}}\big\|_X = \sup\Big\{\big\|\big(\sum_{i=1}^n |x_i \cdot x_i^*|^r\big)^{\frac{1}{r}}\big\|_{L^1(\mu)} : \big\|\big(\sum_{i=1}^n |x_i^*|^s\big)^{\frac{1}{s}}\big\|_{X^*} \le 1\Big\}.$$

Proof. The inequality " $\ge$ " is a consequence of Lemma 2.6. For the equality, consider $\{x_i\}_{i=1}^n \subset X$. By the Hahn-Banach Theorem, we can take $x^* \in B_{X^*}$ such that

$$\big\|\big(\sum_{i=1}^n |x_i|^p\big)^{\frac{1}{p}}\big\|_X = \langle x^*, \big(\sum_{i=1}^n |x_i|^p\big)^{\frac{1}{p}}\rangle.$$

For $i = 1, \dots, n$ let

$$x_i^*(\omega) = \begin{cases} \frac{|x_i(\omega)|^{(p-r)/r}\, x^*(\omega)}{\big(\sum_{i=1}^n |x_i(\omega)|^p\big)^{1/s}} & \text{if } \sum_{i=1}^n |x_i(\omega)|^p \neq 0, \\ 0 & \text{otherwise.} \end{cases}$$

Note that since

$$|x_i(\omega)|^{(p-r)/r} \leq \big(\sum_{i=1}^n |x_i(\omega)|^p\big)^{\frac{1}{s}},$$

it follows that $x_i^* \in X^*$ for $i = 1, \dots, n$. Note also that

$$\int \big(\sum_{i=1}^n |x_i \cdot x_i^*|^r\big)^{\frac{1}{r}} d\mu = \int \big(\sum_{i=1}^n |x_i|^p\big)^{\frac{1}{p}} x^* d\mu = \big\|\big(\sum_{i=1}^n |x_i|^p\big)^{\frac{1}{p}}\big\|_X.$$

Moreover, taking into account that

$$\big(\frac{p-r}{r}\big)s = \frac{sp}{r} - s = p + s - s = p,$$

we obtain

$$\big(\sum_{i=1}^n |x_i^*|^s\big)^{\frac{1}{s}} = \Big(\sum_{i=1}^n \Big|\frac{|x_i|^{(p-r)/r}\, x^*}{\big(\sum_{i=1}^n |x_i|^p\big)^{\frac{1}{s}}}\Big|^s\Big)^{\frac{1}{s}} = x^* \in B_{(X(\mu))^*}.$$

This proves the result. □

Consider a pair of Banach function spaces X, Y over (Ω, Σ, μ) and (Ω', Σ', ν) respectively, such that X and Y^* are order continuous. If we take a linear operator $T: X \to Y$ and $1/r \leq 1/q + 1/s$, we can define a bilinear operator

$$P_T : X(\Omega, \Sigma, \mu; \ell^q)_0 \times Y^*(\Omega', \Sigma', \nu; \ell^s)_0 \to L^1(\nu, \ell^r)$$

by means of the product defined in (2) for vector valued functions, as follows:

$$P_T\big(\sum_{i=1}^n x_i e_i, \sum_{i=1}^n y_i^* e_i\big)(\omega') := \big(\sum_{i=1}^n [Tx_i](\omega') e_i\big) \cdot \big(\sum_{i=1}^n y_i^*(\omega') e_i\big) = \sum_{i=1}^n [Tx_i](\omega') \cdot y_i^*(\omega') e_i.$$

The continuity of such a bilinear map is equivalent to the existence of a constant $C > 0$ such that

$$\big\|\sum_{i=1}^n Tx_i\, y_i^* e_i\big\|_{L^1(\mu, \ell^r)} \leq C \,\big\|\sum_{i=1}^n x_i e_i\big\|_{X(\Omega, \Sigma, \mu; \ell^q)} \cdot \big\|\sum_{i=1}^n y_i^* e_i\big\|_{Y^*(\Omega', \Sigma', \nu; \ell^s)}.$$

In this case, by Lemma 2.5, P_T uniquely extends to a continuous bilinear map on the space $X(\Omega, \Sigma, \mu; \ell^q) \times Y^*(\Omega', \Sigma', \nu; \ell^s)$.

Proposition 2.8. *Let X, Y be Banach function spaces over (Ω, Σ, μ) and (Ω', Σ', ν) respectively, such that X and Y^* are order continuous. Consider an operator $T : X \to Y$. Let $1/r = 1/p + 1/s$ and $q \leq p$. The following assertions are equivalent.*

- (i) *T is (p, q)-regular.*
- (ii) *The bilinear map P_T is continuous from $X(\Omega, \Sigma, \mu; \ell^q)_0 \times Y^*((\Omega', \Sigma', \nu; \ell^s)_0$ to $L^1(\nu, \ell^r)$.*

Moreover, the continuity constant of the bilinear map P_T equals $\rho_{p,q}(T)$.

Proof. (i) $\Rightarrow$ (ii): Let $\{x_i\}_{i=1}^n \subset X$ and $\{y_i^*\} \subset Y^*$. By Lemma 2.6 we have

$$\int \big(\sum_{i=1}^n |Tx_i y_i^*|^r\big)^{\frac{1}{r}}\, d\nu \le \big\|\big(\sum_{i=1}^n |Tx_i|^p\big)^{\frac{1}{p}}\big\|_Y \big\|\big(\sum_{i=1}^n |y_i^*|^s\big)^{\frac{1}{s}}\big\|_{Y^*}$$
$$\le \rho_{p,q}(T)\big\|\big(\sum_{i=1}^n |x_i|^q\big)^{\frac{1}{q}}\big\|_X \big\|\big(\sum_{i=1}^n |y_i^*|^s\big)^{\frac{1}{s}}\big\|_{Y^*}.$$

This gives (ii), with $C \le \rho_{p,q}(T)$.

(ii) $\Rightarrow$ (i): Suppose there is $C > 0$ such that

$$\int \big(\sum_{i=1}^n |Tx_i y_i^*|^r\big)^{\frac{1}{r}}\, d\nu \le C\big\|\big(\sum_{i=1}^n |x_i|^q\big)^{\frac{1}{q}}\big\|_X \big\|\big(\sum_{i=1}^n |y_i^*|^s\big)^{\frac{1}{s}}\big\|_{Y^*}$$

for every $\{x_i\}_{i=1}^n \subset X$ and $\{y_i^*\} \subset Y^*$. By Lemma 2.7, it follows that

$$\big\|\big(\sum_{i=1}^n |Tx_i|^p\big)^{\frac{1}{p}}\big\|_Y = \sup\Big\{\big\|\big(\sum_{i=1}^n |Tx_i \cdot y_i^*|^r\big)^{\frac{1}{r}}\big\|_{L^1(\nu)} : \big\|\big(\sum_{i=1}^n |y_i^*|^s\big)^{\frac{1}{s}}\big\|_{Y^*} \le 1\Big\}$$
$$\le C\big\|\big(\sum_{i=1}^n |x_i|^q\big)^{\frac{1}{q}}\big\|_X.$$

This also gives that $\rho_{p,q}(T) \le C$. □

3. Lattice tensor norms and duality for (p,q)-regular operators

In this section we introduce some specific topologies for tensor products of Banach lattices, following a similar procedure as in the case of the Lapresté tensor norms [7]. This approach allows us to relate (p,q)-regular operators with other classical operator ideals by using standard tensor product duality.

Let us start by recalling the definitions and notation of well-known tensor norms. Given Banach spaces X and Y, for $z \in X \otimes Y$ and $1 \le p \le \infty$ set

$$\varepsilon(z) = \sup\Big\{x^* \otimes y^*(z) : x^* \in B_{X^*},\, y^* \in B_{Y^*}\Big\},$$
$$\pi(z) = \inf_{z=\sum x_i \otimes y_i}\Big\{\sum \|x_i\|\|y_i\|\Big\},$$
$$g_p(z) = \inf_{z=\sum x_i \otimes y_i}\Big\{\Big(\sum \|x_i\|^p\Big)^{1/p} \cdot \sup_{y^* \in B_{Y^*}}\Big(\sum |\langle y_i, y^*\rangle|^{p'}\Big)^{1/p'}\Big\},$$
$$d_p(z) = \inf_{z=\sum x_i \otimes y_i}\Big\{\sup_{x^* \in B_{X^*}}\Big(\sum |\langle x_i, x^*\rangle|^{p'}\Big)^{1/p'} \cdot \Big(\sum \|y_i\|^p\Big)^{1/p}\Big\},$$
$$w_p(z) = \inf_{z=\sum x_i \otimes y_i}\Big\{\sup_{x^* \in B_{X^*}}\Big(\sum |\langle x_i, x^*\rangle|^{p}\Big)^{1/p} \cdot \sup_{y^* \in B_{Y^*}}\Big(\sum |\langle y_i, y^*\rangle|^{p'}\Big)^{1/p'}\Big\}.$$

These are the injective, projective and some particular cases of the Lapresté tensor norms (Chevet-Saphar tensor norms, see [7, 12.5]).

Definition 3.1. Let X, Y be Banach lattices and $1 \le q \le p \le \infty$. For $z \in X \otimes Y$, let us define the positively homogeneous function

$$\phi_{p,q}(z) := \inf\Big\{\big\|\big(\sum |x_i|^q\big)^{1/q}\big\|_X \cdot \big\|\big(\sum |y_i|^{p'}\big)^{1/p'}\big\|_Y \,:\, z = \sum_{i=1}^n x_i \otimes y_i\Big\}$$

and the seminorm

$$r_{p,q}(z) := \inf \Big\{ \sum_{j=1}^{m} \phi_{p,q}(z_j) : \, z = \sum_{j=1}^{m} z_j \Big\}.$$

Proposition 3.2. *Let $1 \le q \le p \le \infty$, X and Y Banach lattices.*

1. *For $z \in X \otimes Y$, $\varepsilon(z) \le r_{p,q}(z) \le \pi(z)$. Consequently, $r_{p,q}$ is a norm.*
2. *If X is q-convex and Y is p'-convex (with $M^{(q)}(X) = 1 = M^{(p')}(Y)$), then $\phi_{p,q}$ is a quasi-norm with constant $2^{\frac{1}{t}-1}$ where $1/t = 1/q + 1/p'$. In particular, for $p = q$, $r_{p,p} = \phi_{p,p}$ is a norm.*

Proof. (1) For the inequality $\varepsilon(z) \le r_{p,q}(z)$ it is enough to prove that $\varepsilon(z) \le \phi_{p,q}(z)$ for every $z \in X \otimes Y$. For every representation $z = \sum_{i=1}^n x_i \otimes y_i$ and $x^* \in B_{X^*}$, $y^* \in B_{Y^*}$, we have

$$\begin{aligned} \sum_{i=1}^{n} \langle x_i, x^* \rangle \langle y_i, y^* \rangle &\le (\sum_{i=1}^{n} |\langle x_i, x^* \rangle|^p)^{1/p} \cdot (\sum_{i=1}^{n} |\langle y_i, y^* \rangle|^{p'})^{1/p'} \\ &\le (\sum_{i=1}^{n} |\langle x_i, x^* \rangle|^q)^{1/q} \cdot (\sum_{i=1}^{n} |\langle y_i, y^* \rangle|^{p'})^{1/p'} \\ &\le \big\| \big(\sum |x_i|^q \big)^{1/q} \big\|_X \cdot \big\| \big(\sum |y_i|^{p'} \big)^{1/p'} \big\|_Y . \end{aligned}$$

This gives $\varepsilon(z) \le r_{p,q}(z)$. For $r_{p,q}(z) \le \pi(z)$, just note that for each representation $z = \sum_{j=1}^m x_j \otimes y_j$,

$$r_{p,q}(z) \le \sum_{j=1}^{m} \phi_{p,q}(x_j \otimes y_j) \le \sum_{j=1}^{m} \|x_j\|_X \, \|y_j\|_Y .$$

(2) Let $1/t = 1/q + 1/p'$, and note that $t \le 1$. Given $z_1, z_2 \in X \otimes Y$ and $\varepsilon > 0$, let $z_1 = \sum_{i=1}^n x_i^1 \otimes y_i^1$ and $z_2 = \sum_{i=1}^n x_i^2 \otimes y_i^2$ such that

$$\big\| \sum_{i=1}^{n} |x_i^1|^q \big\|_{X_{[q]}}^{1/q} = \big\| \big(\sum_{i=1}^{n} |x_i^1|^q \big)^{1/q} \big\|_X \le (\phi_{p,q}(z_1) + \varepsilon)^{t/q},$$

$$\big\| \sum_{i=1}^{n} |y_i^1|^{p'} \big\|_{Y_{[p']}}^{1/p'} = \big\| \big(\sum_{i=1}^{n} |y_i^1|^{p'} \big)^{1/p'} \big\|_Y \le (\phi_{p,q}(z_1) + \varepsilon)^{t/p'},$$

and

$$\big\| \sum_{i=1}^{n} |x_i^2|^q \big\|_{X_{[q]}}^{1/q} = \big\| \big(\sum_{i=1}^{n} |x_i^2|^q \big)^{1/q} \big\|_X \le (\phi_{p,q}(z_2) + \varepsilon)^{t/q},$$

$$\big\| \sum_{i=1}^{n} |y_i^2|^{p'} \big\|_{Y_{[p']}}^{1/p'} = \big\| \big(\sum_{i=1}^{n} |y_i^2|^{p'} \big)^{1/p'} \big\|_Y \le (\phi_{p,q}(z_2) + \varepsilon)^{t/p'}.$$

Then

$$\begin{aligned}
\phi_{p,q}(z_1+z_2) &\le \Big\| \big(\sum_{i=1}^n |x_i^1|^q + \sum_{i=1}^n |x_i^2|^q\big)^{1/q}\Big\|_X \Big\| \big(\sum_{i=1}^n |y_i^1|^{p'} + \sum_{i=1}^n |y_i^2|^{p'}\big)^{1/p'}\Big\|_Y \\
&= \Big\| \sum_{i=1}^n |x_i^1|^q + \sum_{i=1}^n |x_i^2|^q \Big\|_{X_{[q]}}^{1/q} \Big\| \sum_{i=1}^n |y_i^1|^{p'} + \sum_{i=1}^n |y_i^2|^{p'} \Big\|_{Y_{[p']}}^{1/p'} \\
&\le \Big(\| \sum_{i=1}^n |x_i^1|^q \|_{X_{[q]}} + \| \sum_{i=1}^n |x_i^2|^q \|_{X_{[q]}} \Big)^{1/q} \Big(\| \sum_{i=1}^n |y_i^1|^{p'} \|_{Y_{[p']}} + \| \sum_{i=1}^n |y_i^2|^{p'} \|_{Y_{[p']}} \Big)^{1/p'} \\
&\le \Big((\phi_{p,q}(z_1)+\varepsilon)^t + (\phi_{p,q}(z_2)+\varepsilon)^t \Big)^{1/q} \Big((\phi_{p,q}(z_1)+\varepsilon)^t + (\phi_{p,q}(z_2)+\varepsilon)^t \Big)^{1/p'} \\
&\le \Big(2^{1-t}(\phi_{p,q}(z_1)+\phi_{p,q}(z_2)+2\varepsilon)^t \Big)^{1/q} \Big(2^{1-t}(\phi_{p,q}(z_1)+\phi_{p,q}(z_2)+2\varepsilon)^t \Big)^{1/p'} \\
&= 2^{\frac{1}{t}-1}(\phi_{p,q}(z_1)+\phi_{p,q}(z_2)+2\varepsilon).
\end{aligned}$$

As $\varepsilon>0$ was arbitrary, it follows that $\phi_{p,q}(z_1+z_2) \le 2^{\frac{1}{t}-1}(\phi_{p,q}(z_1)+\phi_{p,q}(z_2))$ as claimed. □

Now we can provide the representation theorem for (p,q)-regular operators. Recall that the trace duality allows us to identify $(X\otimes Y^*)^*$ with a certain subspace of $L(X,Y^{**})$: for $\varphi \in (X\otimes Y^*)^*$, take $T_\varphi : X \to Y^{**}$ given by

$$\langle T_\varphi(x), y^*\rangle = \varphi(x\otimes y^*)$$

for $x\in X$ and $y^*\in Y^*$.

Theorem 3.3. *Let $1\le q\le p\le\infty$. Then*

$$\mathcal{R}_{p,q}(X,Y) = \Big(X \otimes_{r_{p,q}} Y^*\Big)^* \cap \mathcal{L}(X,Y).$$

isometrically.

Proof. To see the inclusion $\subseteq$ just take a (p,q)-regular operator $T: X\to Y$ and consider the trace duality with a tensor $z=\sum_{j=1}^m\sum_{i=1}^n x_i^j\otimes y_i^{j*} \in X\otimes Y^*$. We have by Proposition 2.8 for $r=1$ (and so $s=p'$) that

$$\begin{aligned}
\langle T,z\rangle &= \sum_{j=1}^m\sum_{i=1}^n \langle T(x_i^j), y_i^{j*}\rangle \le \sum_{j=1}^m\sum_{i=1}^n |\langle T(x_i^j), y_i^{j*}\rangle| \\
&\le \sum_{j=1}^m \rho_{p,q}(T)\, \big\|\big(\sum |x_i^j|^q\big)^{1/q}\big\|_X \cdot \big\|\big(\sum |y_i^{j*}|^{p'}\big)^{1/p'}\big\|_{Y^*}.
\end{aligned}$$

Since this holds for all representations of z, it follows that the functional φ_T defined by T as $\varphi_T(x\otimes y^*) = \langle T(x), y^*\rangle$ satisfies that

$$\|\varphi_T\| \le \rho_{p,q}(T).$$

For the converse inclusion, take a functional $\varphi: X\otimes_{r_{p,q}} Y^* \to \mathbb{R}$ and define the operator $T_\varphi : X\to Y$ by $\langle T_\varphi(x), y^*\rangle = \varphi(x\otimes y^*)$. Let $\{x_i\}_{i=1}^n \subset X$. For every $\varepsilon>0$, there is a tensor $z=\sum_{i=1}^n x_i^*\otimes y_i^{0*}$ with $\|(\sum_{i=1}^n |y_i^{0*}|^{p'})^{\frac{1}{p'}}\|_{Y^*}\le 1$ such that

$$\sup\Big\{\sum_{i=1}^n \langle T_\varphi(x_i), y_i^*\rangle : \big\|\big(\sum_{i=1}^n |y_i^*|^{p'}\big)^{\frac{1}{p'}}\big\|_{Y^*}\le 1\Big\} \le \sum_{i=1}^n \langle T_\varphi(x_i), y_i^{0*}\rangle + \varepsilon.$$

By Proposition 2.8 for $r=1$, it follows that

$$\Big\|\Big(\sum_{i=1}^n |T_\varphi(x_i)|^p\Big)^{\frac{1}{p}}\Big\| - \varepsilon \le \sum_{i=1}^n \langle T_\varphi(x_i), y_i^{0*}\rangle = \varphi\big(\sum_{i=1}^n x_i\otimes y_i^{0*}\big) \le \|\varphi\| r_{p,q}(z)$$
$$\le \|\varphi\| \big\|\big(\sum_{i=1}^n |x_i|^q\big)^{1/q}\big\|_X \big\|\big(\sum_{i=1}^n |y_i^{0*}|^{p'}\big)^{\frac{1}{p'}}\big\|_{Y^*}$$
$$\le \|\varphi\| \big\|\big(\sum_{i=1}^n |x_i|^q\big)^{1/q}\big\|_X.$$

Therefore, T_φ is (p,q)-regular and $\rho_{p,q}(T_\varphi)\le\|\varphi\|$. This finishes the proof. □

This approach fits actually with a more general framework, in which analogous tensor product representations are also possible for the case of p-convex and p-concave operators. In order to compare these with classical operator ideals, we introduce the following.

Definition 3.4. Let $1\le q\le p\le\infty$, and X, Y Banach lattices. For $z\in X\otimes Y$, define the positively homogeneous functions

$$\delta_{p,q}(z) := \inf\Big\{\big(\sum \|x_i\|_X^q\big)^{1/q}\big\|\big(\sum |y_i|^{p'}\big)^{1/p'}\big\|_Y : z=\sum_{i=1}^n x_i^j\otimes y_i^j\Big\}$$

$$\iota_{p,q}(z) := \inf\Big\{\big\|\big(\sum |x_i|^q\big)^{1/q}\big\|_X\big(\sum \|y_i\|_Y^{p'}\big)^{1/p'} : z=\sum_{i=1}^n x_i^j\otimes y_i^j\Big\},$$

and the corresponding seminorms

$$h_{p,q}(z) := \inf\Big\{\sum_{j=1}^m \delta_{p,q}(z_j) : z=\sum_{j=1}^m z_j\Big\}.$$

$$k_{p,q}(z) := \inf\Big\{\sum_{j=1}^m \iota_{p,q}(z_j) : z=\sum_{j=1}^m z_j\Big\}.$$

Let us write $\mathcal{CX}_{p.q}(X,Y)$ for the space of (p,q)-convex operators from the Banach space X to the Banach lattice Y, and $\mathcal{CC}_{p.q}(X,Y)$ for the space of (p,q)-concave operators from the Banach lattice X to the Banach space Y. The following facts can be proved arguing as in Proposition 3.2 and Theorem 3.3.

Proposition 3.5. *Let $1\le q\le p\le\infty$, X and Y Banach lattices.*

1. *For $z\in X\otimes Y$, $\varepsilon(z)\le h_{p,q}(z)\le\pi(z)$ and $\varepsilon(z)\le k_{p,q}(z)\le\pi(z)$. Consequently, $h_{p,q}$ and $k_{p.q}$ are norms.*
2. *If Y is p'-convex (constant 1), then $\delta_{p,q}$ is a quasi-norm with constant 2^{1-t} where $1/t=1/q+1/p'$. In particular, $h_{p,p}=\delta_{p,p}$ is a norm.*
3. *If X is q-convex (constant 1), then $\iota_{p,q}$ is a quasi-norm with constant 2^{1-t} where $1/t=1/q+1/p'$. In particular, $k_{p,p}=\iota_{p,p}$.*

Theorem 3.6. *Let $1\le q\le p\le\infty$. Then we have the following isometric identities:*

1. $\mathcal{CX}_{p,q}(X,Y)=\Big(X\otimes_{h_{p,q}}Y^*\Big)^*\cap\mathcal{L}(X,Y)$,
2. $\mathcal{CC}_{p,q}(X,Y)=\Big(X\otimes_{k_{p,q}}Y^*\Big)^*\cap\mathcal{L}(X,Y)$.

This point of view allows us to state the relations among (p,q)-regular, (p,q)-convex and (p,q)-concave operators in a straightforward way by comparing the norms appearing in the corresponding representations.

Proposition 3.7. *Let $1 \le q \le p \le \infty$. Then we have the following relations.*

(1) (i) *If X is q-convex, then $\mathcal{R}_{p,q}(X,Y) \subseteq \mathcal{CX}_{p,q}(X,Y)$.*
(ii) *If X is q-concave, then $\mathcal{CX}_{p,q}(X,Y) \subseteq \mathcal{R}_{p,q}(X,Y)$.*
(iii) *If X is an L^q-space, then $\mathcal{R}_{p,q}(X,Y) = \mathcal{CX}_{p,q}(X,Y)$.*

(2) (i) *If Y is p-concave, then $\mathcal{R}_{p,q}(X,Y) \subseteq \mathcal{CC}_{p,q}(X,Y)$.*
(ii) *If Y is p-convex, then $\mathcal{CC}_{p,q}(X,Y) \subseteq \mathcal{R}_{p,q}(X,Y)$.*
(iii) *If Y is an L^p-space, then $\mathcal{R}_{p,q}(X,Y) = \mathcal{CC}_{p,q}(X,Y)$.*

(3) $\mathcal{L}(L^q,L^p) = \mathcal{R}_{p,q}(L^q,L^p) = \mathcal{CC}_{p,q}(L^q,L^p) = \mathcal{CX}_{p,q}(L^q,L^p)$.

Proof. The proofs of (1) and (2) are direct consequences of the previous results and duality arguments. For the proof of (3) just note that $p' \le q'$ and for every representation of a tensor as $z = \sum_{i=1}^n x_i \otimes y_i^*$ we have

$$\begin{aligned}\pi(z) &\le \sum_{i=1}^n \|x_i\|\,\|y_i^*\| \le \Big(\sum_{i=1}^n \|x_i\|^q\Big)^{1/q}\Big(\sum_{i=1}^n \|y_i^*\|^{q'}\Big)^{1/q'}\\ &\le \Big(\sum_{i=1}^n \|x_i\|^q\Big)^{1/q}\Big(\sum_{i=1}^n \|y_i^*\|^{p'}\Big)^{1/p'}.\end{aligned}$$

□

Besides this, more can be said about the coincidence of $(p.q)$-regular operators between L^r-spaces. This will be addressed in Section 4.

Some direct consequences of the general theory of summability of Banach lattices can also be stated in this framework. Using the representation theorem for maximal operator ideals (see [7, p.203]) and the tensor norms associated to the ideals of p-summing and p-dominated operators, we obtain the following:

Proposition 3.8. *Let X, Y be Banach lattices and $1 \le p \le \infty$. The following relations hold:*

(i) *$w_p \le r_{p,p}$ and $\mathcal{D}_p(X,Y) \subseteq \mathcal{R}_{p,p}(X,Y)$, where $\mathcal{D}_p$ is the ideal of p-dominated operators.*
(ii) *$g_p \le h_{p,p}$ and $\Pi_{p'}^*(X,Y) \subseteq \mathcal{CX}_{p,p}(X,Y)$, where $\Pi_{p'}^*(X,Y)$ is the adjoint to the ideal of p'-summing operators.*
(iii) *$d_p \le k_{p',p'}$ and so $\Pi_{p'}(X,Y) \subseteq \mathcal{CC}_{p',p'}(X,Y)$.*

Proof. For (i), use $w_p = w_{p'}^t$ ([7, p.152]) and the fact that the ideal of p'-dominated operators is associated to the norm w_p^*. Then $(w_p^*)' = w_{p'}$. A direct calculation using that for $x_1, \dots, x_n \in X$,

$$\sup_{x^* \in B_{X^*}} \Big(\sum |\langle x_i, x^*\rangle|^p\Big)^{1/p} \le \Big\|\Big(\sum |x_i|^p\Big)^{1/p}\Big\|,$$

gives $w_{p'} \le r_{p',p'}$. Thus the above comments give the required inclusion.

For (ii), use that $\Pi_{p'}^*$ is associated to g_p' ([7, p.211] and [7, §.17.9]). The inequality $g_p \le h_{p,p}$ is given by using the same inequality as in (i). Similar arguments show (iii). □

In the particular case when we deal with L_p spaces, the so called Chevet-Persson-Saphar inequalities (see [7, 15.10]), provide a useful tool for relating the tensor norms we just introduced with other classical tensor norms:

$$d^*_{p'}(z) \leq d_p(z) \leq \Delta_p(z) \leq g^*_{p'}(z) \leq g_p(z), \quad z \in L^p \otimes Y.$$

Moreover, in the case when Y is also an L^p-space, we actually get

$$d^*_{p'}(z) = d_p(z) = \Delta_p(z) = g^*_{p'}(z) = g_p(z), \quad z \in L^p(\mu) \otimes L^p(\nu) \tag{3}$$

for arbitrary measures μ and ν.

Let us now focus on the topological properties of the tensor product $L^p(\mu) \otimes L^{p'}(\nu)$ endowed with the $r_{p,p}$-norm, and compare them with other classical topologies. We will comment on $p = 2$ and the general case separately:

(1) For $p = 2$, the equalities given in (3) show that $L^2(\mu) \otimes_\alpha L^2(\nu)$ cannot be isomorphic to $L^2(\mu) \otimes_{r_{2,2}} L^2(\nu)$ for $\alpha = d^*_2 = d_2 = \Delta_2 = g^*_2 = g_2$, since by Corollary 3.10, we have that $r_{2,2}$ is equivalent to π in this tensor product.
(2) However, we can easily see that $r_{2,2}$ is equivalent to d_2 on the tensor product $X \otimes \ell^2$ for $X = \ell^\infty$ or $X = \ell^1$. This is a direct consequence of the so called Little Grothendieck Theorem (see [7, 17.14]) and Corollary 3.10.
(3) Similarly, d_∞ is also equivalent to $r_{2,2}$, in this case as a consequence of Grothendieck Theorem (see [7, 17.14]) and Corollary 3.10.

For the general case ($p \neq 2$), the Chevet-Persson-Saphar inequalities yield some positive results about (s, q)-regularity for certain well-known operators. The following are just a sample.

(1) For every Banach lattice Y, $d^*_{p'} \leq d_p \leq \Delta_p \leq g^*_{p'} \leq g_p \leq r_{s,q}$ for every $1 \leq q \leq p \leq s$ in the tensor product $L^p(\mu) \otimes Y$. This is a direct consequence of the p-concavity of L^p.
(2) The norm Δ_p concerns spaces of Bochner integrable functions, and can also be related to operators $T : L^p(\mu) \to Y$ defined as Y-valued integral by means of the formula $f \mapsto \int \phi \, f \, d\mu \in Y$ for a certain function $\phi \in L^{p'}(\mu, Y) \hookrightarrow (L^p(\mu) \otimes_{\Delta_p} Y^*)^*$. Thus, the comparison of Δ_p and $r_{s,q}$ provides also some meaningful results.

Using the tensor product representation of maximal operator ideals as dual spaces of topological tensor products (see [7, 17.5]), the above arguments yield the following.

Corollary 3.9. *Let Y be a Banach lattice, $1 \leq q \leq r \leq p$ and $T : L^p(\mu) \to Y$.*

(1) *Every r-integral and r-summing operator from $L^r(\mu)$ to Y, as well as operators belonging to their associated dual ideals, are (p, q)-regular.*
(2) *Every operator $T : L^r(\mu) \to Y$ defined as the Y-valued integral $T(\cdot) = \int \phi \, (\cdot) \, d\mu$ for a function $\phi \in L^{r'}(\mu, Y)$ is (p, q)-regular.*

Let us recall that an equivalent form of Grothendieck's Theorem can be given in terms of tensor products of $C(K)$-spaces [21, Theorem 3.1]: for each pair of compact Hausdorff spaces K_1 and K_2 and every $z = \sum_{i=1}^n x_i \otimes y_i \in C(K_1) \otimes C(K_2)$, it holds that

$$\pi(z) \leq K_G \, \|(\sum_{i=1}^n |x_i|^2)^{1/2}\|_{C(K_1)} \cdot \|(\sum_{i=1}^n |y_i|^2)^{1/2}\|_{C(K_2)},$$

where K_G is Grothendieck's constant.

Krivine's version of Grothendieck's Theorem (cf. [17, Theorem 1.f.14]) states that $\mathcal{R}_{2,2}(X,Y)=\mathcal{L}(X,Y)$, and the corresponding constants are related as $\|T\|\leq \rho_{2,2}(T)\leq K_G\|T\|$. The following result is the pre-dual version of this fact, and so is also equivalent to Grothendieck's Theorem.

Corollary 3.10. *Let X and Y be Banach lattices. Then $\pi\leq K_G\, r_{2,2}\leq K_G\,\pi$ on $X\otimes Y$.*

Proof. Take a tensor $z=\sum_{i=1}^n x_i\otimes y_i\in X\otimes Y$, and consider $x_0=(\sum_{i=1}^n |x_i|^2)^{1/2}\in X$ and $y_0=(\sum_{i=1}^n |y_i|^2)^{1/2}\in Y$. Take the ideals $I(x_0)$ and $I(y_0)$ generated by these elements in the corresponding lattices, and endow both of them with AM-norms:

$$\|x\|_{X,\infty}=\inf\Big\{\lambda:\,|x|\leq\lambda\frac{x_0}{\|x_0\|}\Big\}\quad\text{and}\quad \|y\|_{Y,\infty}=\inf\Big\{\lambda:\,|y|\leq\lambda\frac{y_0}{\|y_0\|}\Big\}.$$

Note that $\|x\|_X\leq\|x\|_{X,\infty}$ and $\|y\|_Y\leq\|y\|_{Y,\infty}$, and so the inclusion maps $J_X:\overline{I}(x_0)\to X$ and $J_Y:\overline{I}(y_0)\to Y$ acting in the closure of these ideals satisfy that $\|J_X\|\leq 1$ and $\|J_Y\|\leq 1$, respectively. By Kakutani's theorem these can be considered as $C(K)$ spaces (cf. [17, Theorem 1.b.6]. Note also that

$$\|x_0\|_{X,\infty}=\|x_0\|_X=\|(\sum_{i=1}^n|x_i|^2)^{1/2}\|_X\quad\text{and}\quad\|y_0\|_{Y,\infty}=\|y_0\|_Y=\|(\sum_{i=1}^n|y_i|^2)^{1/2}\|_Y.$$

Since π satisfies the metric mapping property (see for example [7, 3.2]), we have that

$$\big\|J_X\otimes J_Y:\overline{I}(x_0)\otimes_\pi\overline{I}(y_0)\to X\otimes_\pi Y\big\|\leq\|J_X\|\cdot\|J_Y\|\leq 1.$$

Now, we apply Grothendieck's Theorem for tensor products of $C(K)$ spaces to obtain

$$\begin{aligned}\pi(\sum_{i=1}^n x_i\otimes y_i)&=\pi(\sum_{i=1}^n J_X(x_i)\otimes J_Y(y_i))\leq K_G\,\big\|(\sum_{i=1}^n|x_i|^2)^{1/2}\big\|_{X,\infty}\cdot\big\|(\sum_{i=1}^n|y_i|^2)^{1/2}\big\|_{Y,\infty}\\&=K_G\,\big\|(\sum_{i=1}^n|x_i|^2)^{1/2}\big\|_X\cdot\big\|(\sum_{i=1}^n|y_i|^2)^{1/2}\big\|_Y.\end{aligned}$$

The result follows by convexity. □

4. (p,q)-regular operators between L_r-spaces and Marcinkiewicz-Zygmund inequalities

In this section we will center our attention in the case of operators defined between L_r-spaces, in relation with the Marcinkiewicz-Zygmund type inequalities presented by A. Defant and M. Junge in [9]. By means of the so called Maurey-Rosenthal factorization theory (see for instance [6, 8]), we will be able to extend these results to the case of operators acting in r-convex function lattices and with values in r-concave Banach function lattices. In particular, we study the requirements to reduce the study of (p,q)-regular operators between Banach lattices to the properties of such operators between L_r-spaces.

For the case $p=q$, a Maurey-Rosenthal factorization theorem for p-regular operators holds under the usual convexity/concavity requirements. The following result is similar to Theorem 3.1 in [8]. However, the reader must notice that the requirements on the operator T are different. We sketch the proof showing the argument; see the proof of Theorem 3.2 in [8] for more details.

Theorem 4.1. *Let $1 \leq s \leq p < \infty$. Let X be a p-convex order continuous Banach function space over (Ω, Σ, μ) and Y be an s-concave Banach function space over (Ω', Σ', ν) with Y^* order continuous. Let $T : X \to Y$ be a p-regular operator. Then T factors as*

$$\begin{array}{ccc} X & \xrightarrow{\quad T \quad} & Y \\ {\scriptstyle M_f}\downarrow & & \uparrow{\scriptstyle M_g} \\ L_p(\mu) & \xrightarrow{\quad \hat{T} \quad} & L_s(\nu) \end{array}$$

for suitable functions f and g. Here, $\hat{T}$ is a linear and continuous operator.

Proof. Note first that $1/p + 1/s' \leq 1$. Take $r \geq 1$ such that $1/r = 1/p + 1/s'$. For the aim of simplicity we assume that the p-convexity and s-concavity constants involved are equal to 1; note that, being s-concave, Y is order continuous. For $\{x_i\}_{i=1}^n \subset X$ and $\{y_i^*\}_{i=1}^n \subset Y^*$, using the generalized Hölder inequality (1) and the fact that T is p-regular, it holds that

$$\Big(\sum_{i=1}^n \|Tx_i y_i^*\|_{L_1(\nu)}^r\Big)^{\frac{1}{r}} \leq \int \Big(\sum_{i=1}^n |Tx_i|^p\Big)^{\frac{1}{p}} \Big(\sum_{i=1}^n |y_i^*|^{s'}\Big)^{\frac{1}{s'}} d\nu$$
$$\leq C \big\| \big(\sum_{i=1}^n |x_i|^p\big)^{\frac{1}{p}} \big\|_X \big\| \big(\sum_{i=1}^n |y_i^*|^{s'}\big)^{\frac{1}{s'}} \big\|_{Y^*}.$$

Since X is p-convex and Y^* is s'-convex, by Young's inequality, it follows that

$$\sum_{i=1}^n \|Tx_i y_i^*\|_{L_1(\nu)}^r \leq C^r \big\| \sum_{i=1}^n |x_i|^p \big\|_{X_{[p]}}^{\frac{r}{p}} \big\| \sum_{i=1}^n |y_i^*|^{s'} \big\|_{(Y^*)_{[s']}}^{\frac{r}{s'}}$$
$$\leq C^r \Big(\frac{r}{p} \big\| \sum_{i=1}^n |x_i|^p \big\|_{X_{[p]}} + \frac{r}{s'} \big\| \sum_{i=1}^n |y_i^*|^{s'} \big\|_{(Y^*)_{[s']}}\Big).$$

We now make a standard application of Ky Fan's Lemma (see for example [19, E.4;p.40] for the Lemma and the requirements to apply it, and [6, Theorem 1] for a proof similar to the one presented here). It provides functions $f_0 \in (X(\mu)_{[p]})^* = \mathcal{M}(X(\mu), L^p(\mu))_{[p]}$ (where $\mathcal{M}$ denotes the space of multiplication operators), and $g_0 \in \big((Y(\nu)^*)_{[s']}\big)^* = \mathcal{M}(Y(\nu)^*, L^{s'}(\nu))_{[s']}$ such that

$$\int |Tx\, y^*| d\nu \leq C\Big(\int |x|^p f_0\, d\mu\Big)^{1/p} \Big(\int |y^*|^{s'} g_0\, d\nu\Big)^{1/s'}.$$

This gives the inequality

$$\Big(\int \Big|\frac{Tx}{g_0^{1/s'}}\Big|^s d\nu\Big)^{1/s} \leq C\Big(\int |x|^p f_0\, d\mu\Big)^{1/p}$$

for all $x \in X(\mu)$. This provides the desired factorization, using the associated multiplication operators and T to define the operator $\hat{T}$ (see the argument given in [8, Th.3.2]). The decomposition of the operator is given by

$$M_{g_0^{1/s'}} \circ g_0^{-1/s'} T(\cdot/f_0^{1/p}) \circ M_{f_0^{1/p}} = T,$$

that is, $f = f_0^{1/p}$, $\hat{T} = g_0^{-1/s'} T(\cdot/f_0^{1/p})$ and $g = g_0^{1/s'}$. $\square$

Remark 4.2. The statement of the previous result excludes the fundamental case in which $Y = L_1$, since the dual of such space is not order continuous. It must be mentioned here that this was one of the most relevant instances of the original factorization of Maurey and provides some of its main applications, for example regarding the structure of reflexive subspaces of L^1-spaces (see II.H.13 in [25]). We will not consider this case for the aim of simplicity. However, the result is expected to be true also in this case, since the separation argument can be extended for this case using a nowadays well-known procedure that is explained in [6].

Remark 4.3. The result above assures that the operator factors through an operator $\hat{T} : L_p(\mu) \to L_s(\nu)$ for $1 < s \le p < \infty$. However, as can be seen in [9, Corollary on page 282], not every operator from L_p to L_s is p-regular. In fact, regarding operators between L_p-spaces, we trivially have the following: Every operator $T : L_r(\mu) \to L_t(\nu)$, for $1 \le r \le t \le \infty$, is (t,r)-regular.

Indeed, let $\{x_i\}_{i=1}^n \subset L_r(\mu)$.

$$\Big\| \big(\sum_{i=1}^n |Tx_i|^t \big)^{1/t} \Big\|_{L_t(\nu)} = \big(\sum_{i=1}^n \|Tx_i\|_{L_t(\nu)}^t \big)^{1/t} \le \|T\| \big(\sum_{i=1}^n \|x_i\|_{L_r(\nu)}^t \big)^{1/t}$$
$$\le \|T\| \big(\sum_{i=1}^n \|x_i\|_{L_r(\nu)}^r \big)^{1/r} = \|T\| \Big\| \big(\sum_{i=1}^n |x_i|^r \big)^{1/r} \Big\|_{L_r(\nu)}.$$

In the rest of this section we will analyze what happens when $r = t$ ($s = p$ above).

Theorem 4.1 does not give a priori any information on the regularity properties of $\hat{T}$; of course, T need not be positive, otherwise the results regarding this property are trivial. Next result characterizes the existence of a factorization for (p,q)-regular operators through L_r-spaces preserving the property of being (p,q)-regular. The requirements are formally more restrictive than the ones needed in Theorem 4.1, and they involve a new vector norm inequality that suggests some mixed norms for Banach function spaces.

Theorem 4.4. *Let X be an r-convex Banach function space over (Ω, Σ, μ), and let Y be an r-concave Banach function space over (Ω', Σ', ν) such that X and Y^* are order continuous. Let $T : X \to Y$ be an operator. The following are equivalent.*

(i) *There is a constant $K > 0$ such that for each pair of matrices of elements $(x_{i,j})_{i=1,j=1}^{n,m}$ and $(y^*_{i,j})_{i=1,j=1}^{n,m}$ in X and Y^*, respectively, the following inequality holds.*

$$\sum_{i=1}^n \sum_{j=1}^m |\langle Tx_{i,j}\, y^*_{i,j} \rangle| \le K \Big\| \big(\sum_{i=1}^n \big(\sum_{j=1}^m |x_{i,j}|^q \big)^{\frac{r}{q}} \big)^{\frac{1}{r}} \Big\|_X \Big\| \big(\sum_{i=1}^n \big(\sum_{j=1}^m |y^*_{i,j}|^{p'} \big)^{\frac{r'}{p'}} \big)^{\frac{1}{r'}} \Big\|_{Y^*}$$

(ii) *There is a constant $K > 0$ such that for each matrix of elements $(x_{i,j})_{i=1,j=1}^{n,m}$ in X, the following inequality holds.*

$$\Big\| \big(\sum_{i=1}^n \big(\sum_{j=1}^m |Tx_{i,j}|^p \big)^{\frac{r}{p}} \big)^{\frac{1}{r}} \Big\|_Y \le K \Big\| \big(\sum_{i=1}^n \big(\sum_{j=1}^m |x_{i,j}|^q \big)^{\frac{r}{q}} \big)^{\frac{1}{r}} \Big\|_X$$

(iii) *There are functions f and g such that T factors as*

$$\begin{array}{ccc} X & \xrightarrow{\quad T \quad} & Y \\ {\scriptstyle M_f}\downarrow & & \uparrow{\scriptstyle M_g} \\ L_r(\mu) & \xrightarrow{\quad \hat{T} \quad} & L_r(\nu) \end{array}$$

where $\hat{T}$ is (p,q)-regular.

Proof. It can be seen as a consequence of the lemmata of the introductory sections that (i) and (ii) are equivalent. Let us prove that (i) $\Rightarrow$ (iii). The argument is similar to the one in the proof of Theorem 4.1. Since Y^* is also r'-convex (with constant 1), and writting the inequalities in (i) in the form

$$\sum_{i,j=1}^{n,m} \big\|Tx_{i,j}\, y^*_{i,j}\big\|_{L_1(\nu)} - K\frac{r}{p}\,\Big\| \sum_{i=1}^{n} \big(\sum_{j=1}^{m} |x_{i,j}|^q \big)^{\frac{r}{q}} \Big\|_{X_{[r]}}$$

$$-K\frac{r}{q}\,\Big\| \sum_{i=1}^{n} \big(\sum_{j=1}^{m} |y^*_{i,j}|^{p'} \big)^{\frac{r'}{p'}} \Big\|_{(Y^*)_{[r']}} \leq 0,$$

we can define a set of functions $\phi : B_{(X_{[r]})^*} \times B_{((Y^*)_{[r']})^*} \to \mathbb{R}$ as

$$\phi(f,g) := \sum_{i=1}^{n}\sum_{j=1}^{m} \big\|Tx_{i,j}\, y^*_{i,j}\big\|_{L_1(\nu)} - K\frac{r}{p}\int \big(\sum_{i=1}^{n} \big(\sum_{j=1}^{m} |x_{i,j}|^q \big)^{\frac{r}{q}} \big)\, f\, d\mu$$
$$- K\frac{r}{q}\int \big(\sum_{i=1}^{n} \big(\sum_{j=1}^{m} |y^*_{i,j}|^{p'} \big)^{\frac{r'}{p'}} \big)\, g\, d\nu.$$

We can apply Ky Fan's Lemma now. The arguments are similar to the ones that can be found in the proofs of Theorem 1 and Theorem 2 in [6]. The requirements, which can be found in [19, E.4;p.40], are that the set of functions must be concave (see the definition in [19, E.4.1.2], the functions themselves must be convex and (lower semi-)continuous for the product of the weak* topologies, and the required inequality must be satisfied in a (maybe different) point for each function. Indeed, the above defined functions are continuous when the product of the weak* topologies is defined in the product $B_{(X_{[r]})^*} \times B_{((Y^*)_{[r']})^*}$, and convex. A simple computation shows that the family of all the functions (defined for each couple of finite matrices $(x_{i,j})_{i=1,j=1}^{n,m}$ and $(y^*_{i,j})_{i=1,j=1}^{n,m}$), is concave. So, by Ky Fan's Lemma we find two functions $f_0 \in B_{(X_{[r]})^*}$ and $g_0 \in B_{((Y^*)_{[r']})^*}$ such that, in particular,

$$\Big\| \sum_{j=1}^{m} |Tx_j\, y^*_j| \Big\|_{L_1(\nu)} \leq K\frac{r}{p}\int \big(\sum_{j=1}^{m} |x_j|^q \big)^{\frac{r}{q}} f_0\, d\mu + K\frac{r}{q}\int \big(\sum_{j=1}^{m} |y^*_j|^{p'} \big)^{\frac{r'}{p'}} g_0\, d\nu.$$

Taking into account the homogeneity of this expression, multiplying by positive constants α and β such that $\alpha \cdot \beta = 1$ —see for example the proof in [7, 19.2]—, we can find a minimum in this sum in order we get the inequality

$$\Big\| \sum_{j=1}^{m} |Tx_j\, y^*_j| \Big\|_{L_1(\nu)} \leq K\Big(\int \big(\sum_{j=1}^{m} |x_j|^q \big)^{\frac{r}{q}} f_0\, d\mu \Big)^{\frac{1}{r}} \cdot \Big(\int \big(\sum_{j=1}^{m} |y^*_j|^{p'} \big)^{\frac{r'}{p'}} g_0\, d\nu \Big)^{\frac{1}{r'}}.$$

By duality, and using Lemma 2.7, we find the inequality

$$\Big\|\Big(\sum_{j=1}^{m}\Big|\frac{Tx_j}{g_1}\Big|^p\Big)^{\frac{1}{p}}\Big\|_{L_r(\nu)} \le K\Big(\int\Big(\sum_{j=1}^{m}|x_j|^q\Big)^{\frac{r}{q}} f_1\,d\mu\Big)^{\frac{1}{r}} = K\Big\|\Big(\sum_{j=1}^{m}|x_j|^q\Big)^{\frac{1}{q}} f_1\Big\|_{L_r(\mu)}$$

for functions f_1 and g_1 defined as adequate powers of f_0 and g_0. This gives the factorization —obtained for the case when the inequality is considered just for $m=1$ and all possible vectors x— and also the (p,q)-regularity of operator $\hat{T}$ from $L_r(\mu)$ to $L_r(\nu)$.

The converse is straightforward, and so the proof is finished. □

A factorization for T as the one given in (iii) of Theorem 4.4 is usually called a *strong factorization of T through L^r-spaces.*

The characterization of Marcinkiewicz-Zygmund type inequalities given in [9] provides the following.

Corollary 4.5. *Assume that $1 \le r_1, r_2, p, q \le \infty$. Then $\mathcal{R}_{p,q}(L_{r_1}, L_{r_2}) = L(L_{r_1}, L_{r_2})$ in the following cases.*

- *If $q \le r_1 = r_2 \le p$.*
- *If $r_1 = r_2 = 1$ and $1 \le q \le p \le \infty$.*
- *If $r_1 = r_2 = \infty$ and $1 \le q \le p \le \infty$.*
- *If $1 \le r_2 \le r_1 < 2$ and there exists t such that $q \le t \le p$ and $r_1 < t \le 2$.*
- *If $2 < r_2 \le r_1 \le \infty$ and there exists t such that $q \le t \le p$ and $2 \le t < r_2$.*
- *If $1 \le r_2 \le 2 \le r_1 \le \infty$ and $q \le 2 \le p$.*

For $r_1 = r_2 = r$ and using Theorem 4.4, we obtain the following result.

Corollary 4.6. *Assume that $1 \le p, q \le \infty$. Let $1 < r < \infty$ and let X be an r-convex Banach function space over (Ω, Σ, μ), and Y an r-concave Banach function space over (Ω', Σ', ν) such that X and Y^* are order continuous. Let $T : X \to Y$ be an operator. Suppose that*

$$[q,p] \cap [\min\{r,2\}, \max\{r,2\}] \neq \emptyset.$$

Then the following assertions are equivalent.

(i) *There is a constant $K > 0$ such that for each matrix of elements $(x_{i,j})_{i=1,j=1}^{n,m}$ in X, the following inequality holds.*

$$\Big\|\Big(\sum_{i=1}^{n}\Big(\sum_{j=1}^{m}|Tx_{i,j}|^p\Big)^{\frac{r}{p}}\Big)^{\frac{1}{r}}\Big\|_Y \le K\Big\|\Big(\sum_{i=1}^{n}\Big(\sum_{j=1}^{m}|x_{i,j}|^q\Big)^{\frac{r}{q}}\Big)^{\frac{1}{r}}\Big\|_X.$$

(ii) *There is a strong factorization of T through L_r-spaces.*

Together with Theorem 4.1, this corollary provides an equivalence among (p,q)-regular-type properties for the operator T, that may be understood as a generalization of Marcinkiewicz-Zygmund inequalities. Let us finish the section with this result.

Corollary 4.7. *Under the assumptions on p, q, r, X and Y in Corollary 4.6, with*

$$[q,p] \cap [\min\{r,2\}, \max\{r,2\}] \neq \emptyset.$$

The following statements are equivalent.

(i) *The operator $T : X \to Y$ is r-regular.*

(ii) *There is a constant $K>0$ such that for each matrix of elements $(x_{i,j})_{i=1,j=1}^{n,m}$ in X, respectively, the following inequality holds.*

$$\Big\|\big(\sum_{i=1}^n\big(\sum_{j=1}^m|Tx_{i,j}|^p\big)^{\frac{r}{p}}\big)^{\frac{1}{r}}\Big\|_Y\le K\Big\|\big(\sum_{i=1}^n\big(\sum_{j=1}^m|x_{i,j}|^q\big)^{\frac{r}{q}}\big)^{\frac{1}{r}}\Big\|_X.$$

5. Extension properties

The definition of (p,q)-regular operator makes also sense for operators defined on a subspace of a Banach lattice: Given X, Y Banach lattices, and $X_0\subset X$ a closed subspace, an operator $T:X_0\to Y$ is (p,q)-regular if there is $K>0$ such that

$$\Big\|\Big(\sum_{i=1}^n|Tx_i|^p\Big)^{\frac{1}{p}}\Big\|\le K\Big\|\Big(\sum_{i=1}^n|x_i|^q\Big)^{\frac{1}{q}}\Big\|,$$

for every $(x_i)_{i=1}^n\subset X_0$ (with the obvious modification when p or q are infinite).

In [20, Theorem 4] it is shown that every ∞-regular operator defined on a closed subspace of a Banach lattice with values in another Banach lattice extends to a ∞-regular operator on the whole Banach lattice. This extension property also holds for (∞,q)-regular operators as the following shows.

Before the proof, let us recall the Calderón product construction (cf. [4]): for Banach lattices X_0, X_1 and $\theta\in(0,1)$ let $X_0^{1-\theta}X_1^\theta$ the space of elements f for which there exist $f_0\in X_0$, $f_1\in X_1$ such that $|f|\le|f_0|^{1-\theta}|f_1|^\theta$ and let

$$\|f\|_{X_0^{1-\theta}X_1^\theta}=\inf\{\|f_0\|^{1-\theta}\|f_1\|^\theta:|f|\le|f_0|^{1-\theta}|f_1|^\theta,\text{ with }f_i\in X_i\}.$$

This expression defines a norm when X_0 and X_1 are Dedekind complete Banach lattices (see [4]). In particular, this is the case for Banach lattices of measurable functions satisfying some convexity assumptions, which are those of interest in this section.

Lemma 5.1. *Let $1\le q\le\infty$, let X be a q-convex Banach lattice of measurable functions on (Ω,Σ,μ) (with q-convexity constant equal to one) and let $X_0\subset X$ be a closed subspace. Every (∞,q)-regular operator $T:X_0\to\ell_q^n$ has an (∞,q)-regular extension $\tilde T:X\to\ell_q^n$ with $\rho_{\infty,q}(\tilde T)\le\rho_{\infty,q}(T)$.*

Proof. We follow a similar approach to that of [20, Theorem 4]. Let Z be the tensor product $\ell_{q'}^n\otimes X$, where $\frac{1}{q}+\frac{1}{q'}=1$, endowed with

$$\|v\|_Z=\inf\Big\{\Big(\sum_{i=1}^n|a_i|^{q'}\Big)^{\frac{1}{q'}}\Big\|\Big(\sum_{i=1}^n|x_i|^q\Big)^{\frac{1}{q}}\Big\|_X:\ v=\sum_{i=1}^n a_ie_i\otimes x_i\Big\}.$$

Let us see that $\|\cdot\|_Z$ indeed defines a norm.

Let E_0 be the space of n-tuples of functions of $L_\infty(\mu)$ endowed with the norm

$$\|(g_i)_{i=1}^n\|_{E_0}=\sum_{i=1}^n\|g_i\|_\infty.$$

Let E_1 be the space of n-tuples of measurable functions $(h_i)_{i=1}^n\subset L_0(\mu)$ such that $|h_i|^{\frac{1}{q}}\in X$ for $i=1,\dots,n$, equipped with

$$\|(h_i)_{i=1}^n\|_{E_1}=\Big\|\Big(\sum_{i=1}^n|h_i|\Big)^{\frac{1}{q}}\Big\|_X^q.$$

This is indeed a norm since X is q-convex with constant 1.

We claim that $\|\cdot\|_Z$ coincides with the norm of the space $E_0^{1-\theta}E_1^{\theta}$ for $\theta = \frac{1}{q}$ under the identification mapping $(f_i)_{i=1}^n \in E_0^{1-\theta}E_1^{\theta}$ to $\sum_{i=1}^n e_i \otimes f_i \in \ell_{q'}^n \otimes X$.

Indeed, note first that for $1 \le i \le n$, there exist $g_i \in L_\infty(\mu)$ and $h_i \in L_0(\mu)$ with $|h_i|^\theta = |h_i|^{\frac{1}{q}} \in X$, such that $|f_i| \le |g_i|^{1-\theta}|h_i|^\theta$, which yield that $f_i \in X$ for $1 \le i \le n$, and in particular $\sum_{i=1}^n e_i \otimes f_i \in \ell_{q'}^n \otimes X$. Now, given $\varepsilon > 0$, let $(g_i)_{i=1}^n$ and $(h_i)_{i=1}^n$ as above so that

$$\|f\|_{E_0^{1-\theta}E_1^{\theta}} \ge \Big(\sum_{i=1}^n \|g_i\|_\infty\Big)^{\frac{1}{q'}} \Big\|\Big(\sum_{i=1}^n |h_i|\Big)^{\frac{1}{q}}\Big\|_X - \varepsilon.$$

Set $a_i = \|g_i\|_\infty^{\frac{1}{q'}}$, and $A_i = \{\omega \in \Omega : g_i(\omega) \ne 0\}$. Since $f_i = f_i\chi_{A_i}$, it follows that

$$|h_i| \ge \frac{|f_i|^q}{|g_i|^{\frac{q}{q'}}}\chi_{A_i}.$$

Hence,

$$\Big\|\Big(\sum_{i=1}^n |h_i|\Big)^{\frac{1}{q}}\Big\|_X \ge \Big\|\Big(\sum_{i=1}^n \frac{|f_i|^q}{|g_i|^{\frac{q}{q'}}}\chi_{A_i}\Big)^{\frac{1}{q}}\Big\|_X \ge \Big\|\Big(\sum_{i=1}^n \frac{|f_i|^q}{a_i^q}\Big)^{\frac{1}{q}}\Big\|_X.$$

Thus,

$$\|f\|_{E_0^{1-\theta}E_1^{\theta}} + \varepsilon \ge \Big(\sum_{i=1}^n a_i^{q'}\Big)^{\frac{1}{q'}} \Big\|\Big(\sum_{i=1}^n \Big|\frac{f_i}{a_i}\Big|^q\Big)^{\frac{1}{q}}\Big\|_X \ge \Big\|\sum_{i=1}^n a_i e_i \otimes \frac{f_i}{a_i}\Big\|_Z = \Big\|\sum_{i=1}^n e_i \otimes f_i\Big\|_Z,$$

and as $\varepsilon > 0$ is arbitrary, the inequality

$$\|f\|_{E_0^{1-\theta}E_1^{\theta}} \ge \Big\|\sum_{i=1}^n e_i \otimes f_i\Big\|_Z$$

follows. For the converse inequality, let $\varepsilon > 0$ and $f = \sum_{i=1}^n a_i e_i \otimes x_i$ with

$$\|f\|_Z + \varepsilon \ge \Big(\sum_{i=1}^n |a_i|^{q'}\Big)^{\frac{1}{q'}} \Big\|\Big(\sum_{i=1}^n |x_i|^q\Big)^{\frac{1}{q}}\Big\|_X.$$

Let $h_i = |x_i|^q$ and $g_i = |a_i|^{q'}\chi_\Omega$. Hence, as $|f_i| = |a_i x_i| \le (|a_i|^{q'})^{\frac{1}{q'}}(|h_i|)^{\frac{1}{q}} = |g_i|^{1-\theta}|h_i|^\theta$, it follows that

$$\|f\|_Z + \varepsilon \ge \Big(\sum_{i=1}^n \|g_i\|_\infty\Big)^{\frac{1}{q'}} \Big\|\Big(\sum_{i=1}^n |h_i|\Big)^{\frac{1}{q}}\Big\|_X \ge \|(g_i)_{i=1}^n\|_{E_0}^{1-\theta}\|(h_i)_{i=1}^n\|_{E_1}^{\theta} \ge \|f\|_{E_0^{1-\theta}E_1^{\theta}}.$$

As $\varepsilon > 0$ is arbitrary, the claim follows.

Now, we claim that $Z^* = R_{\infty,q}(X, \ell_q^n)$ isometrically. Indeed, given $u : X \to \ell_q^n$ we have

$$\begin{aligned}
\rho_{\infty,q}(u) &= \sup\Big\{\Big\|\bigvee_{i=1}^m |ux_i|\Big\|_{\ell_q^n} : \Big\|\Big(\sum_{i=1}^m |x_i|^q\Big)^{\frac{1}{q}}\Big\|_X \le 1,\, m\in\mathbb{N}\Big\}\\
&= \sup\Big\{\Big\|\sum_{k=1}^n \Big(\bigvee_{i=1}^m |\langle e_k^*, ux_i\rangle|\Big)e_k\Big\|_{\ell_q^n} : \Big\|\Big(\sum_{i=1}^m |x_i|^q\Big)^{\frac{1}{q}}\Big\|_X \le 1,\, m\in\mathbb{N}\Big\}\\
&= \sup\Big\{\sum_{k=1}^n a_k \bigvee_{i=1}^m |\langle e_k^*, ux_i\rangle| : \Big\|\sum_{k=1}^n a_k e_k^*\Big\|_{\ell_{q'}^n},\, \Big\|\Big(\sum_{i=1}^m |x_i|^q\Big)^{\frac{1}{q}}\Big\|_X \le 1,\, m\in\mathbb{N}\Big\}\\
&= \sup\Big\{\Big|\sum_{k=1}^n a_k\langle e_k^*, ux_{i_k}\rangle\Big| : \Big(\sum_{k=1}^n |a_k|^{q'}\Big)^{\frac{1}{q'}},\, \Big\|\Big(\sum_{k=1}^n |x_{i_k}|^q\Big)^{\frac{1}{q}}\Big\|_X \le 1, (i_k)_{k=1}^n \subset [1,m]\Big\}\\
&= \sup\{|\langle u, v\rangle| : \|v\|_Z \le 1\} = \|u\|_{Z^*}.
\end{aligned}$$

Finally, consider the subspace $M \subset Z$ formed by all $v = \sum_{k=1}^n a_k e_k \otimes x_k$ such that $x_k \in X_0$ for $k = 1, \ldots, n$. Let $T : X_0 \to Y$ be a (∞, q)-regular operator. Given $v \in M$, and $\varepsilon > 0$, take scalars a_k and $x_k \in X_0$ such that

$$\Big(\sum_{k=1}^n |a_k|^{q'}\Big)^{\frac{1}{q'}} \Big\|\Big(\sum_{k=1}^n |x_k|^q\Big)^{\frac{1}{q}}\Big\| \le \|v\|_Z + \varepsilon.$$

By [17, Proposition 1.d.2.], we have

$$\begin{aligned}
|\langle T, v\rangle| &= \Big|\sum_{k=1}^n \langle T(x_k), a_k e_k\rangle\Big|\\
&\le \langle \bigvee_{k=1}^n |T(x_k)|, \sum_{k=1}^n |a_k| e_k\rangle\\
&\le \Big\|\bigvee_{k=1}^n |T(x_k)|\Big\|_{\ell_q^n} \Big\|\sum_{k=1}^n |a_k| e_k\Big\|_{\ell_{q'}^n}\\
&\le \rho_{\infty,q}(T)(\|v\|_Z + \varepsilon).
\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, it follows that for every $v \in M$ we get $|\langle T, v\rangle| \le \rho_{\infty,q}(T)$. Hence, we can consider a Hahn-Banach extension of $v \in M \mapsto \langle T, v\rangle$ with norm not exceeding $\rho_{\infty,q}(T)$. This extension is clearly of the form $v \in Z \mapsto \langle \tilde{T}, v\rangle$ for some operator $\tilde{T} : X \to \ell_q^n$ with the required properties. □

Theorem 5.2. *Let $1 < q \le \infty$, and measure spaces (Ω, Σ, μ), (Ω', Σ', ν). Given a closed subspace $X_0 \subset L_q(\mu)$ and an (∞, q)-regular operator $T : X_0 \to L_q(\nu)$, there is an (∞, q)-regular extension $\tilde{T} : L_q(\mu) \to L_q(\nu)$ with $\rho_{\infty,q}(\tilde{T}) = \rho_{\infty,q}(T)$.*

Proof. For simplicity, assume $\Omega = \Omega' = [0, 1]$ endowed with Lebesgue measure. The proof can be easily carried over to general measure spaces. For every $n \in \mathbb{N}$, let $P_n : L_q \to \ell_q^{2^n}$ be given by

$$P_n f = 2^{\frac{n}{q'}} \sum_{i=1}^{2^n} \int_{\frac{i-1}{2^n}}^{\frac{i}{2^n}} f d\nu \cdot e_i.$$

Notice that the norm of P_n is less or equal than one. Also, let $J_n : \ell_q^{2^n} \to L_q$ be given by

$$J_n e_i = 2^{\frac{n}{q}} \chi_{[\frac{i-1}{2^n}, \frac{i}{2^n}]},$$

for $1 \leq i \leq 2^n$ and extended linearly. Its norm is equal to one: in fact it is an isometry.

Given a closed subspace $X_0 \subset L_q(\mu)$ and an (∞, q)-regular operator $T : X_0 \to L_q(\nu)$, for each $n \in \mathbb{N}$, consider $T_n = P_n T : X_0 \to \ell_q^{2^n}$. Since P_n is positive, we clearly have $\rho_{\infty,q}(T_n) \leq \rho_{\infty,q}(T)$, so by Lemma 5.1, there is an extension $\tilde{T}_n : L_q \to \ell_q^{2^n}$ with $\rho_{\infty,q}(\tilde{T}_n) \leq \rho_{\infty,q}(T)$.

Given $f \in L_q(\mu)$, the sequence $(J_n \tilde{T}_n f)_{n\in\mathbb{N}}$ is bounded in L_q-norm. Since $1 < q \leq \infty$, balls in L_q are weak-$*$ compact. Let $\mathcal{U}$ be a free ultrafilter in $\mathbb{N}$ and for each $f \in L_q(\mu)$, let

$$\tilde{T} f = \lim_{n\in\mathcal{U}} J_n \tilde{T}_n f,$$

the limit taken in the weak-$*$ topology along the ultrafilter $\mathcal{U}$.

It is clear that $\tilde{T}$ defines a bounded linear operator on L_q. We claim that $\tilde{T}$ is the required extension.

Indeed, for $f \in X_0$, note that

$$J_n \tilde{T}_n f = J_n P_n T f \underset{n\to\infty}{\longrightarrow} T f,$$

which implies in particular that $\tilde{T} f = T f$ for every $f \in X_0$.

Moreover, for every $(f_i)_{i=1}^m$ and every $g \in L_{q'}$ with $\|g\|_{q'} \leq 1$ we have

$$\begin{aligned}
\Big\langle \bigvee_{i=1}^m |\tilde{T} f_i|, g \Big\rangle &= \lim_{n\in\mathcal{U}} \Big\langle \bigvee_{i=1}^m |J_n \tilde{T}_n(f_i)|, g \Big\rangle \\
&\leq \lim_{n\in\mathcal{U}} \Big\| \bigvee_{i=1}^m |J_n \tilde{T}_n(f_i)| \Big\|_q \\
&\leq \lim_{n\in\mathcal{U}} \Big\| J_n \Big(\bigvee_{i=1}^m |\tilde{T}_n(f_i)| \Big) \Big\|_q \\
&\leq \lim_{n\in\mathcal{U}} \Big\| \bigvee_{i=1}^m |\tilde{T}_n(f_i)| \Big\|_q \\
&\leq \rho_{\infty,q}(T) \Big\| \Big(\sum_{i=1}^m |f_i|^q \Big)^{\frac{1}{q}} \Big\|.
\end{aligned}$$

Thus,

$$\Big\| \bigvee_{i=1}^m |\tilde{T} f_i| \Big\| \leq \rho_{\infty,q}(T) \Big\| \Big(\sum_{i=1}^m |f_i|^q \Big)^{\frac{1}{q}} \Big\|.$$

□

Remark 5.3. The analogous results for extension properties of regular operators on subspaces of L_1 and L_∞ can be found in [16, 20]. However, note that by Corollary 4.5, every operator $T : L_q \to L_q$ is (∞, q)-regular, while the projection onto the span of the Rademacher sequence in L_q is not ∞-regular.

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