Abstract

A dual conformal symmetry, analogous to the dual conformal symmetry observed for the scattering amplitudes of $\mathcal{N} = 4$ Super Yang-Mills theory, is identified in the Regge limit of QCD. Combined with the original two-dimensional conformal symmetry of the theory, this dual symmetry can potentially explain the integrability of the BFKL Hamiltonian. We also give evidence that the symmetry survives when a subset of $1/N_c$ corrections are taken into account by studying briefly the non-planar 2 to $m$ reggeon transition vertices. This provides a hint that the dual conformal symmetry of $\mathcal{N} = 4$ SYM is also present, in some form, at finite $N_c$.

1 Introduction

In the last few years there has been a great deal of progress in the study of gluon scattering amplitudes in the maximally supersymmetric gauge theory in four dimensions, $\mathcal{N} = 4$ Super Yang-Mills (SYM). One of the most surprising developments has been the discovery of a hidden symmetry in the planar ($N_c \to \infty$) limit, coined as “dual super-conformal symmetry” [1 2], different from the original super-conformal symmetry of the lagrangian. This symmetry was uncovered by introducing a new set of variables $x_i$, related to the external (all taken as incoming) gluon momenta $p_i$, $i = 1 \ldots n$, through

$$x_i - x_{i+1} = p_i,$$

and acts on the $x_i$ just as a four-dimensional conformal symmetry acts on spatial coordinates. The presence of this dual symmetry can be understood through the AdS/CFT
correspondence \[3\] since it was shown \[4\] that the problem of calculating a given scattering amplitude can be mapped, through a fermionic T-duality, to that of calculating a light-like Wilson loop with corners at coordinates given by the \(x_i\). This fermionic T-duality maps the string \(\sigma\)-model to itself, and the dual conformal symmetry becomes the ordinary symmetry of the space in which the Wilson loop lives.

Unlike the ordinary conformal symmetry, the dual symmetry is broken by infrared divergences, arising as cusp divergences in the language of Wilson loops. However, the cusp divergences are known to exponentiate, which allows the use of the broken symmetry to impose powerful constraints on the amplitudes in the form of anomalous Ward identities. These identities fix the 4 and 5 point amplitudes completely while the undetermined parts of higher-point amplitudes can only depend on dual-conformal invariants. Also, taken together, the original and dual conformal symmetries generate an infinite-dimensional Yangian symmetry \[5\], ordinarily characteristic of exactly solvable models.

Interesting properties of \(\mathcal{N} = 4\) amplitudes appear also in their high energy (Regge) limit. In a nutshell, Regge theory establishes the structure of scattering amplitudes when the momentum transfer is small compared to the total center-of-mass energy. It turns out that \(\mathcal{N} = 4\) amplitudes exhibit Regge-like behaviour at all orders in the ’t Hooft coupling even outside of the Regge limit. In fact, the 4 and 5 point amplitudes are Regge exact \[2, 6\], meaning that they can always be written in a factorized form characteristic of high energies, irrespective of the values of the kinematical invariants \[3\]. Furthermore, in the Leading Logarithmic Approximation (LLA) of gluon amplitudes the Regge limit is independent of the gauge theory, so \(\mathcal{N} = 4\) can give insight into the high energy behaviour of QCD.

In the Regge limit amplitudes are dominated by the \(t\)-channel exchange of reggeized gluons (a reggeized gluon is a collective state of ordinary gluons projected on a colour octet). The bound state of two reggeized gluons when projected on a colour singlet in the \(t\)-channel is known as the hard (or perturbative) pomeron. The interaction between reggeized gluons is governed by the Schrödinger-like BFKL integral equation \[8\] where the invariant mass of \(s\)-channel gluons can be interpreted as the time variable and its kernel as an effective Hamiltonian living on the two-dimensional transverse space. This Hamiltonian is free from infrared singularities and carries a yet to be understood integrability \[10\].

The question that arises is if the integrable structures present in \(\mathcal{N} = 4\) SYM can shed some light on the integrability found in the Regge limit. In this region the dynamics of the

\[1\] Also, a proposal for the undetermined part of the 6 point amplitude, having the correct Regge behaviour and given in terms of conformal cross-ratios, is given in the latest version of \[7\].

\[2\] Integrability also appears when the gluon composite states are projected onto the adjoint representation \[11\].
theory is reduced to the transverse plane, where a two-dimensional conformal symmetry was found in the effective Hamiltonian \([9]\). Given the emergence of the Yangian in the four-dimensional case, and that one can heuristically interpret this \(SL(2, C)\) as a reduction of the four-dimensional ordinary conformal symmetry, it would then seem natural to look for a dual \(SL(2, C)\) symmetry in the high-energy limit\(^4\). It is this question that we address in this letter, showing that BFKL indeed exhibits covariance under such a dual symmetry.

The study of the high-energy limit may also give some information back to \(\mathcal{N} = 4\) SYM. At present, the origin of dual conformal symmetry is only understood for planar amplitudes, since the fermionic T-duality is only well-defined in this case. For this reason we will study a set of non-planar corrections to BFKL, in the form of \(2 \to m\) reggeized gluon transition vertices, which also turn out to be dual \(SL(2, C)\)-covariant. This result is also important for high-energy QCD, since the inclusion of such vertices is necessary, at sufficiently high energies, to fulfill unitarity in all channels.

2 The dual \(SL(2, C)\) symmetry

![BFKL integral equation](image1)

Figure 1: The BFKL integral equation for the four-point reggeized gluon Green function.

The scattering amplitude for the \(2 \to 2\) reggeized gluons process in the Regge limit has an iterative structure dominated by the exchange in the \(t\) channel of a colour singlet. This implies that in the LLA the corresponding 4 point gluon Green function can be written as the solution to an integral equation, the BFKL equation, shown in Fig.1. Written in terms

\(^4\)Hints in this direction have already appeared in the litterature. A similar dual symmetry was exploited in \([11]\) in order to map supersymmetric multiparticle amplitudes in multiregge kinematics to an integrable open spin chain. In fact, the octet kernel, after subtraction of infrared divergences, can be written in a form manifestly invariant under this symmetry. Also, the BFKL Hamiltonian has holomorphic separability into two pieces which can be written such that they are invariant under the duality transformation \(p_i \to \rho_i - \rho_{i+1} \to p_{i+1}\), similar to the change of variables \((1.1)\), with the \(\rho\) being the gluon transverse coordinates in complex notation \([12]\).
of $\omega$, the Mellin conjugate variable of the center-of-mass energy (which can be translated into the rapidity, $Y$, of the emitted particles in the $s$-channel), and the incoming two dimensional momenta it reads

$$\omega F(\omega, k_A, k_B, q) = \delta^2(k_A - k_B) + \int d^2k' K(k_A, k_A - q; k', k' - q) F(\omega, k', k_B, q)$$

(2.1)

where the kernel $K(k_A, k_A - q; k', k' - q)$ is given by

$$\frac{K_R(k_A, k_A - q; -k' + q, -k')}{{8\pi}^3 k_A^2 (k' - q)^2} + \left[\omega(k_A^2) + \omega((k_A - q)^2)\right] \delta^2(k_A - k').$$

(2.2)

The “real emission” part has the following structure

$$K_R(p_1, p_2; p_3, p_4) = -N_c g^2 \left[ (p_3 + p_4)^2 - \frac{p_2 p_3^2}{(p_2 + p_3)^2} - \frac{p_1^2 p_4^2}{(p_1 + p_4)^2} \right].$$

(2.3)

This notation, with $p_1, \ldots, p_4$ being the cyclically ordered reggeized gluon momenta taken as incoming, will be convenient for the generalization of this vertex to the $2 \rightarrow m$ reggeized gluon transition case as we will see below.

The gluon Regge trajectory reads

$$\omega(q^2) = -\frac{g^2 N_c}{16\pi^3} \int d^2k' \frac{q^2}{k'^2 (k' - q)^2}.$$  

(2.4)

We will now show that the BFKL equation in Eq. (2.1) exhibits formally a dual $SL(2, C)$ symmetry, which, in contrast with the original $SL(2, C)$ symmetry of BFKL, uncovered by Fourier transforming into a coordinate representation, is realized in the transverse momentum space. This new symmetry is closely analogous to the dual conformal symmetry observed in $\mathcal{N} = 4$ SYM for gluon scattering amplitudes, and we will see that it turns out to be broken by infrared effects just as in the four dimensional gauge theory.

Let us now rewrite Eq. (2.1) in terms of dual variables. Taken as incoming, the external momenta are $k_A, -k_A + q, k_B - q$ and $-k_B$ so, introducing the notation $x_{i,j} \equiv x_i - x_j$, we define the new set of variables as

$$p_1 = x_{1,2} = k_A, p_2 = x_{2,3} = q - k_A, p_3 = x_{3,4} = k_B - q, p_4 = x_{4,1} = -k_B.$$  

(2.5)

Equivalently, we could have written $k_A = x_{1,2}, k_B = x_{1,4}, q = x_{1,3}$ with $x_1$ then being a simple shift of the origin for the external momenta.

In these new variables the gluon Regge trajectory is

$$\omega(k_A^2) = \omega(x_{1,2}^2) = -\frac{g^2 N_c}{16\pi^3} \int d^2x_1 \frac{x_{1,2}^2}{x_{1,1}^2 x_{1,2}^2},$$

(2.6)

Due to the optical theorem when the forward $q = 0$ limit is taken this piece in the kernel corresponds to the contribution to multiparticle production from on-shell gluons in the $s$-channel.
where we have introduced $x_I$ through $k' = x_{I,2}$. This expression has a formal two-dimensional conformal symmetry. It is invariant under translations, rotations and scalings of the $x_i$, and also under the conformal inversions $x_i \rightarrow \frac{x_i}{x_i^2}$, since under inversions

$$d^2x_I \rightarrow \frac{d^2x_I}{x_I^2}, \quad x_{i,j}^2 \rightarrow \frac{x_{i,j}^2}{x_i^2 x_j^2}. \quad (2.7)$$

In the same way $\omega((k_B - q)^2) = \omega(x_{2,3}^2)$ is also formally conformally invariant. Now, given that the trajectory is infrared divergent one would expect this symmetry to be broken by the introduction of a regulator. However, in the BFKL equation such divergences cancel allowing for the possibility that the symmetry remains.

Rewriting the full kernel (2.2) in terms of the $x_i$, with $k' = x_{1,I}$, we get

$$K(x_{1,2}, x_{3,2}; x_{1,I}, x_{3,I}) = \frac{K_R(x_{1,2}, x_{3,2}; x_{3,I}, x_{1,I})}{8\pi^3 x_{1,2}^2 x_{2,3}^2} + \left[\omega(x_{1,2}^2) + \omega(x_{2,3}^2)\right] \delta^{(2)}(x_{2,I}) \quad (2.8)$$

where

$$K_R(x_{1,2}, x_{2,3}; x_{3,I}, x_{1,I}) = -N_c g^2 \left[ x_{1,3}^2 - \frac{x_{2,3}^2 x_{1,1}^2}{x_{2,I}^2} - \frac{x_{1,2}^2 x_{3,1}^2}{x_{2,I}^2} \right]. \quad (2.9)$$

Using that $\delta^{(2)}(x_{2,I}) \rightarrow x_{2,I}^2 \delta^{(2)}(x_{2,I})$ under conformal inversions one then finds immediately that the kernel transforms covariantly\(^5\)

$$K(x_{1,2}, x_{3,2}; x_{1,I}, x_{3,I}) \rightarrow x_{2,I}^2 x_{I}^2 K(x_{1,2}, x_{3,2}; x_{1,I}, x_{3,I}). \quad (2.10)$$

Together with translations, rotations and dilatations this forms a dual $SL(2, C)$ symmetry, different from the one previously known. More precisely, dilatations and rotations coincide with the original $SL(2, C)$-symmetry, while translations and inversions will be different.

Applied to the BFKL equation (2.1) and using that the integration measure transforms according to (2.1) one finds that a factor of $\frac{x_2^2}{x_I^2}$ is produced inside the integral. Consequently, if the Green function $F(\omega, x_{1,2}, x_{1,4}, x_{1,3})$ were to produce a factor of $x_2^2$ upon inversion, then its convolution with the kernel $K \otimes F$, would transform in the same way as $F$ itself. Now, at lowest order, $F$ is simply given by the delta function, which indeed transforms in this way, $\delta^{(2)}(k_A - k_B) = \delta^{(2)}(x_{2,4}) \rightarrow x_2^2 x_4^2 \delta^{(2)}(x_{2,4})$. Since $F$ can be constructed through iterated convolution with the kernel, it follows that the Green function should have the same conformal properties as the delta function.

We can obtain a formal expression for the Green function having the correct conformal properties by iteration. Introducing the short-hand notation

$$\omega_0(k_A, q) \equiv \omega(k_A^2) + \omega((k_A - q)^2), \xi(k, k_A, q) \equiv K_R(k_A, k_A - q; -k + q, -k) \quad \frac{8 \pi^3 k_A^2 (k - q)^2}{(2.11)}$$

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\(^5\)This is again similar to the dual conformal symmetry of $N = 4$ SYM, under which scattering amplitudes transform covariantly, as opposed to the ordinary conformal symmetry which leave them invariant.
one finds (with $k_0 \equiv k_A$):

$$F (\omega, k_A, k_B, q) = \frac{\delta^{(2)} (k_A - k_B) + \sum_{n=1}^{\infty} \prod_{i=1}^{n} \int d^2 k_i \frac{\delta^{(2)} (k_i - k_{i-1})}{\omega - \omega_0 (k_i, q)} \delta^{(2)} (k_n - k_B)}{\omega - \omega_0 (k_A, q)}.$$  

(2.12)

Rather than $\omega$ it is more natural to use the rapidity difference, $Y$, between the external particles as the evolution variable. To this end we perform the inverse Mellin transform

$$\mathcal{F} (k_A, k_B, q, Y) = \int_{a-i\infty}^{a+i\infty} \frac{d\omega}{2\pi i} e^{\omega Y} F (\omega, k_A, k_B, q).$$

(2.13)

The formula $\int_{a-i\infty}^{a+i\infty} \frac{d\omega}{2\pi i} e^{\omega Y} \prod_{i=0}^{n-1} \frac{1}{\omega - \omega_i} = e^{\omega_0 Y} \prod_{i=1}^{n} \int_0^{\omega_i} d\omega_i \omega_i e^{\omega_i, i-1, y_0}$ for $n > 0$, with $\omega_i \equiv \omega_i - \omega_j, y_0 \equiv Y$, is useful to obtain the final expression, written in dual $x$-variables:

$$\mathcal{F} (x_{12}, x_{14}, x_{13}, Y) = e^{\omega_{2,1} Y} \left\{ \delta^{(2)} (x_{24}) + \sum_{n=1}^{\infty} \prod_{i=1}^{n} \int_0^{y_{i-1}} dy_i \int d^2 x_{i, i-1} e^{\omega_{i, i-1, y_i} \delta^{(2)} (x_{i, i-1})} \right\},$$

(2.14)

where

$$\omega_{i, i-1} = \omega_0 (x_{i, i}, x_{i-1}) - \omega_0 (x_{i, i-1}, x_{i-1}),$$

(2.15)

$$\xi_{i, i-1} = \frac{\bar{\alpha}_s}{2\pi} \left\{ \frac{x_{i, i}^2 x_{i-1, i-1}^2 + x_{i-1, i} x_{i, i-1}^2 - x_{i, i-1}^2 x_{i-1, i}^2}{x_{i, i}^2 x_{i-1, i} x_{i-1, i}^2} \right\}.$$  

(2.16)

This representation preserves the transformation properties of the original equation. In the forward case, where the momentum transfer is zero, the same structure remains with

$$\omega_{i, i-1} = 2 (\omega (x_{i, i}) - \omega (x_{i, i-1})) , \xi_{i, i-1} = \frac{\bar{\alpha}_s}{\pi} \frac{1}{x_{i-1, i}^2}.$$  

(2.17)

In this case the solution also has a formal dual $SL(2, C)$ covariance. This should be contrasted with the original $SL(2, C)$-invariance of the BFKL kernel, which does not appear in the forward case.

Before ending this section, it is noteworthy to mention that this dual $SL(2, C)$ covariance is present in the same form for all color projections in the $t$-channel since they only differ by a different factor in front of $K_R$: with $N_c = 3, c_1 = 1, c_{8a} = c_{8s} = 1/2, c_{10} = c_{10} = 0$, etc.

3 The effect of IR divergences

In $\mathcal{N} = 4$ SYM infrared divergences break the dual conformal symmetry. For BFKL, such divergences cancel, opening the possibility that the dual $SL(2, C)$-symmetry remains
exact. However, this turns out not to be the case. Perhaps the simplest way to see this is by studying the forward case. If \( F \) has the transformation properties of the delta function it can be written as
\[
F = F_1 \delta^{(2)}(k_A - k_B) + \frac{1}{(k_A - k_B)^2} F_2 ,
\] (3.1)
where \( F_1 \) and \( F_2 \) are dual conformally invariant, since \((k_A - k_B)^{-2}\) is the only other function that transforms correctly. When \( q = 0, x_1 = x_3 \) and no non-trivial conformal invariant can be formed from the three remaining \( x_i \). \( F_2 \) can thus only be a function of \( \omega \) (or equivalently the rapidity \( Y \)), and the coupling. But when forming physical quantities one integrates over \( k_A \) and \( k_B \) and the divergences at \( k_A = k_B \) must cancel between \( F_1 \) and \( F_2 \). The factor \((k_A - k_B)^{-2}\) is singular enough to cancel one factor of the trajectory, but \( F_1 \) is obtained by repeated application of the trajectory part of the kernel so, starting from the second iteration, products of two or more trajectories will appear and the divergences will fail to cancel.

One can also observe the breakdown of the dual \( SL(2, C) \) symmetry directly by regularizing the integrals and cancelling the divergences explicitly when performing the iteration. One then finds that the first iteration respects the symmetry, while the second iteration produces a contribution to \( F_2 \) (when \( q = 0 \)) proportional to an anomalous factor of the form \( \ln \left( \frac{(k_A - k_B)^4}{k_A^2 k_B^2} \right) \), which breaks the symmetry under inversions. The origin of this factor is the regularization of infrared divergences. For example, using dimensional regularization with \( D = 4 - 2\epsilon \)
\[
\omega(x^2_{1,2}) = -\frac{g^2 N_c}{16\pi^3} (4\pi \mu)^{2\epsilon} \int d^{2-2\epsilon} x_1 \frac{x^2_{1,2}}{x^2_{1,1} x^2_{1,2}} \approx -\frac{g^2 N_c}{8\pi^2} (4\pi e^{\gamma})^\epsilon \left( \ln \frac{x^2_{1,2}}{\mu^2} - \frac{1}{\epsilon} \right) .
\] (3.2)
The divergences will cancel between the trajectories and the real emission part of the kernel, but factors such as \( \ln x^2_{1,2} \) will add up giving a non-vanishing anomalous term. So, even though BFKL is infra-red finite, a remnant of the divergences remains in the form of the breaking of the dual \( SL(2, C) \) symmetry.

Further insight can be gained by studying a standard representation of the Green function in the forward case, obtained by diagonalizing the BFKL kernel. It is
\[
\mathcal{F}(x_{1,2}, x_{1,4}, Y) = \sum_{n=-\infty}^{\infty} \int \frac{d\gamma}{2\pi i} \frac{x^2_{1,2}}{x^2_{1,1} x^2_{1,2}} \gamma^{-\frac{1}{2}} e^{\alpha_n(\gamma) Y + in\theta_{2,4}} \frac{\chi_n(\gamma)}{\pi \sqrt{x^2_{1,2} x^2_{1,4}}} ,
\] (3.3)
with \( \chi_n(\gamma) = 2\Psi(1) - \Psi \left( \gamma + \frac{|n|}{2} \right) - \Psi \left( 1 - \gamma + \frac{|n|}{2} \right) \) and \( \cos \theta_{2,4} = \frac{x_{1,2} x_{1,4}}{\sqrt{x^2_{1,2} x^2_{1,4}}} \). In this representation any dependence on an IR cutoff has canceled explicitly, and one can check that the covariance under conformal inversions is lost.
4 Non-planar vertex

The BFKL amplitude will violate bounds imposed by unitarity at sufficiently high energies. In order to restore unitarity, one of the new elements that must be introduced is a vertex in which the number of reggeized gluons in the \( t \)-channel is not conserved. As shown in Fig. 2 we choose to write this \( 2 \rightarrow m \) vertex (see, for example, Eq. (3.57) of [13]) using a convenient assignment of the momentum indeces.

\[
K_{2 \rightarrow m}^{(b-a)}(p_2, p_3, p_4, \ldots, p_{m+2}, p_1) = f_{a_1 b_1 c_1} f_{c_1 a_2 c_2} \cdots f_{c_{m-1} a_m b_2} g^m
\times \left[ (p_4 + \cdots + p_1)^2 - \frac{p_5^2 (p_5 + \cdots + p_1)^2}{(p_3 + p_4)^2} - \frac{p_6^2 (p_4 + \cdots + p_{m+2})^2}{(p_1 + p_2)^2} + \frac{p_5^2 p_6^2 (p_5 + \cdots + p_{m+2})^2}{(p_1 + p_2)^2 (p_3 + p_4)^2} \right],
\]

where the \( a_1, b_1 \) etc are the color indeces of the reggeized gluons and \( f_{ijk} \) the structure constants of \( SU(N_c) \).

![Figure 2: The 2 \rightarrow m reggeized gluon vertex. All momenta are taken as ingoing.](image)

Written in terms of \( x \) variables this becomes

\[
K_{2 \rightarrow m}^{(b-a)}(x_{23}, x_{34}, x_{45}, \ldots, x_{m+2,1}, x_{12}) = f_{a_1 b_1 c_1} f_{c_1 a_2 c_2} \cdots f_{c_{m-1} a_m b_2} g^m \left[ x_{24}^2 - \frac{x_{34}^2 x_{25}^2}{x_{35}^2} - \frac{x_{24}^2 x_{14}^2}{x_{13}^2} + \frac{x_{23}^2 x_{34}^2 x_{15}^2}{x_{13}^2 x_{35}^2 x_{15}} \right],
\]

and is manifestly conformally covariant. The assignment of the momenta in [14] was chosen so that the vertex takes a form independent of \( m \) when written in terms of the \( x_i \). Note that the last term vanishes when \( m = 2 \) since then \( x_1 = x_5 \), and one recovers the corresponding term in the BFKL kernel.

In [14] it was shown that the \( 2 \rightarrow 4 \) reggeized gluon vertex exhibited the same coordinate representation \( SL(2, C) \)-invariance as the BFKL equation. This was taken to indicate...
that a unitary, two-dimensional CFT describing scattering amplitudes in the Regge limit should exhibit this $SL(2, C)$-invariance. Our results would seem to indicate that such a theory should also be covariant under the dual $SL(2, C)$.

5 Conclusions

We have shown that not only does the LLA BFKL kernel, and its extension in the form of the $2 \rightarrow m$ reggeized gluon vertex, exhibit the ordinary $SL(2, C)$-symmetry, found by Lipatov but also a dual $SL(2, C)$, analogous to the dual conformal symmetry of $\mathcal{N} = 4$. It is tempting to interpret these symmetries as reductions to the transverse plane of the conformal and dual conformal symmetries of the supersymmetric theory, although it is not clear exactly how such a reduction should be carried out. Purely transverse versions of the conformal algebras are not symmetries of the 4-dimensional gauge theory amplitudes, but seem to emerge in the Regge limit.

The dual $SL(2, C)$ covariance of the non-planar $2 \rightarrow m$ reggeized gluon vertex would seem to hint at the existence of dual conformal symmetry beyond the planar limit also in $\mathcal{N} = 4$ SYM. This is interesting since the dual conformal symmetry is motivated, at the moment, by the AdS/CFT correspondence and the existence of a fermionic T-duality. The current understanding of this T-duality is that it is only valid in the planar limit, since it is taken along non-compact directions, but our result would seem to indicate that a non-planar version might exist, at least in the high-energy limit. Also, the dual invariance of the reggeized gluon vertex suggests that a unitary two-dimensional CFT describing high-energy gauge theory should have both $SL(2, C)$ groups. In future work, having identified the dual $SL(2, C)$ one can try to understand the origin of the integrability of the Regge limit in terms of the integrability of $\mathcal{N} = 4$ SYM.

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