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Optimal Crowdfunding Design*

Matthew Ellman† and Sjaak Hurkens†

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Abstract

This paper characterizes profit- and welfare-maximizing reward-based crowdfunding, defined by an aggregate funding threshold for production. We disentangle crowdfunding's selling and funding roles, locating its key benefit in its market test role of adapting production to demand. Multiple prices prove necessary for effective learning and adaptation, even with relatively large crowds. Mechanism design proves general optimality in our baseline and shows the value of limiting reward quantities. Funding is not fundamental and crowdfunding may even complement traditional finance. We characterize welfare consequences, model price dynamics and identify platform designs and regulations that enhance innovation and social benefits.

Keywords: Crowdfunding, mechanism design, entrepreneurial finance, market-testing, adaptation, rent-extraction, threshold commitment.
JEL Classifications: C72, D42, L12.

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†Institute for Economic Analysis (CSIC) and Barcelona GSE.
Emails: matthew.ellman@iae.csic.es, sjaak.hurkens@iae.csic.es.
‡Ellman and Hurkens (2016), henceforth EH2016, contains supplementary results.

1 Introduction

Crowdfunding is a rapidly growing phenomenon with a major promise: to bring more socially beneficial projects to fruition. Online crowdfunding platforms have sharply reduced entrepreneurs' costs of pitching their projects to a wide range of potential funders before sinking the costs of production, C. We study the prominent case of reward-based crowdfunding where funders are compensated with the project's product. So the funders are buyers. Each buyer chooses a bid after the entrepreneur sets a funding *threshold*, T and a *minimum price*, p. Production occurs in the "success" event where the aggregate funds or sum of bids reaches the threshold.[1] That is, a simple aggregate fund threshold (AFT) fully determines production. The entrepreneur then receives these funds and has to sink C and deliver her product to all buyers who bid at least p. Buyers can rest assured that (1) they pay nothing when funding fails and (2) they pay exactly their bids when it succeeds; so bids are prices. Together with crowdfunding's defining characteristic AFT, these reassuring properties explain why so many small funders are willing to participate.[2]

We characterize optimal crowdfunding design with two buyer types and both for-profit and not-for-profit entrepreneurs. Using mechanism design, we find the general optimum (Propositions 1 and 2 in Section 3) and then show how crowdfunding implements it (Propositions 3 and 4 in Section 4). Our analysis disentangles crowdfunding's selling and funding roles. We locate its main benefit as a market test that adapts production to revealed demand: by only producing when the threshold is reached, crowdfunding avoids sinking C when demand is low. The threshold's implicit threat of non-production also improves rent-extraction. Proposition 5 derives the welfare implications.

Funding is not fundamental here: even an entrepreneur with no credit constraints uses crowdfunding for adapting to demand and extracting rent. With credit constraints, crowdfunding can, as its name suggests, substitute for traditional finance (Corollary 3), but we identify complementarities when entrepreneurs have not-for-profit motivations (Corollary 4) and when campaigns cannot reach all potential buyers (Proposition 12).

We now describe the main results in our baseline model with profit-maximization where a finite number of buyers have independent valuations that are either high (v_H) or low (v_L). Optimal crowdfunding design, like traditional selling, hinges on whether the high type frequency exceeds the ratio of low to high valuations. Above this critical value (high frequency), price is set at the high valuation, $p = v_H$, which implies that low types are excluded (E), while below it (low frequency), $p = v_L$ and low types are included (I). In the high frequency (exclusion) case, crowdfunding adapts production to demand via

[1] Our analysis applies more generally to include discrete investment problems with pre-ordering or contribution pledges and an aggregate threshold, as used for new product designs (*e.g.*, jet engines) and excludable public goods (*e.g.*, R&D in the pharmaceutical industry, see Sahm (2015)).

[2] Massolution (2015) provide statistics for 2014: 3.3 million backers pledged 529 million dollars, generating over 22,000 successfully financed projects on the major reward-based platform Kickstarter, alone. Overall, global crowdfunding raised 16.2 billion dollars with large future increases predicted.

a threshold set at cost ($T = C$) to preclude losses. This induces production when the number of high types exceeds a cutoff which we call the pivot, n_E. The low frequency (inclusion) case is more interesting. Crowdfunding again adapts production, but now sets $T > C$ to extract rent from high types by making them sometimes pivotal for production. The optimal threshold trades off higher rents against lower success rates. Each high type pays less than with exclusion but thanks to the payments from low types under inclusion, the pivotal number of highs needed for production is lower ($n_I \leq n_E$). So production is more likely despite the higher threshold. Crowdfunding's adaptation of production to demand raises welfare in general, but excessively high thresholds in the low frequency (inclusion) case waste some production opportunities and can lower welfare.

Illustration 1. *Esther wants to produce a CD but recording costs 2650. True fans value her music at 20, others are only willing to pay 5 and most simply have no interest. She targets 500 people who have some interest (friends, family and followers), whose values are i.i.d. draws from* $\{5, 20\}$ *with the high value,* 20*, having (i) probability,* $q = 0.3$ *(high frequency) or (ii)* $q = 0.2$ *(low frequency). With traditional selling, Esther only produces in case (i). She then sells at 20, generating an expected profit and welfare of 350. In case (ii), her best posted price of 5 would give a loss of 150. Crowdfunding helps in both cases by adapting production to actual demand:*

(i) When $q = 0.3$*, a minimum price of 20 and threshold of 2650 raise maximized expected profit to 353.47 by avoiding losses when demand is low (below* n_E *=133 fans).*

(ii) When $q = 0.2$*, a minimum price of 5 and 2726.80 threshold motivate fans to pay 7.10, a premium of 2.10, to raise the chance of project success and getting the CD (*n_I *=108 fans are needed to reach the threshold). This raises expected profit to 17.30 and welfare to 307.82.*

Crowdfunding's profit gains are small in this illustration, because the relatively large crowd and *i.i.d.* valuations restrict per capita demand uncertainty.[3] Even so, the impacts on project success rates and social welfare are substantial, especially when motives extend beyond short-run profits, as we explain below. Notice that, despite the low pivotality motive that limits rent-extraction, the resulting multiple prices are crucial for profit and welfare gains. Imposing a single crowdfunding price in case (ii) would induce an exclusive strategy with a minimum price of 20, reducing project success rates to just 0.02% and decimating profits and welfare to 0.01. Sections 5.4 and 6 show that crowdfunding is important for pricing and profits in arbitrarily large crowds when buyers' valuations are correlated since demand uncertainty then remains substantial.

The not-for-profit solution is a little more involved. We model the idealized case where they maximize expected welfare under the constraint that expected profits cannot

[3] Per capita profits rise 9-fold on downscaling by a factor of 10 to a typical crowd size of 50.

be negative.[4] At low frequency, they maintain inclusion but reduce the threshold and corresponding production pivot as far as possible. In case (ii) of Illustration 1, Esther lowers her threshold to 2644.84, needing only 102 fans (paying 6.42 each). This lower production pivot more than doubles the success rate to 42.9% and welfare to 632.17.

At high frequency, welfare-maximizers still want to include low types, but deficit-avoidance may then require a production pivot above n_E. In case (i) of Illustration 1, Esther can raise welfare to 1227.59 by setting minimum price 5 and threshold 2647.51, but this requires at least 149 fans (paying 5.99 each). Our general mechanism design solution indicates that such entrepreneurs should instead adopt a partially inclusive solution where low types get the good with a probability ϕ less than one. We now describe two ways crowdfunding can implement the partially inclusive solution for case (i) by adding a modified reward. First, Esther could sell lottery tickets giving a $\phi = 0.74$ chance of winning a CD, at 3.70, as well as selling the CD at price 9. Combined with a threshold of 2554.90, this raises expected welfare from 1227.59 to 1589.91. Second, Esther could limit the set of low-priced rewards: setting threshold 2487 and offering an unlimited number of CDs at a price of 9 but only 258 at price 5 rations low types with probability $1 - \phi$.

Multiple rewards are an important feature of crowdfunding platforms and a recurring theme of our study. First, while our baseline treats a single product, the two prices in our inclusive solution are often coordinated via two rewards: the basic product and the product plus some token of appreciation to thank the high price bidder. Second, welfare-maximization may add stochastic or limited rewards for low types as just described. Third, in Section 5.2 we prove that product differentiation and crowdfunding are orthogonal tools for price discrimination (Proposition 8): crowdfunding does not change optimal differentiation, though differentiating optimally, by raising profitability, does weakly lower the optimal crowdfunding pivot.

Moving beyond the focus on entrepreneurs and buyers, Section 5 also analyzes the role of crowdfunding platforms, who create value by matching those two sides. One important platform decision is whether to ban or allow self-bidding. Entrepreneurs have incentives to bid on their own projects when funds exceed cost but fall short of the threshold. Self-bidding undermines "threshold-commitment", the commitment not to produce when the threshold is not reached, and thus affects rent-extraction. We characterize optimal crowdfunding under no commitment (Propositions 6 and 7). Platforms charge fees proportional to revenues and we characterize when platforms are biased towards promoting and attracting profit-maximizers or not-for-profits (Proposition 9).

Section 6 extends the model to allow for post-crowdfunding sales and uncertain project quality and derives profit-maximizing crowdfunding design (Proposition 11). Ex-post sales are a key source of complementarity between crowdfunding and traditional finance (Proposition 12). Project quality shocks generate correlation and imply substan-

[4] The analysis for career-motivated entrepreneurs and success- and audience-maximizers is similar.

tial crowdfunding benefits for arbitrarily large markets (see also Proposition 10).[5] This extension explains observed pricing dynamics and shows how reward rationing also raises post-crowdfunding price credibility.

Section 7 characterizes the general solution with multiple types and discusses how crowdfunding can and must be adjusted to implement it. Section 8 concludes.

Related Literature

The field of crowdfunding has become quite crowded (Agrawal et al. (2014) and Belleflamme et al. (2015) survey early empirics and theory) but our work stands out: we characterize the *optimal* crowdfunding mechanism, demonstrating its adaptation value as an incentive-compatible market test and the role of multiple prices.

Our analysis harks back to the public goods literature on heterogenous private contributions towards a common goal: (a) the abstract general mechanism approach (as in Cornelli, 1996) neglects AFT which is central to crowdfunding; (b) the simple intuitive, contribution and subscription, games that *do* impose AFT have only been solved fully with complete information or two players – hardly a crowd![6] We make progress on (a) and (b) by characterizing the general optimum for a binary type space and any crowd size, and then showing that crowdfunding can implement this optimum. By explicitly characterizing welfare-maximizing entrepreneurs as well as profit-maximizers, we identify a role for stochastic inclusion and pivots. Our results deviate from the well-known efficiency results (e.g., d'Aspremont and Gérard-Varet (1979) and d'Aspremont et al. (2004)) on Bayesian implementation in the public goods literature. This is because the crowdfunding context requires interim individual rationality whereas efficiency only holds under the weaker restriction of ex-ante individual rationality. Finally, our indirect mechanism analysis is vital for studying the role of threshold commitment in crowdfunding.

Among crowdfunding models, ours is the only paper to address multiple prices; all other papers impose a unique crowdfunding price. Multiple prices are salient in practice and matter in theory: they can substantially increase efficiency by enhancing demand adaptation; they introduce the risk of excessive extraction that underlies our threshold commitment results; they reduce the need for credit and they facilitate learning demand.

The prior work of Belleflamme et al. (2014) assumed crowdfunders enjoy warm-glow proportional to their consumer valuations. High types pay a crowdfunding premium over a regular ex-post price. We show warm-glow is unnecessary (but compatible), by moving to a finite crowd where high types pay a premium to pivot into production.

Two contemporaneous papers also model pivotality-based price discrimination. Kumar et al. (2015) study a continuum of consumers contradicting their pivotality claim and

[5] Indeed, larger crowds provide more accurate profitability signals for financiers and ex-post pricing.

[6] See *e.g.,* Cornelli (1996); Ledyard and Palfrey (2007); Schmitz (1997) on (a) and Alboth et al. (2001); Bagnoli and Lipman (1989); Barbieri and Malueg (2010); Menezes et al. (2001) on (b).

precluding aggregate demand uncertainty to which to adapt. Finiteness is necessary to obtain the pivotality motive. Sahm (2015) does treat a finite crowd, but his mechanism is suboptimal (multiple crowdfunding prices improve outcomes) and he claims pivotality breaks down with any more than a few buyers which we contradict (recall Illustration 1). Also, he assumes traditional finance and crowdfunding are mutually exclusive, obliging at-cost thresholds despite the ex-post revenues, but we derive complementarities. Chang (2015) treats fraud in a pure common value environment. More recently, Strausz (2017) models fraud in a simplified version of our baseline ($v_L = 0$). He characterizes crowdfunding that prevents fraud.[7] Chemla and Tinn (2016) consider the same problem but with correlated demand and two trading periods, similar to our Section 6. Both papers share the simplest version of our adaptation result,[8] but setting $v_L = 0$ confines them to the exclusion case of our general two-type analysis. This rules out our richer inclusive results and the lessons for reward design and pricing dynamics, and precludes any distinction between profit- and welfare-maximizing entrepreneurs. To focus on those insights, we abstract from fraud. Taking funds and not fulfilling rewards is empirically rare (Mollick, 2014). Both reputation concerns, from future careers and social networks, and ex-post sales (Section 6) plausibly motivate entrepreneurs to sink fixed costs and deliver.

2 Baseline model

We present a streamlined model that generalizes Illustration 1; we defer justification and extensions. A single entrepreneur has a project for producing a good at fixed cost $C > 0$ and constant marginal cost normalized to zero. $\mathcal{N} = \{1, \ldots, N\}$ is the set of buyers, who have unit demands for the good with private values drawn independently from the 2-type distribution: probability $q > 0$ on v_H and $1 - q > 0$ on $v_L < v_H$. We assume $C < Nv_H$, otherwise production is never desirable. The entrepreneur and the buyers interact through a market mechanism that determines production, consumption and payment outcomes. We contrast two types of mechanism. First, the traditional selling benchmark is an ex-post mechanism where the entrepreneur interacts with buyers by posting a price only *after* she decides to produce. Second, we describe the potential for gains from an ex-ante selling mechanism where the entrepreneur and buyers interact *before* production, as in crowdfunding. In our baseline model, the entrepreneur can fully commit to any ex-ante selling mechanism.

We consider both profit and not-for-profit motivated entrepreneurs; neither discounts time. The not-for-profits are idealized: they maximize expected welfare under the constraint of *No Deficit in Expectation* (NDE) requiring non-negative expected profits; for generic uniqueness, we assume they lexicographically maximize profit among welfare-

[7] Ellman and Hurkens (2019) show that tolerating some fraud generates even higher welfare.
[8] Strausz (2017) and Chemla and Tinn (2016) respectively refer to "screening" and "real value option".

equivalent options. This NDE baseline represents an entrepreneur with unlimited access to a risk-neutral source of credit or unlimited wealth and compulsion to break even on average. Under complete information, the first-best has production whenever $\sum_{i\in\mathcal{N}} v_i \geq C$. The challenge is to characterize the second-best distortions caused by private information and budget restrictions.

Notationally, $V = \{v_L, v_H\}^{\mathcal{N}}$ denotes the set of type profiles with typical element $v \in V$; v_i denotes the type of bidder i and v_{-i} the type profile of all bidders other than i. $V_{-i} = \{v_L, v_H\}^{\mathcal{N}\setminus\{i\}}$ denotes the set of all such profiles. For brevity, we also say that bidder i is of type L when $v_i = v_L$ and of type H when $v_i = v_H$, and sometimes indicate this by writing $v = (v_L; v_{-i})$ or $v = (v_H; v_{-i})$. Let $n(v)$ be the number of H-types: $n(v) = \#\{j \in \mathcal{N} : v_j = v_H\}$. With some abuse of notation, we let $q(v)$ denote the probability of type profile v: $q(v) = q^{n(v)}(1-q)^{N-n(v)}$. Similarly, $q_{-i}(v_{-i}) = q^{n(v_{-i})}(1-q)^{N-1-n(v_{-i})}$.[9] Since $n(v)$ proves to be a sufficient statistic for v, we define $f_k^N(q) = \sum_{\{v\in V: n(v)=k\}} q(v)$, the probability that exactly k of N buyers have type H, and call this k, the demand state. Following the conventions $\binom{M}{k} = 0$ if $k < 0$ or $k > M$ and $\binom{M}{0} = 1$, we define the equivalent, generalized expression, later sometimes suppressing the q,

$$f_k^M(q) = \binom{M}{k} q^k (1-q)^{M-k} \tag{1}$$

We define the probability that at least n out of M buyers have type H as

$$S_n^M = \sum_{k=n}^{M} f_k^M \tag{2}$$

and the hazard rate faced by a putative bidder in crowdfunding as

$$h_n = \frac{f_{n-1}^{N-1}}{S_{n-1}^{N-1}} \tag{3}$$

Lemma A.1 in the Appendix states some useful mathematical results relating f_k^M, S_n^M, and h_n. Finally, we define $\hat{q} = v_L/v_H$.

2.1 Traditional selling

In traditional selling (TS), the product is only sold after production. If the entrepreneur decides to produce, C is sunk. A single posted price p is then optimal.[10] She gets expected revenue Np from $p \leq v_L$, qNp from $p \in (v_L, v_H]$ and 0 from higher p. So she chooses between the "exclusive" price $p = v_H$ that excludes L-types, extracting all H-type rent, and the "inclusive" price $p = v_L$ that includes L-types, leaving some rent to H-types. For

[9] $n(v_{-i}) = \#\{j \in \mathcal{N} \setminus \{i\} : v_j = v_H\}$.

[10] Independent valuations imply no role for probabilistic offers, nor for interpersonal bundling where price offers depend on other buyers' choices.

a profit-maximizing entrepreneur, exclusion is optimal if $q > \hat{q}$ and inclusion is optimal if $q \leq \hat{q}$. She indeed produces if $C \leq \max\{Nv_L, qNv_H\}$. So traditional selling earns her the expected profit,

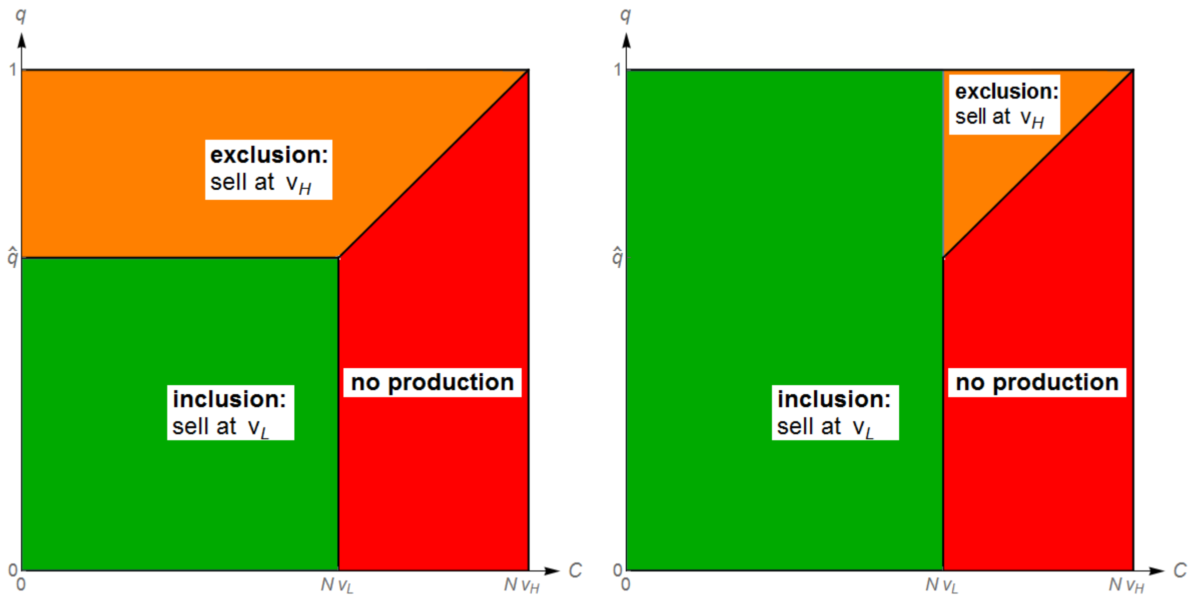
$$\pi^{TS} = \begin{cases} (Nv_L - C)^+ & \text{if } q \leq \hat{q} \\ (qNv_H - C)^+ & \text{if } q > \hat{q} \end{cases} \tag{4}$$

where x^+ denotes $\max\{x, 0\}$ for any x. Notice that NDE does not affect this: both profit expressions are non-negative, so there is never an expected deficit.

For a welfare-maximizing entrepreneur, production and inclusion is optimal if $Nv_L - C \geq 0$, production and exclusion is optimal if $Nv_L - C < 0 \leq qNv_H - C$, and no production is optimal otherwise. So traditional selling then generates welfare

$$W^{TS} = \begin{cases} (1-q)Nv_L + qNv_H - C & \text{if } C \leq Nv_L \\ qNv_H - C & \text{if } Nv_L < C \leq qNv_H \\ 0 & \text{otherwise} \end{cases} \tag{5}$$

Fig. 1 illustrates these benchmarks.



(a) Profit-maximizer

(b) Welfare-maximizer

Figure 1: TS for profit- and welfare-maximizers

2.2 Crowdfunding and other ex-ante selling mechanisms

For any strictly positive fixed cost C, the entrepreneur can improve on traditional selling (posted prices) by using an ex-ante mechanism that makes production contingent on revealed preferences. First, she can adapt her fixed cost production decision to actual, instead of only expected, demand, by eliciting buyer preferences via incentive-compatible, pre-production sales contracts. Second, she can make non-production threats to induce some buyers to pay more so that the probability of trade rises. We focus on a particularly simple ex-ante mechanism: in reward-based crowdfunding, the entrepreneur chooses an Aggregate Funds Threshold T and offers the good at a reserve price p. Buyers submit a bid for the good. Production occurs if and only if the sum of bids exceeds threshold T, in which case all bidders pay their bid and all who bid above p consume the product. This threshold already allows the entrepreneur to both adapt production to actual demand and to threaten non-production but we first characterize in Section 3 the generally optimal outcome among all ex-ante mechanisms; there, production, consumption and monetary transfers can be contingent on the full vector of buyers' expressed demands. We then show in Section 4 how reward-based crowdfunding can implement this general optimum. We derive and illustrate key properties of optimal reward-based crowdfunding and analyze welfare effects relative to traditional selling (TS).

3 General mechanism design

An optimal general mechanism maximizes the objective (profit or welfare) *without* imposing crowdfunding's AFT restriction. Cornelli (1996) solves this for profit-maximization with a *continuum* of buyer types using Myerson's (1981) virtual valuation approach. She also considers welfare-maximization with an exact budget balance constraint, equivalent to assuming NDE binds; she identifies conditions for implementing the first-best, but does not characterize the second-best. We provide *explicit* solutions for both profit- and welfare-maximization in our two-type setting. In Section 4, we show how a simple direct mechanism – reward-based crowdfunding – can implement these general optima.

By the revelation principle, we can restrict attention to incentive compatible and individually rational, direct mechanisms $\mathcal{M} = (\rho, p, t)$, where bidders report their types and the mechanism determines the probability $\rho(v)$ that the entrepreneur produces and the vectors of production-contingent consumption probabilities $p(v) = (p_1(v), \ldots, p_N(v))$ and transfers $t(v) = (t_1(v), \ldots, t_N(v))$; i pays the entrepreneur $t_i(v)$ and consumes with probability $p_i(v)$ given production.[11] If i reports type v_θ and all other buyers report truthfully, i gets the good with probability P^i_θ, paying expected transfer τ^i_θ, where $P^i_\theta = \sum_{v_{-i}\in V_{-i}} p_i(v_\theta; v_{-i})\rho(v_\theta; v_{-i})q_{-i}(v_{-i})$ and $\tau^i_\theta = \sum_{v_{-i}\in V_{-i}} t_i(v_\theta; v_{-i})q_{-i}(v_{-i})$.

$\mathcal{M}$ is individual rational for types L and H if for all i

$$U^i_L \equiv P^i_L v_L - \tau^i_L \geq 0 \qquad (IR_L)$$

$$U^i_H \equiv P^i_H v_H - \tau^i_H \geq 0 \qquad (IR_H)$$

[11] Production-conditional transfers do not expand the implementable set. Nor do correlations between the conditional consumption probabilities $p_i(v)$. We omit them for notational convenience.

$\mathcal{M}$ is incentive compatible for types L and H if for all i

$$U_L^i = P_L^i v_L - \tau_L^i \geq P_H^i v_L - \tau_H^i \tag{IC_L}$$
$$U_H^i = P_H^i v_H - \tau_H^i \geq P_L^i v_H - \tau_L^i \tag{IC_H}$$

$\mathcal{M}$ satisfies No Deficit in Expectation when expected transfers cover expected costs:

$$\pi \equiv \sum_{v \in V} \left(\sum_{i=1}^{N} t_i(v) - C\rho(v) \right) q(v) \geq 0 \tag{NDE}$$

(NDE) guarantees non-negative expected profits, so the entrepreneur participates.

3.1 Profit-maximizers

We maximize the entrepreneur's expected profit π (just defined) from mechanism $\mathcal{M}$, subject to (IR_L) and (IC_H), and show that the solution satisfies all other constraints. As (IC_H) must bind, Myerson's (1981) techniques let us express aggregate transfers as,

$$\sum_{i=1}^{N} \sum_{v \in V} t_i(v) q(v) = \sum_{i=1}^{N} \sum_{v \in V} (w_i p_i(v) \rho(v) - U_L^i) q(v) \tag{6}$$

where w_i is bidder i's virtual valuation: $w_L = (v_L - q v_H)/(1-q)$ for L-types, $w_H = v_H$ for H's. So

$$\pi = \sum_{v \in V} \left(\left(\sum_{i=1}^{N} w_i p_i(v) - C \right) \rho(v) - \sum_{i=1}^{N} U_L^i \right) q(v) \tag{7}$$

which is pointwise maximized subject to (IR_L) by setting $U_L^i = 0$ and $p_i(v) = 0$ if $w_i < 0$, $p_i(v) = 1$ if $w_i \geq 0$ and $\rho(v) = 1$ if $\sum_{i=1}^{N} w_i^+ - C \geq 0$ and $\rho(v) = 0$ otherwise. Note that $w_H = v_H > 0$ while $w_L = (v_L - q v_H)/(1-q) \geq 0$ if and only if $q \leq \hat{q} = v_L/v_H$. So, on $q \leq \hat{q}$, $p_i(v) = 1$ for all i,v: given production, both types of bidders get to consume. We call this the *inclusive* solution.[12] Meanwhile, when $q > \hat{q}$, only H-types ever consume; L-types are *excluded* from consumption. In the exclusive case, production occurs when $n(v) \geq n_E \equiv \lceil \tilde{n}_E \rceil$ where

$$\tilde{n}_E \equiv \frac{C}{v_H} \tag{8}$$

and in the inclusive case, when $(N - n(v))w_L + n(v)v_H \geq C$, that is, when $n(v) \geq n_I \equiv \lceil \tilde{n}_I \rceil$ where

$$\tilde{n}_I \equiv \frac{C - N v_L + q(N v_H - C)}{v_H - v_L} \tag{9}$$

We call n_E and n_I the production pivots. Note that $\tilde{n}_I \leq \tilde{n}_E$ on $q \leq \hat{q}$, so that $n_I \leq n_E$.

[12] Note that we break the tie at $q = \hat{q}$ in favour of the inclusive solution, as this raises welfare.

Proposition 1. *The profit-maximizing outcome is characterized by* $U_L = 0$ *and*

(i) For $q \leq \hat{q} = v_L/v_H$*, both* L *and* H*-types get the good when* $n(v)$ *weakly exceeds pivot* $n_I \equiv \lceil \tilde{n}_I \rceil$*, defined by (9).* H*-types receive a rent:* $U_H^I = P_L^I(v_H - v_L) = S_{n_I}^{N-1}(v_H - v_L)$*.*

$$\pi_{n_I}^I = S_{n_I}^N \Big(Nv_L - C + qNh_{n_I}(v_H - v_L)/(1-(1-q)h_{n_I}) \Big)$$

(ii) For $q > \hat{q} = v_L/v_H$*,* L*-types are excluded and* H*-types get the good when the number* $n(v)$ *of* H*-types weakly exceeds the pivot* $n_E \equiv \lceil C/v_H \rceil$*. Neither gets a rent:* $U_L^E = U_H^E = 0$*.*

$$\pi_{n_E}^E = S_{n_E}^N \Big(qNv_H/(1-(1-q)h_{n_E}) - C \Big)$$

The choice (immaterial when $n_I = N$) between (i) inclusion and (ii) exclusion is the same as in traditional selling: $q \leq \hat{q}$ triggers inclusion in both scenarios.

3.2 Not-for-profits

Expected total welfare under mechanism $\mathcal{M} = (\rho, p, t)$ equals

$$W = \sum_{v \in V} \left(\sum_{i=1}^N v_i p_i(v) - C \right) \rho(v) q(v)$$

Without a budget constraint, the entrepreneur should of course produce and sell to all whenever $n(v)v_H + (N - n(v))v_L \geq C$, that is, whenever $n(v) \geq n^* \equiv \lceil \tilde{n}^* \rceil$ where

$$\tilde{n}^* \equiv \frac{C - Nv_L}{v_H - v_L} \tag{10}$$

This can be implemented if the entrepreneur is willing and able to cover an expected deficit out of her own pocket, but we maximize W subject to (IR_L), (IC_H) *and* (NDE), so that in expectation the entrepreneur does not run a deficit.[13]

We can again take (IC_H) binding, apply (6) and maximize the Lagrangian

$$\mathcal{L}(p, \lambda, \mu) = \sum_{v \in V} \left(\sum_{i=1}^N (v_i + \lambda w_i) p_i(v) - (1+\lambda)C \right) \rho(v) q(v) - \sum_{i=1}^N (\lambda - \mu_i) U_L^i$$

where $\lambda \geq 0$, $\mu_i \geq 0$ are the Lagrange multipliers on (NDE) and each i's (IR_L) constraint.

As $v_H + \lambda w_H > 0$, it is optimal to set $p_i(v_H, v_{-i}) = 1$. Also $p_i(v_L, v_{-i}) = 1$ if $v_L + \lambda w_L > 0$, and $p_i(v_L, v_{-i}) = 0$ if $v_L + \lambda w_L < 0$. On $q \leq \hat{q}$, $w_L \geq 0$, so $v_L + \lambda w_L > 0$, immediately implying inclusion. But on $q > \hat{q}$, $w_L < 0$ so inclusion tightens the budget constraint and is only optimal if: $v_L + \lambda w_L \geq 0$, that is, $\lambda \leq \bar{\lambda} \equiv (1-q)v_L/(qv_H - v_L) = (1-q)\hat{q}/(q - \hat{q})$.

If strict, $\lambda < \bar{\lambda}$, inclusion is full: all buyers get the good when production occurs

[13] NDE is relevant given credit access; EH2016 treat the case imposing no ex-post deficits.

which is when $n(v) \geq n$ for a pivot $n \leq n_E$ determined by λ. If that solution violates (NDE), full-inclusion is infeasible (see below).

If instead the budget constraint is relatively tight, $\lambda > \bar{\lambda}$, the optimal production pivot is n_E but excluding all L-types then generates strict profits. So, interestingly, welfare-maximization can require an interior inclusion probability $\phi \in [0,1]$ (making (NDE) bind) as well as a production pivot n: $p_i(v_L, v_{-i}) = \phi$ for all v. Letting $A_{n,\phi}$ denote the maximal expected aggregate funds from such mechanisms, we have

$$A_{n,\phi} \equiv \sum_{i=1}^{N} \sum_{v \in V} w_i p_i(v) \rho(v) q(v) = S_n^N \left(N\phi w_L + \frac{qN(v_H - \phi w_L)}{1-(1-q)h_n} \right) \tag{11}$$

In sum, the set of feasible, deterministic pivots under (NDE) with full inclusion is

$$\mathcal{I} = \left\{ n \in \mathbb{N} : n^* \leq n \leq n_E \text{ and } A_{n,1} \geq S_n^N C \right\} \tag{12}$$

If $\mathcal{I} \neq \varnothing$, letting $n_I^W \equiv \min \mathcal{I}$, either $n_I^W = n^*$ and the first-best is feasible or $n = n_I^W > n^*$ and it is optimal to produce with probability $\rho(v) = \rho^W$ when $n(v) = n_I^W - 1$ and with probability 1 when $n(v) \geq n_I^W$, where $\rho^W \in [0,1)$ makes (NDE) bind:

$$\rho^W (A_{n-1,1} - S_n^N C) + (1-\rho^W)(A_{n,1} - S_{n-1}^N C) = 0 \tag{13}$$

If $\mathcal{I} = \varnothing$, full inclusion is not feasible and partial inclusion is optimal. The optimal inclusion probability, denoted ϕ^W, is the largest satisfying (NDE):

$$\phi^W = \max\{\phi \in [0,1] : A_{n_E,\phi} \geq S_{n_E}^N C\} \tag{14}$$

Proposition 2. *The welfare-maximizing outcome under (NDE), is characterized by:*
(i) Full inclusion whenever feasible ($\mathcal{I} \neq \varnothing$). Production is then determined by a pivot given by a lottery between n_I^W (with probability $1-\rho^W$) and $n_I^W - 1$ (with probability ρ^W). Restricting to a deterministic production pivot, the optimal pivot is simply n_I^W.
(ii) Otherwise ($\mathcal{I} = \varnothing$), inclusion is partial. The production pivot is then $n_E = \lceil C/v_H \rceil$ and the inclusion probability is $0 < \phi^W < 1$.
Both (i) and (ii) feature H-type rents: $U_L = 0$ and $U_H = P_L(v_H - v_L)$, where $P_L = \rho^W S_{n_I^W-1}^{N-1} + (1-\rho^W) S_{n_I^W}^{N-1}$ in case (i) and $P_L = \phi^W S_{n_E}^{N-1}$ in case (ii).

This has the intuitive implication that a welfare-maximizer is weakly more inclusive than a profit-maximizer, as case (i) of Proposition 2 contains case (i) of Proposition 1.

Corollary 1. *Welfare-maximizers produce more and more often than profit-maximizers.*

4 Optimal reward-based crowdfunding

Section 3 characterized the general optimal outcome among all ex-ante mechanisms. There is a continuum of possible transfers, dependent on the full type profile, but we identify a unique set of transfers satisfying a specially simple structure: the bidder only pays when receiving the good and his price only depends on his own type. This allows us to show how, in our two-type setting, reward-based crowdfunding with refunds can implement the general optimal outcome: in its simplest form, the entrepreneur chooses an Aggregate Funds Threshold T and offers a *single* reward (one unit of the good) at a *minimum* price $p \geq 0$. Buyers submit bids, paid as prices in return for the reward whenever the sum of bids exceeds the threshold. Absent production, bids are refunded. Kickstarter emphasizes refunds as they help attract funders. We will generalize beyond this format, but we maintain as fundamental the near universal aggregate fund threshold:

AFT: *Production occurs if and only if the sum of bids exceeds threshold T.*

Bids $b \in (0, p)$ sometimes lead to paying for nothing and are weakly dominated by $b = 0$ (equivalent to not participating). The timing is as follows:

1. The entrepreneur chooses her offer (T, p)
2. Buyers learn their private values and simultaneously choose their bids from $\mathbb{R}_+$
3. AFT with refunds determine production, consumption and transfers

This reward-based crowdfunding can implement any production pivot n and conditional inclusion probability ϕ, and hence any optimal general mechanism outcome by Propositions 1 and 2.[14] In optimal outcomes, $U_L = 0$ so L-types should bid $b_L = \phi v_L$. As (IC_H) binds, H-types should bid b_H defined by $(v_H - b_H)S_{n-1}^{N-1} = (\phi v_H - b_L)S_n^{N-1}$. Using hazard ratio (3), $b_H = v_H - \phi(v_H - v_L)(1 - h_n)$. If the entrepreneur sets $T = nb_H + (N - n)b_L$, production occurs exactly when $n(v) \geq n$. So it suffices to verify that this T and some p induce a BNE in which H-types bid b_H and L-types bid b_L. In fact, these bidding strategies form the unique Pareto undominated BNE, so reward-based crowdfunding *fully implements* the general optimum (see Ledyard and Palfrey, 2007). We start with the cases where the optimum has $\phi = 0$ (full exclusion) or $\phi = 1$ (full inclusion). Let $p = v_L$ if $\phi = 1$ and $p = v_H$ if $\phi = 0$.

L-types cannot do better than to bid $b_L = \phi v_L$. H-types are indifferent between bidding b_L and b_H by the definition of b_H: as he may be pivotal, he is willing to bid the higher price b_H to raise the probability of production. Bidding $b \in (b_L, b_H)$ is strictly worse than bidding b_L as b_L obtains the good with the same probability ϕS_{n-1}^{N-1} at a lower price. Clearly, bidding $0 \leq b < b_L$ cannot benefit the bidder either. Bidding strictly more than b_H *does* raise the probability of reaching the threshold but this does not improve an

[14] Crowdfunding imposes a deterministic pivot.

H-type's expected payoff. (See the proof of Proposition 3, also showing that any other BNE is Pareto dominated.) Defining $b_n = h_n v_H + (1 - h_n)v_L$, $T_n = nb_n + (N - n)v_L$ and $T_n^E = nv_H$, we have

Proposition 3. *Reward-based crowdfunding can implement the general profit-maximizing outcome of Proposition 1 as a unique Pareto undominated BNE by setting:*
(i) For $q \leq \hat{q}$, minimum price $p = v_L$ and threshold $T = T_{n_I}$; H-types bid b_{n_I}, L's bid v_L.
(ii) For $q > \hat{q}$, minimum price $p = v_H$ and threshold $T = T_{n_E}^E$; here H-types bid v_H.

The optimal inclusion threshold T_{n_I} is unique. Any threshold in $(T_{n_E}^E - v_H, T_{n_E}^E)$ generates the optimal exclusive outcome but $T = C$ is a natural focus.

The above analysis applies directly for implementing fully inclusive welfare optima, but to implement with partial inclusion ($0 < \phi < 1$) requires an interesting twist: crowdfunding must offer a menu of rewards and minimum prices, a standard practice. In addition to the (unit) reward at minimum price $p = b_H$, the entrepreneur now offers lottery tickets at minimum price $p' = \phi v_L$. Each ticket gives probability ϕ of winning one unit of the good. L-types are just willing to bid ϕv_L for such a ticket, while, again by definition of b_H, H-types are indifferent between the two rewards. So we have

Proposition 4. *Reward-based crowdfunding can implement the deterministic pivot, general welfare-maximizing outcome of Proposition 2 as a unique Pareto undominated BNE. In case (i), set minimum price $p = v_L$ for the standard reward and threshold $T^W = T_{n_I^W}$, inducing L-type bids v_L and H-type bids $b_{n_I^W}$;*
In case (ii), set minimum prices $p = b^W \equiv \phi^W b_{n_E} + (1 - \phi^W)v_H$ for the standard reward and $p' = \phi^W v_L$ for a ϕ^W probability reward, alongside threshold $T^W = \phi^W T_{n_E} + (1 - \phi^W)T_{n_E}^E$, inducing L-type bids $\phi^W v_L$ and H-type bids b^W.

Implementation via reward limits: Instead of lottery tickets, the entrepreneur could approximate by offering a limited number $\bar{n}_L$ of standard rewards at the low price, and an unlimited number of rewards at a higher price b^W. L-types are then rationed (and refunded) when $N - k > \bar{n}_L$. L-types are willing to bid $p = v_L$. Given production, they get the good with probability

$$\phi(\bar{n}_L) = \left(\sum_{k=n_E}^{N-1} f_k^{N-1} \min\{1, \bar{n}_L/(N-k)\} \right) / S_{n_E}^{N-1}$$

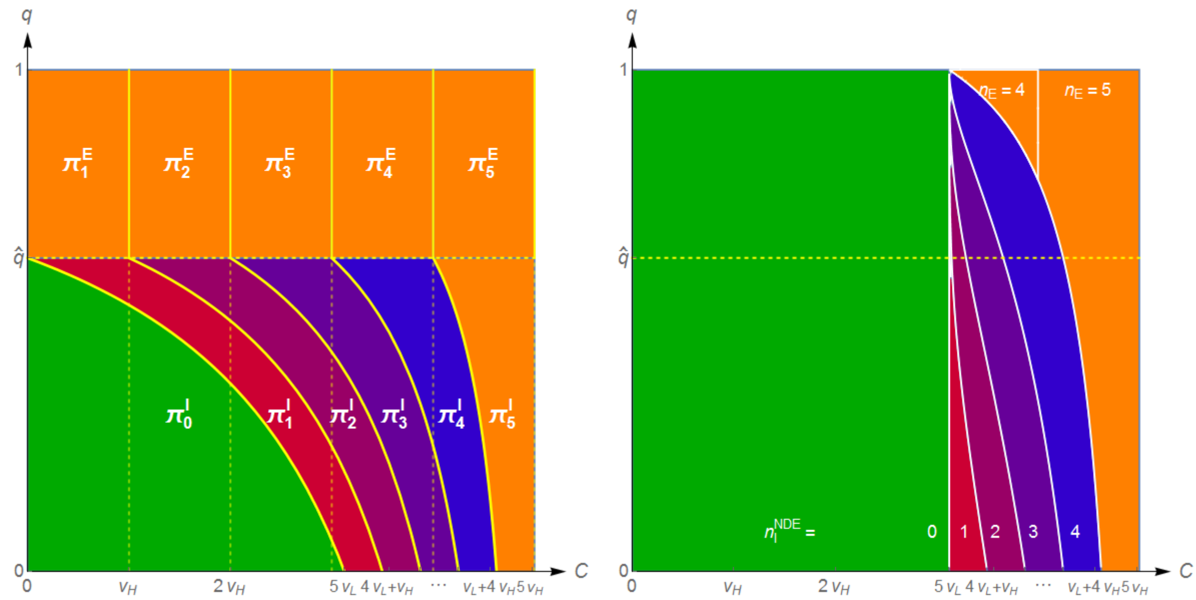
Clearly, $\phi(\bar{n}_L)$ increases from 0 to 1 as $\bar{n}_L$ increases from 0 to N. The optimal deterministic capacity constraint (subject to the integer restriction) is $\bar{n}_L = \max_{\bar{n}}\{\bar{n} : \phi(\bar{n}) \leq \phi^W\}$.

4.1 Illustration

The following parameter range generates a representative set of outcomes.

Example 1. $N = 5$, $v_L = 5$, $v_H = 8$ *(so* $\hat{q} = 0.625$*)*, $0 < q < 1$, $0 < C < 40$.

Fig. 2(a) shows the regions in (C, q)-space where each strategy type, n with exclusion E or inclusion I, is profit-maximizing in Example 1; e.g., $n_I = 4$ is optimal on the (blue) region marked π_4^I. The following properties hold generally. We see exclusion E above $q = \hat{q}$ and I below it. Pivots n_E, n_I increase in C, are equal on $q = \hat{q}$; n_I rises with q with concave boundaries. $\pi_5^I = \pi_5^E$ as $b_N = v_H$ and L-types never get to buy at $n_I = N$.



(a) Profit-maximizer (b) Welfare-maximizer

Figure 2: Optimal crowdfunding in Example 1

The project success rate S is $S_{n_E}^N(q)$ on $q > \hat{q}$ and $S_{n_I}^N(q)$ on $q \leq \hat{q}$. S is decreasing in C. S is not monotonic in q owing to upward jumps in n_I. Nevertheless, we prove

Corollary 2. *The entrepreneur's maximized profit strictly falls with* C *and rises with* q.

Fig. 2(b) depicts welfare-maximizers' optimal design, illustrating their greater use of inclusion: the orange region is a strict subset of the exclusion region in Fig. 2(a) and becomes partially inclusive. In the fully inclusive regions, prices are as before. The large rectangle (in green) with zero pivot shows not-for-profits gain nothing from crowdfunding at low costs, even at high q, but they gain more from crowdfunding at higher costs.

4.2 Welfare gains and losses from crowdfunding

We study crowdfunding's welfare impacts compared to traditional selling (TS)'s posted prices. The first-best welfare-maximizing benchmark had all buyers consume whenever production occurs, namely, in all states $k \geq n^* = \lceil \tilde{n}^* \rceil$ (see (10)). From Section 2.1 on

TS, both types of entrepreneur (for- *and* not-for-profits) produce if $C \leq Nv_L$ and $q \leq \hat{q}$ (setting $p = v_L$) and if $Nv_L < C \leq qNv_H$ (setting $p = v_H$). When $C \leq Nv_L$ and $q > \hat{q}$, they both produce but set different prices.

With not-for-profit entrepreneurs, crowdfunding unambiguously raises payoffs of both consumers and entrepreneurs, strictly so on $C > Nv_L$, and neutral otherwise. With profit-maximizers, the welfare effects are more involved. Fig. 3 illustrates. Crowdfunding raises welfare on the orange exclusion region $q \geq \hat{q}$ by adapting production perfectly to demand, avoiding production in low demand states but producing in high ones. Consumer surplus remains zero here. Welfare is also raised on the inclusive blue rectangle with $q \leq \hat{q}$ and $C > Nv_L$ where there is no production in TS. Consumer surplus strictly increases unless $n_I = N$. However, on $q \leq \hat{q}$ and $C \leq Nv_L$, TS maximizes welfare by always producing. Here, crowdfunding can only do harm. On the convex green, low (C, q) region, crowdfunding sets $n_I = 0$ and is welfare neutral. But on the adjacent purple region, crowdfunding lowers welfare and consumer surplus by restricting production.

Proposition 5. *(a) With for-profit entrepreneurs, crowdfunding's welfare impact versus traditional selling (TS) is: (i) strictly positive if* $C > Nv_L$*, (ii) strictly positive for any* C *if* $q > \hat{q}$*, (iii) strictly negative if* $q \leq \hat{q}$, $C \leq Nv_L$, $n_I > 0$*, (iv) neutral otherwise; (b) With not-for-profits, its impact is: (i) strictly positive if* $C > Nv_L$*, (ii) neutral otherwise.*

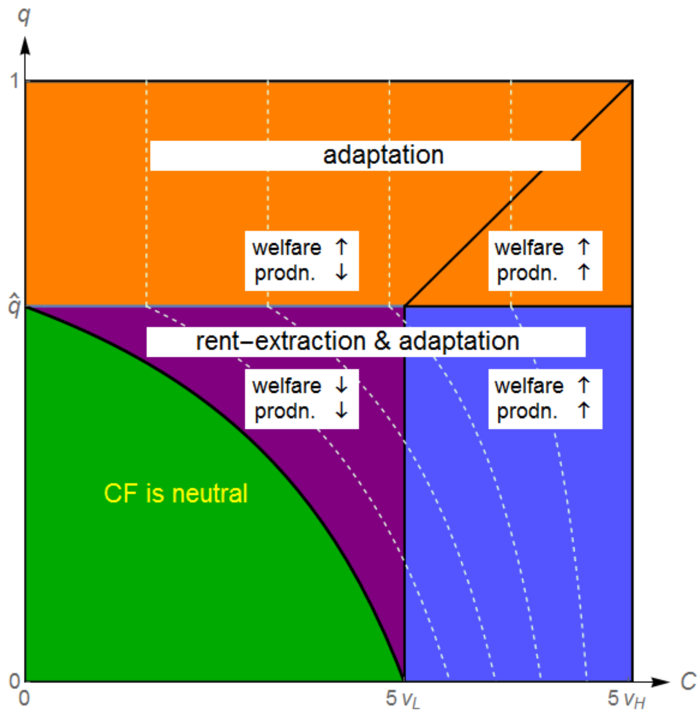


Figure 3: Welfare effects of profit-maximizing crowdfunding relative to TS, indicating adaptation to produce in higher demand states and rent-extraction where thresholds induce price discrimination.

Crowdfunding has the same qualitative effect on consumer surplus as on welfare, except in having a null consumer effect when exclusive (or inclusive with $n = N$). Unsurprisingly, under crowdfunding, not-for-profits give strictly higher consumer surplus as well as welfare than do for-profits, whenever their strategies differ.

4.3 The role of funding in the baseline

Propositions 3 and 4 show that crowdfunding allows the entrepreneur to adapt her production decision to actual demand and extract rent optimally. Funding played no role in those results because we assumed the entrepreneur had enough wealth or credit to sink her cost C provided funds covered costs *in expectation*. Here, we set the ground for a richer study of crowdfunding and credit in Section 6, by explaining how crowdfunding can implement the optimal outcomes even without wealth or traditional sources of credit.

We begin with a profit-maximizing entrepreneur since she can directly cover her cost from sales during crowdfunding provided the campaign reaches all potential buyers.

Corollary 3. *The entrepreneur's optimized profit is non-negative in every demand state.*

This is obvious for exclusion where $T = C$ is a solution. We now prove that $T_{n_I} > C$ since: (1) $T_N = Nv_H > C$; (2) $n_I b_{n_I} + (N - n_I)v_L \leq C$ with $n_I < N$ would contradict n_I's optimality since $n_I + 1$ could then avoid producing in unprofitable state n_I and strictly increase profits in higher states.

The case of a welfare-maximizing entrepreneur is different because typically $T^W < C$; whenever (NDE) binds, revenues equal cost in expectation, so given $n < N$, they fall short of cost when the threshold is just reached. However, adding an investment-based element, crowdfunding can still implement the general welfare-maximizing outcome, either by borrowing money from buyers or by issuing them shares as follows. With R denoting the expected funds conditional on reaching the threshold, the entrepreneur's expected profit is $S_n^N(R - C) \geq 0$, where $n = n_I^W$ or n_E. The entrepreneur offers a fraction $1 - \kappa$ of the firm to raise $C - T > 0$ when pricing so that H-types are just willing to buy.[15]

Production occurs *and* shares are issued only if the new threshold $T' = T + C - T = C$ is reached, guaranteeing that the entrepreneur has enough cash when the campaign succeeds. Since all shares are sold, the new threshold is reached if and only if $k \geq n$, so total welfare is as in the general optimum. The entrepreneur's expected profit then equals $S_n^N \kappa(R - C + C - T)$. Setting the entrepreneur's retained fraction at $\kappa = (R - C)/(R - T)$ yields her the same expected profit as in the general optimum $S_n^N(R - C)$; $1 > \kappa \geq 0$ since $R \geq C > T$ by (NDE). Consumer surplus is then also the same, which proves that H-types are indeed just willing to acquire all shares.

Corollary 4. *Pure reward-based crowdfunding cannot implement the welfare-maximizing optimum given credit-constraints if $T^W < C$, but can do so when combined with equity.*

[15] H-types value shares more than do L's since, given production, they expect a higher total number of H's: $\mathbb{E}[k|k \geq n - 1] + 1 > \mathbb{E}[k|k \geq n]$ when $n > 0$. When $n = 0$, $C \leq Nv_L$ so there is no credit problem.

This can explain why some early crowdfunding platforms like Sellaband used financial as well as material rewards (see Agrawal et al., 2015). Tighter regulation of firms selling financial products arguably frustrated mixed equity and reward-based crowdfunding, forcing credit-constrained entrepreneurs to raise their pivots above n_I^W in our model (though pure investment-based platforms have grown since the U.S. JOBS Act 2012).

5 Crowdfunding in practice

This section examines robustness and fit with crowdfunding practice. We relax the entrepreneur's assumed power to commit to a threshold. Then we analyze how crowdfunding interacts with product differentiation and the role of platforms. Finally, we relate our results to observed crowdsize patterns and model the effect of correlation in large crowds.

5.1 Threshold commitment

The general mechanism design approach assumes full commitment. In particular, the entrepreneur commits not to produce unless a sufficient number of buyers reveal themselves to be H-types. When she does so via a crowdfunding threshold, she must commit not to bid on her own project, despite her incentive to do so whenever aggregate funds from true backers exceed her cost C but fall short of the threshold. However, enforcement is costly. Some platforms even fully preclude threshold commitment by explicitly allowing self-bidding (see Crosetto and Regner, 2018).[16]

We now characterize optimal crowdfunding when the entrepreneur lacks threshold commitment, first for profit-maximizers and later for welfare-maximizers. We denote this by NC for no commitment. Under NC, a profit-maximizing entrepreneur produces precisely when the total sum of bids covers the fixed cost. So NC effectively forces $T = C$. If only one price was ever paid, $T = C$ is anyway optimal. Exclusive (n_E) and corner inclusive ($n_I = 0$ and N) solutions have unique prices (v_H, v_L and v_H, respectively), so we begin with profit-maximization when $q \leq \hat{q}$ and $0 < n_I < N$.

NC strictly lowers profits, because an H-type can now shave his bid without lowering the success rate (recall $T_{n_I} > C$). The entrepreneur can partly counterbalance this effect by *lowering* her minimum price *below* the low type valuation to foster high type bids.[17] But if q is high enough, she instead raises p to exclude. Overall, we prove that NC raises production and total and consumer welfare if and only if inclusion remains optimal.

The *local* incentive compatibility condition that prevents H-types from shaving down

[16] Threshold commitment is also implausible in pre-ordering contexts where the seller of a new product such as a jet engine sets a demand threshold, without third party support.

[17] Surprisingly, extracting maximal rents from L-types via $p = v_L$ (so $b_L = v_L$) is no longer optimal. Hansmann (1981) and Baumol and Bowen (1968) offer evidence for such "underpricing."

from b_H requires equality in the n-pivotality condition (any $n > 0$): $nb_H + (N-n)b_L = C$.[18]

Lowering p to p' substitutes for the inability to raise T above C, because $b_L = p' < v_L$ raises the gap $T - Nb_L = C - Nb_L$ that motivates H-type bids. Denoting the resulting b_H by b'_n and letting $\delta'_n = b'_n - p'$, we have $nb'_n + (N-n)p' = C$ or equivalently

$$\delta'_n = (C - Np')/n \tag{15}$$

which falls with p'. The entrepreneur's profit for any $n > 0$ is now

$$\pi'_n = \sum_{k=n}^{N} f_k^N((N-k)p' + kb'_n - C) = S_n^N \left(\mathbb{E}[k|k \geq n] - n\right) \delta'_n$$

For a given $n > 0$, the entrepreneur chooses p' to maximize δ'_n, subject to L-type individual rationality, $p' \leq v_L$, and incentive compatibility of H-types not deviating to bid p',

$$S_n^{N-1}(v_H - p') \leq S_{n-1}^{N-1}(v_H - b'_n)$$

which can be rewritten as

$$\delta'_n \leq h_n(v_H - p') \tag{IC$'_H$}$$

Solving (15) and the binding (IC$'_H$) gives the optimal p':

$$p'_n = \frac{C - nh_n v_H}{N - nh_n} \tag{16}$$

which is readily seen to decrease with n. This n-type strategy is only feasible if $p'_n \leq v_L$, or equivalently, $C \leq Nv_L + nh_n(v_H - v_L) = T_n$.

For $n = 0$, inclusion simply yields $\pi'_0 = Nv_L - C$. The profit-maximizing inclusive pivot denoted n'_I trades off rent-extraction and project success. The resulting inclusive payoff is compared to $\pi^E_{n_E}$.

Proposition 6. *Removing commitment power only affects maximized profits in the price discriminating case, $q \leq \hat{q}$ and $0 < n_I < N$. It then strictly lowers profits and:*
(i) if inclusion remains optimal, $n'_I \leq n_I \leq n_E$, the minimum price is strictly lower, consumer surplus strictly higher and total surplus and the success rate rise, strictly if $n'_I < n_I$;
(ii) if exclusion becomes optimal, consumer surplus is strictly lower and total surplus and the success rate also fall, strictly if $n_I < n_E$.

Intuitively, new constraint (15) encourages the profit-maximizing entrepreneur to lower n under inclusion. The option to lower p complicates the proof that indeed $n'_I \leq n_I$,

[18] Restricting bids to $\{v_L, b_H\}$ would prevent this bid-shaving. Bid restrictions have no impact under threshold commitment but can improve outcomes under NC (see EH2016).

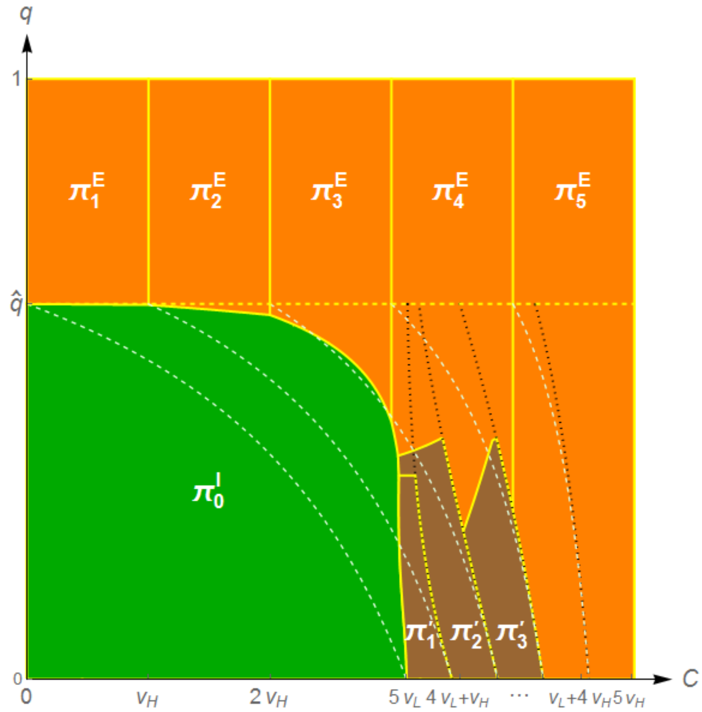


Figure 4: Profit-maximizing crowdfunding with no commitment (NC) in Example 1.

but that still holds since reducing p directly reduces revenues per L-type and the optimal value p'_n falls with n.[19]

This reduced minimal number of H-types reveals that, if inclusion remains optimal, non-commitment weakly raises total welfare. The entrepreneur cannot gain from lost commitment power and she strictly loses in the price discriminating case. So consumer surplus must be strictly higher. Even L-types now get a strict positive surplus because they pay less than v_L. So total and consumer welfare rise provided that inclusion remains optimal, which holds for low values of q and C. If instead q or C is relatively high, exclusion becomes optimal and consumer surplus falls to zero.

Fig. 4 illustrates these results. The dotted black curves, representing loci $C = T_1$ through $C = T_4$, indicate feasibility of inclusive strategies with no commitment; $n = 4$ is often feasible but exclusion always dominates it. Clearly, in the orange subregion below $q = \hat{q}$ where exclusion becomes optimal, consumer surplus falls to zero and profits strictly fall as $\pi^I_{n_I}$ was strictly preferred. In all other regions below $q = \hat{q}$, consumer surplus is strictly higher as both types pay less, H-types strictly, and the success rate rises.[20]

Turning now to welfare-maximizers, commitment is much less of an issue. NC does not

[19]It also introduces multiple Pareto undominated equilibria: H-types prefer the equilibria with higher p and weakly lower n, while L-types prefer lower p. We now assume the entrepreneur selects her preferred equilibrium, as standard in mechanism design, but we do not need this for our key results: that $n \leq n_I$ and NC raises welfare provided the entrepreneur does not switch to exclusion.

[20]Generically, L-types pay strictly less when $n'_I \geq 1$: L-types pay v_L only along the dotted black curve, $C = T_{n'_I}$. In the green region marked π^I_0, all buyers pay v_L so NC does not benefit L-types strictly.

affect total welfare though it can shift the distribution of surplus from the entrepreneur to funders. When (NDE) binds, NC has no effect at all because self-bidding would then break (NDE). So the partial inclusion solution is unaffected. The full inclusion solution of Proposition 4(i) is affected but only with a deterministic production pivot $n_I^W > 0$, in which case, $C > Nv_L$. Under NC, the equilibrium of Proposition 4(i) would induce H-types to shave their bids from $b_{n_I^W}$, knowing that the entrepreneur could and would still produce in the same set of states without breaking (NDE). Instead, the equilibrium under NC has H-types bidding the lower amount b'_H, defined below, which just ensures that (NDE) binds. This can be implemented without the entrepreneur ever self-bidding or revising her threshold downwards if she simply sets minimum price $p = v_L$ and threshold $T = n_I^W b'_H + (N - n_I^W)v_L$, where

$$b'_H = v_L + \frac{(1-(1-q)h_{n_I^W})(C - Nv_L)}{qN}$$

so that expected profit $\pi'_{n_I^W} = S^N_{n_I^W}\left(E[k|k \geq n_I^W](b'_H - v_L) + Nv_L - C\right) = 0$.

Proposition 7. *Removing commitment power does not affect not-for-profit entrepreneurs' maximized welfare, although it weakly lowers their profit.*

5.2 Vertical product differentiation

We have shown how crowdfunding can explain multiple pricing with a single product, but entrepreneurs do often offer diverse rewards that vary the quality or quantity of the product. For instance, musicians often offer signed CDs, special editions and a variety of quantity packs at non-linear prices. Some reward variants essentially serve as tokens of gratitude or help to coordinate bidding strategies, but others are substantive. We develop two results on the strategic interaction between profit-maximizing crowdfunding and product differentiation. We prove that crowdfunding actually has no impact on optimal product differentiation, but an ability to product differentiate lowers the optimal crowdfunding pivot since improved rent-extraction encourages production.

We build on Mussa and Rosen's (1978) canonical model of vertical differentiation: multiplicative quality x at strictly convex cost $c(x)$ ($c(0) = c'(0) = 0$) gives utility $v_\theta x_\theta$ to type $\theta \in \{L, H\}$ buyers who buy a reward of quality x_θ. Profit (7) adjusts to:

$$\pi = \sum_{v\in V}\left(\left(\sum_{i=1}^{N}[w_i x_i - c(x_i)]p_i(v) - C\right)\rho(v) - \sum_{i=1}^{N} U_L^i\right) q(v) \tag{17}$$

$$= \sum_{k\geq n}\Big((N-k)\,[w_L x_L - c(x_L)] + k\,[v_H x_H - c(x_H)] - C\Big) f_k^N \tag{18}$$

The only difference between (17) and (7) is that the virtual valuation w_i becomes the "net virtual valuation" $[w_i x_i - c(x_i)]$, net of individual quality costs. This expression is independent of the demand state v, so optimal qualities x do not depend on v nor on the crowdfunding pivot n. Optimal differentiation remains as in traditional selling (TS), which corresponds to $n = 0$. The L-type quality level x_L is distorted down to $x_L^{**} = (c')^{-1}\left(w_L^+\right)$ from L's first-best quality $x_L^* = (c')^{-1}(v_L)$, while $x_H^{**} = x_H^* = (c')^{-1}(v_H)$.

We now use (18) to analyze how product differentiation affects optimal crowdfunding. First, notice that product differentiation does not change the optimality of inclusion in crowdfunding; this again depends on the sign of w_L, or equivalently whether $q \leq \hat{q}$; exclusion is simply the extreme of downward distorting x. Second, downward distortion raises the first square bracket (x_L^{**} maximizes $[w_L x_L - c(x_L)]$) and has no other effects. As the optimal pivot n is the least k making the large parenthesis in (18) non-negative, the production pivot must (weakly) fall and differentiation raises the probability of production. Low types get the good more often but at a distorted quality and their rent is always zero. High types lose from the direct effect of improved price discrimination but gain from the lower pivot; their rent is $P_L x_L^{**}(v_H - v_L)$ and x_L^{**} is lower while P_L is higher.

Proposition 8. *Crowdfunding does not change optimal product differentiation but optimal product differentiation lowers crowdfunding's optimal production pivot.*

This asymmetry owes to crowdfunding's roles: adapting production to demand and extracting rent as a device for price discrimination. Product differentiation is a device for the latter only and they operate orthogonally, but rent-extraction affects adaptation.

5.3 The role of platforms

Platforms play a vital role in crowdfunding, reducing transaction costs between entrepreneurs and buyers. They support trust in commitments by leveraging social networks and defining clear obligations: buyers cannot withdraw bids once the threshold is reached and typically pay in advance, and entrepreneurs are obliged to fulfill promised rewards once funds are transferred.[21] Not-for-profit entrepreneurs naturally do their best to fulfill these rewards. Profit-maximizers tend to do so too, thanks to potential ex-post sales and reputational concerns, enhanced by platform's feedback forums.

Platforms typically charge entrepreneurs a share of revenues on successful projects, so they aim to maximize expected revenues, and thus care about the number and size of successful projects. They value high revenues even if accompanied by high costs. Their revenue share can bias their interests towards those of entrepreneurs, but they must also attract buyers to have any successes. We apply lessons from two-sided markets to understand platform strategies.[22]

[21]Platforms warn of the inherent risks in unfinished products, but avoid direct contractual responsibility for reward delivery. This might change if fraud became at all frequent. Fraud is currently rare but delivery delays from unforeseen technical and logistical problems are quite common (Mollick, 2014).

[22]See Agrawal et al. (2014); Belleflamme et al. (2015).

First, an important strategic choice for platforms affecting revenues is whether to provide threshold commitment or not. We apply Proposition 6 to shed light on this central platform design choice. NC always lowers entrepreneur profits, and if NC leads to more exclusion, all actors, including the platform, are better off with threshold commitment. However, if inclusion remains optimal under NC, prices are lower and consumers are better off. So the optimal choice for the platform depends on the relative elasticities of entrepreneurial and consumer participation. This is consistent with diversity in platform strategies: Kickstarter exerts effort to enforce threshold commitment by prohibiting self-bidding, inhibiting the use of pseudonyms and precluding adjustment of thresholds once set, while Startnext explicitly allows self-bidding and Indiegogo's "Keep-it-All" format also corresponds to NC in our setup (see EH2016).

Second, platforms can influence outcomes by using project rankings to draw buyers' attention to particular projects. Their revenue share biases them towards projects with high expected revenues, but does this bias them towards profit-maximizing entrepreneurs to the detriment of not-for-profits? In general, platforms care more for success than profit-maximizing entrepreneurs because expected revenue equals expected net profit *plus* expected cost expenditure. By Section 4, they favour not-for-profits as follows

Proposition 9. *(a) When $C \leq Nv_L$, there exist $\bar{q}(C) \geq \hat{q} \geq \underline{q}(C)$ such that the revenue-maximizing platform strictly prefers (i) not-for-profits if $\underline{q}(C) < q < \bar{q}(C)$, (ii) profit-maximizers when $q > \bar{q}(C)$ and (iii) is indifferent on $q \leq \underline{q}(C)$.*
(b) When $C > Nv_L$, the platform strictly prefers (i) not-for-profits when $q \leq \hat{q}$ if $n_I^W < n_I$ and (ii) profit-maximizers when $q > \hat{q}$ and $n_I^W = n_E$.

In essence, the platform biases towards profit-maximizers at high q, but maximizes welfare by selecting not-for-profits at low q. The intuition is clearest in case (a): profits and success rates with not-for-profits are then independent of q, but are both increasing in q with profit-maximizers. In an empirical study of Kickstarter, Pitschner and Pitschner-Finn (2014) show that not-for-profit motivated entrepreneurs have higher success rates, consistent with our finding that $n_I^W \leq n_I$. Excluding the top 1%, they also find not-for-profits generate more revenues. Warm-glow (see Belleflamme et al., 2014) could explain this if such crowdfunders prefer not-for-profits, consistent with anecdotal evidence of funders upset by entrepreneurs raking in profits. However, for-profit platforms may well promote not-for-profit entrepreneurs, even when crowdfunders are purely self-interested. They raise expected revenues via success rates (Proposition 9). They also raise inclusivity (Section 4.2) and reduce minimum prices (Footnote 17), which attract crowdfunders.

5.4 Crowd size and correlated valuations

In our *i.i.d.* baseline, the private and social benefits of crowdfunding through demand-adaptation and rent-extraction are always positive, but could be small. In particular, they shrink away as the crowd grows, because *per capita* demand uncertainty and the relevant hazard rate both fall to zero.[23] So entrepreneurs with very large crowds should opt for traditional selling given that crowdfunding involves platform fees and added costs associated with pre-production campaigns. We still predict a vibrant future for crowdfunding.

First, rent-extraction and especially demand adaptation benefits may be substantial for surprisingly large crowds, as Illustration 1 with 500 bidders shows.[24] Also typical projects are far smaller: on Kickstarter, the average number of funders is 101.3, falling to 56.2 on excluding the top one per cent.[25] 41% were successfully funded, with 75% raising less than $10,000. The many project failures show market-testing in action: crowdfunding filters out projects with too little demand, only sinking costs in viable projects. Second, consumer tastes are generally correlated within groups. Per capita aggregate uncertainty then persists for arbitrarily large crowds. So, by adapting production to demand, crowdfunding still yields substantial private and social benefits over TS.

We consider a group-preference model where each crowd has N groups with m members; valuations are perfectly correlated within groups while each group's valuation is drawn independently from a well-behaved distribution G on $\mathbb{R}_+$. Fixing N, aggregate demand uncertainty remains substantial for arbitrarily large m. Rent-extraction via crowdfunding's price discrimination becomes ineffective so to simplify, we restrict to a single price p, allowing the extension beyond our two-type baseline.

With price p, demand is $k \cdot m$ with probability $f_k^N(q(p))$ where $q(p) = 1 - G(p)$ and k now represents the number of groups with a valuation weakly above p. The entrepreneur's profit is then,

$$\pi(p) = \sum_{k=n(p)}^{N} f_k^N(kmp - C)$$

where $n(p) = \lceil C/mp \rceil$. Inside region $n(p) \equiv n$, price p satisfies the first-order condition,

$$0 = q(p) + pq'(p) + h_n\big[(n-1)p - cN\big]q'(p)$$

where we fix the *per capita* cost $c = C/Nm$. Inside each such region, p strictly increases with c, unlike standard independence of fixed costs. At region boundaries, n rises discretely and p jumps down to mitigate the resulting fall in success. Under TS, the entrepreneur sets $p^{TS} = \arg\max pq(p)$ if producing, so crowdfunding's price is strictly higher: by avoiding production in loss-making, low demand states, crowdfunding alleviates the negative demand effect of raising p, reflected in $q = 1 - G(p)$.

Proposition 10. *With group-preferences, even single-price crowdfunding benefits a profit-maximizing entrepreneur over traditional selling (TS) for arbitrarily large crowds*

[23] Norman (2004) proves that traditional selling is asymptotically as good as the optimal general mechanism fixing $c = C/N$; this result also holds in our discrete type setting.

[24] Extraction depends on the hazard rate, not the probability of pivotality, as bids are efficient transfers.

[25] We exclude the 1.3% of projects with over 1,000 funders in U.C. Berkeley's Fung Institute's Kickstarter data 2009-2014, `http://rosencrantz.berkeley.edu/crowdfunding/index.php.`

(Nm). She sets $T = C$ and $p > p^{TS}$, making a strict per-capita gain, independent of m.

Consistent with this, concert and sports events often use advance sales, sometimes inflating prices and cancelling when demand proves insufficient. Committing to a threshold T adds little when crowds are large so the crowdfunding format is rarely salient.

6 Ex-post sales: credit and price dynamics

In our baseline model, the entrepreneur can contact all potential buyers from the start, but in practice crowdfunding often has *limited reach* in that many potential buyers only learn about the product *after* production, forcing the entrepreneur to sell to these new buyers ex-post. In this section, we model limited reach and derive the implications for crowdfunding design, price dynamics and credit. We also allow for prior uncertainty over the probability q of high valuation v_H to capture how crowdfunding performance can inform the entrepreneur about ex-post demand. We distinguish two classes of buyers: N_1 crowdfunding participants or "funders" who can fund by buying in period one, ex-ante, or wait and possibly buy later, and N_2 "new buyers" who can only buy in period two, ex-post.[26] The timing of our model with ex-post demand is as follows:

Timing with two selling periods. (0) Nature determines project quality $\theta \in \{B, G\}$ with probability γ^θ and each buyer's (privately observed) conditionally independent type: $\text{prob}(v_H|\theta) = q^\theta$, where $q^B \leq q^G$. (1) The entrepreneur sets crowdfunding offer (T, p). (2) Funders choose bids. (3) If funds do not reach T, the game ends with no production and no payments.[27] If funds reach T, the entrepreneur receives the funds, sinks her fixed cost C, delivers the goods to funders who bid and the game continues. (4) The entrepreneur sets her ex-post price p_2. (5) New buyers and funders who did not buy ex-ante decide whether to purchase at p_2.

We focus on three implications. First, the type correlation resulting from prior quality uncertainty adjusts optimal crowdfunding design but we can still solve by providing an algorithm. Second, we show how prices may rise, despite the downward logic of durable good monopoly. Third, we show when limited reach transforms crowdfunding and traditional finance from substitutes into mutual complements.

6.1 Profit-maximizing crowdfunding with ex-post demand

We focus on optimal profit-maximizing choices of (T, p) and p_2 when the entrepreneur borrows at a zero market interest rate (online Appendix B provides the details).

[26] Capacity constraints with demand uncertainty (*e.g.*, Dana, 2001; Gale and Holmes, 1993) or consumers learning preferences (*e.g.*, Courty and Li, 2000; Möller and Watanabe, 2010; Nocke et al., 2011) can also explain price rises but our entrepreneurial learning resonates best with crowdfunding evidence.

[27] Platforms certainly want to prevent fee evasion and can gain by providing this commitment power.

With correlated valuations, unrestricted general mechanism design would enable the decision maker to extract the efficient surplus from funders (see Cremer and McLean, 1988) but simple reward-based crowdfunding cannot implement this, so we solve directly.

We start with optimal ex-post pricing. If the entrepreneur learns that exactly k funders are H-type, she updates her belief about project quality being G to

$$\eta_k^{N_1} = \frac{\gamma^G f_k^{N_1}(q^G)}{\gamma^G f_k^{N_1}(q^G) + \gamma^B f_k^{N_1}(q^B)} \tag{19}$$

and her belief that a new buyer is type H to $\bar{q}_k^{N_1} = \eta_k^{N_1} q^G + (1-\eta_k^{N_1})q^B$. These beliefs are increasing in k so that the optimal ex-post price equals v_H for sufficiently high k. The pivot on k for setting $p_2 = v_H$ depends on whether crowdfunding is inclusive or not.

Lemma 1. *(i) If the entrepreneur observes k H-type funders and all funders buy in the crowdfunding stage, the optimal ex-post price is $p_2^I(k) = v_H$ for all $k \geq m_I$ and $p_2^I(k) = v_L$ otherwise, where $m_I = \min(\{k : \bar{q}_k^{N_1} > \hat{q}\} \cup \{N_1 + 1\})$.*

(ii) If the entrepreneur observes k H-type funders and only H-type funders buy in the crowdfunding stage, the optimal ex-post price is $p_2^E(k) = v_H$ for all $k \geq m_E$ and $p_2^E(k) = v_L$ otherwise, where $m_E = \min(\{k : \bar{q}_k^{N_1} N_2 v_H > (N_2 + N_1 - k)v_L\} \cup \{N_1 + 1\}) \geq m_I$.

Next we analyze the crowdfunding stage. Note that H-type funders are more optimistic: they assign probability $\tilde{\gamma}^G = \eta_1^1 \geq \gamma^G$ to project quality G and the relevant hazard rate becomes $\tilde{h}_n$.[28] The entrepreneur now anticipates her ex-post sales revenues and has beliefs $\bar{f}_k^{N_1} = \gamma^G f_k^{N_1}(q^G) + \gamma^B f_k^{N_1}(q^B)$. These are the only modifications to the optimal inclusive solution. With inclusive pivot $n > 0$, H-types are willing to pay $\tilde{b}_n^I = \tilde{h}_n v_H + (1 - \tilde{h}_n)v_L > v_L$, so the entrepreneur learns the number k of H-type funders and updates her beliefs to $\eta_k^{N_1}$. She adapts the price, setting $p_2^I(k)$ as defined in Lemma 1(i). Note that with inclusion, price can only rise.

The optimal exclusive solution is more involved because H-type funders have the option to wait and buy ex-post at a potentially lower price. To dissuade waiting, the entrepreneur must set p low enough: Under exclusion with pivot n, H-type funders will pay at most $\tilde{b}_n^E = \tilde{h}_n v_H + (1 - \tilde{h}_n)\mathbb{E}[p_2]$, where $\mathbb{E}[p_2]$ denotes the price an H-type expects to pay given he waits and crowdfunding succeeds. Writing $\mathbb{E}[p_2]$ as $\alpha_n v_H + (1 - \alpha_n)v_L$, $\tilde{b}_n^E = \alpha_n v_H + (1 - \alpha_n)\tilde{b}_n^I$. Only when $\alpha_n = 1$ are H-type funders willing to pay v_H. When $\alpha_n = 0$, $\tilde{b}_n^E = \tilde{b}_N^I$ and the exclusive and inclusive strategies are equivalent: L-type funders are then not really excluded – they all buy later. When $0 < \alpha_n < 1$, exclusion of L-type funders is partial, suggesting that exclusion is rather ineffective compared to our baseline model without ex-post sales. After partial exclusion, the price can go up or down.

[28] $\tilde{h}_n = \tilde{f}_{n-1}^{N_1-1} / \tilde{S}_{n-1}^{N_1-1}$ where $\tilde{f}_k^{N_1-1} = \eta_1^1 f_k^{N_1-1}(q^G) + (1 - \eta_1^1) f_k^{N_1-1}(q^B)$, $\tilde{S}_n^{N_1-1} = \sum_{k=n}^{N_1-1} \tilde{f}_k^{N_1-1}$.

Proposition 11. *(i) The optimal inclusive strategy for pivot $n > 0$ has $T = (N_1 - n)v_L + n\tilde{b}_n^I$, $p = v_L$ and ex-post price $p_2^I(k)$, where $\tilde{b}_n^I = \tilde{h}_n v_H + (1 - \tilde{h}_n)v_L$. Profit equals*

$$\pi_n^I = \sum_{k=n}^{N_1} \bar{f}_k^{N_1}(k\tilde{b}_n^I + (N_1 - k)v_L - C) + \sum_{k=n}^{N_1} \bar{f}_k^{N_1} \max\left\{N_2 v_L, \bar{q}_k^{N_1} N_2 v_H\right\}$$

(ii) The optimal exclusive strategy for pivot n has $T = np$, $p = \tilde{b}_n^E$ and ex-post price $p_2^E(k)$, where $\tilde{b}_n^E = \alpha_n v_H + (1 - \alpha_n)\tilde{b}_n^I$ and $\alpha_n = \tilde{S}_{m_E}^{N_1-1}/\tilde{S}_n^{N_1-1}$. Profit equals

$$\pi_n^E = \sum_{k=n}^{N_1} \bar{f}_k^{N_1}(k\tilde{b}_n^E - C) + \sum_{k=n}^{N_1} \bar{f}_k^{N_1} \max\left\{(N_2 + N_1 - k)v_L, \bar{q}_k^{N_1} N_2 v_H\right\}$$

(iii) Let $n_I = \arg\max \pi_n^I$ and $n_E = \arg\max \pi_n^E$. Inclusion is optimal when $\pi_{n_I}^I \geq \pi_{n_E}^E$.

In our timing, ex-post price is chosen after the crowdfunding stage. If the entrepreneur could commit to the ex-post price during crowdfunding, she might commit to a higher price to dissuade waiting to make exclusion more effective: the new price pivot becomes $m_E' \leq m_E$ so L-type funders are excluded with a higher probability and H-type funders pay more. Below we illustrate how price commitment affects optimal crowdfunding design. Full price commitment is not plausible but limiting available rewards indirectly raises commitment. We illustrate the optimal design consequences below.

6.2 Illustration

Using Proposition 11, we now solve specific cases explicitly to illustrate how (i) credit demand can increase relative to TS, (ii) ex-post price can increase or decrease, and (iii) without ex-post price commitment, the entrepreneur includes more often, but can counter this by limiting the number of rewards. To do so, we adapt Illustration 1, so that the project now has $q^G = 0.3$ (case (i)) with probability $\gamma^G = 0.5$ and $q^B = 0.2$ (case (ii)) with probability $\gamma^B = 0.5$. There is an ex-post market with $N_2 = 500$ buyers and we raise the cost by the revenues from selling to the new buyers at 5, so that $C = 2650 + 2500 = 5150$.

Ideally, the entrepreneur would like to exclude in the crowdfunding stage, learn about project quality by observing the number of H-types k, and then sell at v_L or v_H, depending on her posterior. In particular, if the entrepreneur could commit to the ex-post price, she would set $p_2(k) = v_H$ for $k \geq 124$. This induces a high expected price in period two and this allows her to exclude with pivot $n_E = 109$ and price $p = 18.52$, yielding a profit of 429.01. Note that price goes down when $k < 124$ and otherwise rises. Without commitment, however, exclusion is ineffective because $m_E = 401$, inducing a low expected ex-post price. Consequently, optimal exclusion requires a pivot of $n_E = 157$ and H-type funders pay $p = \tilde{b}_{n_E}^E = 6.68$, resulting in profit $\pi_{n_E}^E = 16.27$. Inclusion is optimal, yielding a profit of 181.36 by setting $p = 5$ and $T = 2527.25$, so that $n_I = 109$ and $m_I = 124$

while H-types bid $\tilde{b}^I_{n_I} = 5.25$. Clearly, with inclusion, price cannot fall and it rises when $k \geq 124$. The entrepreneur needs external credit whenever crowdfunding is successful. Credit demand thus increases relative to the case of TS which has no production since expected revenues equal $5000 < C$.

Interestingly, the entrepreneur achieves some price commitment power by restricting the number of units offered in crowdfunding, because rationed H-type funders wait to buy ex-post.[29] The entrepreneur would optimally offer $\bar{n}_H = 120$ rewards destined for H-types at price $b_H = 10.69$ and $\bar{n}_L = 165$ rewards at price $b_L = 5$ destined for L-types. Setting threshold $T = 1990.21$ implies a production pivot of $n = 109$. Note that when all rewards are sold, the entrepreneur does not learn k exactly but only that $120 \leq k \leq 335$. It is then ex-post optimal and credible for her to set $p_2(k) = v_H$, while setting $p_2(k) = v_L$ when $109 \leq k < 120$.[30] The expected ex-post price is higher, allowing her to extract more rent from H-type funders and generate a profit of 309.21. This case readily explains the phenomenon of higher ex-post prices in big successes such as PicoBrew Zymathic's automatic beer brewing appliance that sold a limited number at \$1599 during crowdfunding on Kickstarter and sold ex-post for \$1999 dollars.[31]

6.3 Credit

Section 4.3 established that crowdfunding (CF) fully substitutes for traditional finance (TF) in the baseline model, but with limited reach, revenues from ex-post sales generate a demand for credit that reward-based crowdfunding can only cover partially. Crowdfunding can even be strictly complementary with traditional finance thanks to its credible market test: the funds raised in crowdfunding ex-ante signal profitability of the ex-post market and may convince traditional financiers to step in. This mutual complementarity sheds light on why venture capital and angel investors have been joining forces with entrepreneurs after successful crowdfunding campaigns (see Mollick and Kuppuswamy, 2014, Table 3). We now characterize when CF is a substitute or complement of TF.

In traditional selling (TS), only $N_1 + N_2$ matters. The project is viable if $C \leq \hat{C}^{TS} \equiv (N_1 + N_2)\max\{v_L, \bar{q}v_H\}$, where $\bar{q} = \gamma^B q^B + \gamma^G q^G$. Lacking any ex-ante revenues, an entrepreneur (with no personal funds) must borrow the full fixed cost C whenever she produces, so her credit demand $D^{TS} = C$ if $C \leq \hat{C}^{TS}$ and $D^{TS} = 0$ if $C > \hat{C}^{TS}$.

The demand for credit D^{CF} under crowdfunding is stochastic; it depends on state k through the funds raised, denoted F_k. There are three ranges: (i) for low k, $F_k < T$, there is no production and $D^{CF}_k = 0$; (ii) for intermediate k, $T \leq F_k < C$, there is production and $0 < D^{CF}_k < C$; (iii) for high k, $F_k \geq C$ and $D^{CF}_k = 0$. So we have:

[29] Dana (2001) limits the number of low-priced units to raise rent-extraction from high types. Our entrepreneur also limits high-priced units to make high ex-post pricing credible.

[30] For details, see online Appendix B.

[31] www.kickstarter.com/projects/1708005089/picobrew-zymatic-the-automatic-beer-brewing-applia/description.

Proposition 12. *(a) For high fixed costs $C > \hat{C}^{TS}$, crowdfunding raises credit demand compared to traditional selling. (b) For $C \leq \hat{C}^{TS}$, crowdfunding reduces credit demand.*

In case (a), crowdfunding raises borrowing demand since the adaptation makes production viable and the crowd's funds do not always cover cost. In case (b), crowdfunding lowers credit demand for two reasons. First, crowdfunding is a strategic substitute since it reduces production in relatively unprofitable states. Second, crowdfunding directly substitutes as a source of credit in each production state with $k > 0$.

7 Multiple types

The baseline model restricts to two types of buyer. This allowed us to characterize the optimal reward-based crowdfunding mechanism by showing how an appropriate threshold and minimum price implement the general optimum. We now: (1) generalize to settings with $J > 2$ types and (2) solve for the profit-maximizing general mechanism. We state necessary and sufficient conditions for crowdfunding to achieve the general optimum. Example 2 illustrates how crowdfunding profits can be strictly lower. Finally, we show how a multiple token extension of crowdfunding can implement the general optimum.

7.1 Notation

Each buyer's valuation is now an independent draw from $\mathcal{V} = \{v_1, \ldots, v_J\}$ with probabilities $\mathbf{q} = (q_1, ..., q_J)$ where $\sum_{j=1}^{J} q_j = 1$, each $q_j \in (0, 1)$, and $0 \leq v_1 < \ldots < v_J$; bold letters denote $1 \times J$ vectors. The demand state is now summarized by $\mathbf{k}$ where k_j is the number of buyers with valuation v_j for each $j = 1, ..., J$. We also define: cumulative probabilities, $Q_j = \sum_{j' \leq j} q_{j'}$; the j'th unit vector, $\mathbf{e_j}$; for $M = N-1, N$, $\Omega_M = \{\mathbf{k} \in \mathbb{N}^J : \mathbf{k} \cdot \mathbf{1} = M\}$ where $\mathbf{1} = (1, ..., 1)$.[32] The probability of a state $\mathbf{k} \in \Omega_M$ is given by:

$$f_{\mathbf{k}}^{M}(\mathbf{q}) = \binom{M}{\mathbf{k}} \prod_{j}^{J} q_j^{k_j} \tag{20}$$

where $\binom{M}{\mathbf{k}} = {}^{M!}/_{k_1!\ldots k_J!}$, $\binom{0}{0,\ldots,0} = 1$ and $\binom{M}{\mathbf{k}} = 0$ if any $k_j < 0$ or $> M$. We suppress $\mathbf{q}$ where not confusing and for non-trivial production, we assume: $C < Nv_J$.

7.2 Optimal general mechanism

We again use virtual valuations: for type j, $w_j = v_j - (v_{j+1} - v_j)(1 - Q_j)q_j$. We assume strict monotonicity: $\mathbf{w} = (w_1, ..., w_J)$ with $w_1 < w_2 < ... < w_J$.[33] Noting that $w_J = v_J >$

[32] States in Ω_N represent realized demands; states in Ω_{N-1} are relevant for a buyer estimating how other buyers behave. Both sets lie in $\mathbb{N}^J$ and $\forall j$, $\mathbf{k} \in \Omega_{N-1} \implies \mathbf{k} + \mathbf{e_j} \in \Omega_N$.

[33] Ironing techniques readily deliver similar results for the general case.

0, we define $j^* = \min\{j : w_j \geq 0\}$ and $\mathbf{w}^+$ replaces negative values in $\mathbf{w}$ by 0.

In the general optimal solution, types $j < j^*$ are excluded and pay nothing, higher types get the good in states $\mathbf{k}$ in the production set, $K^* = \{\mathbf{k} \in \Omega_N : \mathbf{w}^+ \cdot \mathbf{k} \geq C\}$. Maximal expected transfers τ_j from types $j \geq j^*$ then follow recursively from incentive compatibility and the equilibrium probabilities P_j with which a type j buyer anticipates getting the good. $P_j = 0$ for all $j < j^*$. For $j \geq j^*$, defining $K^*_{-j} = \{\mathbf{k} \in \Omega_{N-1} : \mathbf{k} + \mathbf{e_j} \in K^*\}$, the set of other buyer demands for which production occurs if j plays the equilibrium,

$$P_j = \sum_{\mathbf{k} \in K^*_{-j}} f_{\mathbf{k}}^{N-1}(\mathbf{q}).$$

Without loss of generality, P_j is strictly increasing on $j \geq j^*$.[34] For all $j < j^*$, individual rationality implies $\tau_j = 0$ as $P_j = 0$. For $j \geq j^*$, τ_j is given recursively by: $\tau_{j^*} = v_{j^*} P_{j^*}$ (individual rationality) and $\tau_{j+1} = \tau_j + (P_{j+1} - P_j) v_{j+1}$ (incentive compatibility that $j+1$ is just willing to not masquerade as type j).

These expected payoffs can always be implemented with buyers stating bids they pay when production occurs. An optimal indirect mechanism has buyers choose bids from the set $\{0, b^*_{j^*}, b^*_{j^*+1}, \ldots, b^*_J\}$ where $\mathbf{b}^*$ is the unique bid strategy $\mathbf{b}$: $b_j = 0$ for $j < j^*$ as $P_j = 0$; $b_{j^*} = v_{j^*}$; $b_j = \tau_j / P_j$ for $j > j^*$. Defining $H_j = 1 - {}^{P_{j-1}}/_{P_j}$ on $j > j^*$, $b_j = b_{j-1}(1 - H_j) + H_j v_j > b_{j-1}$ as P_j's monotonicity implies $H_j > 0$; higher types pay strictly higher prices. The profile of N bids exactly reveals the demand state $\mathbf{k} \in \Omega_N$ and the entrepreneur must produce if and only if $\mathbf{k} \in K^*$, then providing the good to all buyers with a strictly positive bid. But in crowdfunding, some threshold on aggregate funds must determine production. So it is possible to implement the general optimum if and only if $\{\mathbf{k} \in \Omega_N : \mathbf{b}^* \cdot \mathbf{k} \geq T\} = K^*$ for some T; minimum price $p = b_{j^*}$ then allocates goods optimally, given production. If any such T exists, $T = \min_{\mathbf{k} \in K^*}\{\mathbf{b} \cdot \mathbf{k}\}$ is one such threshold. Together with $p = b_{j^*}$, this defines a crowdfunding mechanism that achieves the general optimum. As shown above, this is always possible in our baseline with $J = 2$ but not when $J > 2$, as the following example demonstrates.

Example 2. *Let $J = 3$, $N = 2$, $\mathcal{V} = \{0, 1, 2\}$ and $\mathbf{q} = (1/4, 1/2, 1/4)$, with $1 < C < 2$. Then $\mathbf{w} = (-3, 1/2, 2)$ so the general optimum excludes type 1 and has production in states in $K^* = \{(0,0,2), (1,0,1), (0,1,1)\}$. Type-wise bids are $\mathbf{b} = (0, 1, 7/4)$ and the expected profit is $(18 - 7C)/16$. Crowdfunding cannot implement this outcome because threshold $T = \min_{\mathbf{k} \in K^*}\{\mathbf{b} \cdot \mathbf{k}\} = 7/4$ just reached in state $\mathbf{k} = (1,0,1)$ also generates production in state $(0,2,0)$.*

While simple reward-based crowdfunding cannot implement the general optimum, it is easy to adjust crowdfunding to restore implementability. For simplicity, assume $w_{j^*} > 0$.

[34] Virtual valuations are increasing, so $\mathbf{k} + \mathbf{e_j} \in K^* \Rightarrow \mathbf{k} + \mathbf{e_{j+1}} \in K^*$ so $P_{j+1} \geq P_j \; \forall j \geq j^*$ and if $P_{j+1} = P_j$, combining all $j+1$'s into j's gives identical outcomes.

The entrepreneur sells $J - j^* + 1$ types of tokens. For each $j \geq j^*$, she charges w_j type j tokens for type j's reward and sells each type j token at b_j/w_j euros. She applies her production threshold T to the token aggregate and sets $T = C$. In this mechanism, a type j buyer buys w_j refundable tokens of type j and bids them all on reward j.

8 Concluding remarks

We have characterized the optimal design of crowdfunding in a private value environment. We demonstrated the twin roles of crowdfunding's threshold mechanism in adapting production and pricing to the crowd's revealed demand and its signal of future demand, and in price discrimination that enhances adaptation except when excessive thresholds waste trade opportunities. Even in our *i.i.d.* model, both benefits can be substantial for surprisingly large crowds. Also, with binary types, more general mechanisms relaxing crowdfunding's reassuring constraints cannot deliver higher profits.

We showed funding is not fundamental but Section 6 showed how reward-based crowdfunding can substitute for traditional finance *or* be mutually complementary. Introducing investment-based elements fully substituted for traditional finance in Corollary 4 but we expect economies of scale from centralized monitoring (Diamond, 1984) and expertise (Gompers and Lerner, 2001) to complement the "wisdom of the crowd." Adding financial rewards in the project-quality model of Section 6 creates a common value effect. Our tractable framework could be extended to study *interdependent* prices, an important feature of investment-based crowdfunding, missing in the literature.

Our reward-based insights are also relevant for donation-based crowdfunding. In the model, funders only cared for the good, but the results apply exactly if funders gain recognition or "warm-glow" values v_L, v_H from contributing at least the minimum bid to a charitable cause or public good. Nonetheless, to properly understand the donation environment will require a richer model of preferences and the public good technology.

In sum, crowdfunding enables many entrepreneurs to bring otherwise infeasible projects to life. Its rent-extraction role is particularly important at high costs. Hansmann (1981) marshals the evidence in Baumol and Bowen (1968) to argue that "voluntary price discrimination" is critical to the survival of theatres, museums and opera, since observed contributions cannot be explained by vertical differentiation alone. We proved that product differentiation is orthogonal to rent-extraction in crowdfunding. Thoroughly testing this and other implications of our analysis will require cunning techniques to estimate costs and valuation distributions, but might permit more refined policy conclusions. A good first test could exploit the sequentiality of actual crowdfunding, by examining whether high price rewards are chosen less often after the threshold is reached, extinguishing pivotality motives.

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Appendix A

Mostly omitting argument q, we start by stating some useful mathematical results relating $f_k^M(q)$, $S_n^M(q) = \sum_{k=n}^M f_k^M(q)$ and $h_n(q) = f_{n-1}^{N-1}(q)/S_{n-1}^{N-1}(q)$.

Lemma A.1.

(i) $f_k^N = qf_{k-1}^{N-1} + (1-q)f_k^{N-1}$

(ii) $S_n^N = S_{n-1}^{N-1} - (1-q)f_{n-1}^{N-1}$

(iii) $\sum_{k=n}^N kf_k^N = qNS_{n-1}^{N-1}$ *and* $\mathbb{E}[k|k \geq n] = \frac{qN}{1-(1-q)h_n}$ *for all* $N \geq 1$ *and* $0 \leq n \leq N$.

(iv) $\sum_{k=n}^N (N-k)f_k^N = (1-q)NS_n^{N-1}$, *for all* $N \geq 1$ *and* $0 \leq n \leq N$.

(v) $\frac{\partial f_k^M(q)}{\partial q} = f_k^M \frac{k-Mq}{q(1-q)}$

(vi) $\frac{\partial S_n^N(q)}{\partial q} = Nf_{n-1}^{N-1}$

(vii) h_n *is strictly increasing in* n *for* $0 \leq n \leq N$, *with* $h_0 = 0$ *and* $h_N = 1$.

(viii) *For* $0 < n < N$, $\frac{\partial h_n(q)}{\partial q} < 0$.

(ix) $n(1-q)h_n \geq n - qN$ *where the inequality is strict when* $q > 0$ *and* $n < N$

Proof of Lemma A.1.

(i) is immediate on expanding on any one draw and $N-1$ other independent draws.

(ii) Summing (i) from $k=n$ to N and recalling that $f_N^{N-1}=0$

$$\begin{aligned} S_n^N &= qS_{n-1}^{N-1} + (1-q)S_n^{N-1} \\ &= qS_{n-1}^{N-1} + (1-q)\left(S_{n-1}^{N-1} - f_{n-1}^{N-1}\right) \\ &= S_{n-1}^{N-1} - (1-q)f_{n-1}^{N-1} \end{aligned}$$

(iii)
$$\begin{aligned} \sum_{k=n}^{N} k f_k^N &= \sum_{k=n}^{N} kq^k(1-q)^{N-k}\frac{N!}{(N-k)!k!} \\ &= Nq\sum_{k=n}^{N} q^{k-1}(1-q)^{N-1-(k-1)}\frac{(N-1)!}{(N-1-(k-1))!(k-1)!} \\ &= Nq\sum_{k=n}^{N} f_{k-1}^{N-1} = Nq\sum_{k=n-1}^{N-1} f_k^{N-1} = NqS_{n-1}^{N-1} \end{aligned}$$

From (ii), $S_n^N/S_{n-1}^{N-1} = 1-(1-q)h_n$, so $\mathbb{E}[k|k\geq n] = \frac{qNS_{n-1}^{N-1}}{S_n^N} = \frac{qN}{1-(1-q)h_n}$.

(iv) Using (ii) and (iii),

$$\begin{aligned} \sum_{k=n}^{N}(N-k)f_k^N(q) &= N\left((1-q)\left(S_{n-1}^{N-1} - f_{n-1}^{N-1}\right)\right) - NqS_{n-1}^{N-1} \\ &= N(1-q)(S_{n-1}^{N-1} - f_{n-1}^{N-1}) \\ &= N(1-q)S_n^{N-1} \end{aligned}$$

(v) Differentiating, $$\begin{aligned} \partial f_k^M(q)/\partial q &= \binom{M}{k} q^{k-1}(1-q)^{M-k-1}\Big[k(1-q)-(M-k)q\Big] \\ &= \frac{\binom{M}{k}q^k(1-q)^{M-k}}{q(1-q)}(k-Mq) = f_k^M\frac{k-Mq}{q(1-q)} \end{aligned}$$

(vi) Differentiating the summation that defines S_n^N using (v) gives,

$$\begin{aligned} \partial S_n^N(q)/\partial q &= \sum_{k=n}^{N}(k-Nq)f_k^N\Big/q(1-q) \\ &= \left(NqS_{n-1}^{N-1} - NqS_n^N\right)\Big/q(1-q) \text{ (from (iii))} \\ &= \left(S_n^N + (1-q)f_{n-1}^{N-1} - S_n^N\right)N\Big/(1-q) \text{ (from (ii))} \\ &= Nf_{n-1}^{N-1} \end{aligned}$$

(vii) From the definition it is clear that $h_0=0$ and $h_N=1$. We will show that h_n is strictly increasing by induction. As a first step, note that $h_N = 1 > h_{N-1}$ since $f^N_{N-1} > 0$, for all $q \in (0, 1)$. Now suppose that $h_N > h_{N-1} > ... > h_{n+2} > h_{n+1}$ for $N-1 \geq n+1 \geq 0$. We have to show that $h_{n+1} > h_n$ follows.

Note that

$$h_{n+2} > h_{n+1} \Leftrightarrow \frac{f^{N-1}_{n+1}}{S^{N-1}_{n+1}} > \frac{f^{N-1}_{n}}{S^{N-1}_{n}} \Leftrightarrow \frac{S^{N-1}_{n}}{f^{N-1}_{n}} > \frac{S^{N-1}_{n+1}}{f^{N-1}_{n+1}} \tag{*}$$

Next observe that for any $N-1 \geq k \geq 0$,

$$\frac{f^{N-1}_{k+1}}{f^{N-1}_{k}} = \frac{\binom{N-1}{k+1} q^{k+1}(1-q)^{N-k-2}}{\binom{N-1}{k} q^{k}(1-q)^{N-k-1}} = \frac{q}{1-q}\frac{N-k-1}{k+1}$$

This is clearly decreasing in k so that in particular,

$$\frac{f^{N-1}_{n}}{f^{N-1}_{n-1}} > \frac{f^{N-1}_{n+1}}{f^{N-1}_{n}}$$

Combined with the induction hypothesis expressed as (*), we have

$$\frac{S^{N-1}_{n}}{f^{N-1}_{n-1}} > \frac{S^{N-1}_{n+1}}{f^{N-1}_{n}}$$

Adding 1 to both sides of the inequality yields

$$\frac{S^{N-1}_{n-1}}{f^{N-1}_{n-1}} > \frac{S^{N-1}_{n}}{f^{N-1}_{n}}$$

which is precisely $1/h_n > 1/h_{n+1}$, completing the proof by induction.

(viii) $$\frac{\partial h_n(q)}{\partial q} = \left[(\partial f^{N-1}_{n-1}/\partial q)\sum_{k=n-1}^{N-1} f^{N-1}_k - f^{N-1}_{n-1}\sum_{k=n-1}^{N-1}(\partial f^{N-1}_k/\partial q)\right] \Big/ (S^{N-1}_{n-1})^2$$

$$= f^{N-1}_{n-1}\left[(n-1-(N-1)q)\sum_{k=n}^{N-1} f^{N-1}_k - \sum_{k=n}^{N-1} f^{N-1}_k(k-(N-1)q)\right] \Big/ q(1-q)(S^{N-1}_{n-1})^2$$

$$= \frac{f^{N-1}_{n-1}\sum_{k=n}^{N-1} f^{N-1}_k (n-1-k)}{q(1-q)(S^{N-1}_{n-1})^2} < 0$$

The inequality follows from the facts that the summation is over $k > n-1$, $f^{N-1}_k > 0$ on the summation range and $f^{N-1}_{n-1} > 0$ for $n \geq 1$ and the summation range is non-trivial for $n \leq N-1$. Note that when n takes its extremal values of $n = 0$ and $n = N$, the derivative equals zero since h_n is then fixed at 0 and 1, respectively.

(ix) Clearly the inequality holds when $q = 0$ or $n = N$. Observe next that $\forall n < N$, $n < \mathbb{E}[k|k \geq n] = \frac{qN}{1-(1-q)h_n}$ by Lemma A.1(iii), so $qN > n(1-(1-q)h_n)$ as claimed. ■

Proof of Proposition 1. We first prove (6). Note that (IC_H) must bind because otherwise one raises profits by increasing some $t_i(v_H, v_{-i})$. The ex-ante utility of i is

$$\sum_{v \in V} \big(p_i(v)\rho(v)v_i - t_i(v)\big)q(v) = (1-q)(P_L^i v_L - \tau_L^i) + q(P_H^i v_H - \tau_H^i)$$
$$= P_L^i v_L - \tau_L^i + q(v_H - v_L)P_L^i = U_L^i + q(v_H - v_L)P_L^i$$

where the second equality uses the binding (IC_H) constraint and $U_L^i = P_L^i v_L - \tau_L^i$ denotes the L-type bidder's expected utility. Summing over all bidders i,

$$\sum_{i=1}^{N}\sum_{v\in V} p_i(v)\rho(v)v_i q(v) = \sum_{i=1}^{N}\sum_{v\in V} t_i(v)q(v) + \sum_{i=1}^{N}\big(q(v_H - v_L)P_L^i + U_L^i\big)$$

Substituting for P_L^i,

$$\begin{aligned}
\sum_{i=1}^{N}\sum_{v\in V} t_i(v)q(v) =& \sum_{i=1}^{N}\sum_{v_{-i}\in V_{-i}} (1-q)p_i(v_L, v_{-i})\rho(v_L, v_{-i})v_L q_{-i}(v_{-i}) \\
&+ \sum_{i=1}^{N}\sum_{v_{-i}\in V_{-i}} q p_i(v_H, v_{-i})\rho(v_H, v_{-i})v_H q_{-i}(v_{-i}) \\
&- \sum_{i=1}^{N}\left(\sum_{v_{-i}\in V_{-i}} q(v_H - v_L)p_i(v_L, v_{-i})\rho(v_L, v_{-i})q_{-i}(v_{-i}) + U_L^i\right) \\
=& \sum_{i=1}^{N}\sum_{v_{-i}\in V_{-i}} p_i(v_L, v_{-i})\rho(v_L, v_{-i})[v_L(1-q) - q(v_H - v_L)]q_{-i}(v_{-i}) \\
&+ \sum_{i=1}^{N}\sum_{v_{-i}\in V_{-i}} q p_i(v_H, v_{-i})\rho(v_H, v_{-i})v_H q_{-i}(v_{-i}) - \sum_{i=1}^{N} U_L^i \\
=& \sum_{i=1}^{N}\sum_{v\in V} (w_i p_i(v)\rho(v) - U_L^i)q(v) \qquad (21)
\end{aligned}$$

Next we show that the derived solution satisfies the other constraints. Note from the arguments in the main text that the solution has $p_i(v_L, v_{-i}) \leq p_i(v_H, v_{-i})$, implying that $P_L^i \leq P_H^i$. Given (IC_H) binds, this implies (IC_L). Moreover, $P_L^i \leq P_H^i$ and (IR_L) imply (IR_H). Finally, any profit-maximizing solution clearly satisfies (NDE).

Finally, we derive the profit expressions. Using (7), $U_L = 0$ and Lemma A.1(iii), the profit in the exclusive case equals

$$\pi_{n_E}^E = \sum_{k=n_E}^{N} f_k^N (k v_H - C) = S_{n_E}^N \left(\frac{qNv_H}{1-(1-q)h_{n_E}} - C\right)$$

Using additionally $w_L = (v_L - qv_H)/(1-q)$ so that $Nw_L = Nv_L - qN(v_H - v_L)/(1-q)$

and $v_H - w_L = (v_H - v_L)/(1-q)$, the profit in the inclusive case equals

$$\pi^I_{n_I} = \sum_{k=n_I}^{N} f_k^N((N-k)w_L + kw_H - C) = S^N_{n_I}\left(Nw_L - C + \frac{qN(v_H - w_L)}{1-(1-q)h_{n_I}}\right)$$
$$= S^N_{n_I}\left(Nv_L - C + \frac{qN(v_H - v_L)h_{n_I}}{1-(1-q)h_{n_I}}\right)$$

■

Proof of Proposition 2. We here complement and prove the arguments stated in the main text preceding the Proposition. We distinguish three cases.

Case I: (i) $w_L < 0$ and $\lambda < \bar{\lambda}$ or (ii) $w_L \geq 0$. In both subcases, $v_L + \lambda w_L > 0$, so that $p_i(v) = 1$ for all v. So the solution is fully inclusive here ($\phi = 1$). Note that an increase in $\rho(v)$ strictly raises $\mathcal{L}$ for all v with $n(v) > \tilde{n}_I(\lambda)$ where this (production) pivot is uniquely determined by $\tilde{n}_I(\lambda)(v_H + \lambda w_H) + (N - \tilde{n}_I(\lambda))(v_L + \lambda w_L) = (1+\lambda)C$. That is,

$$\tilde{n}_I(\lambda) \equiv \frac{(1-q+\lambda)(C - Nv_L) + \lambda q(Nv_H - C)}{(v_H - v_L)(1-q+\lambda)} = \frac{C - Nv_L + \lambda(C - Nw_L)}{v_H - v_L + \lambda(v_H - w_L)}$$

Notice that $\tilde{n}_I(0) = \tilde{n}^*$ and $\tilde{n}_I(\bar{\lambda}) = \tilde{n}_E$ and that $\tilde{n}_I(\lambda)$ is increasing in λ. Moreover, when $w_L \geq 0$, for any $\lambda \geq 0$, $\tilde{n}_I(\lambda) < \tilde{n}_I \leq \tilde{n}_E$ where $\tilde{n}_I$ is defined in (9). So λ determines a production pivot, $n \leq n_E$. So the set of feasible pivots under (NDE) with full inclusion is indeed $\mathcal{I}$. When $\mathcal{I} = \varnothing$, $w_L < 0$ and there is no solution with $\lambda < \bar{\lambda}$. Recall $n_I^W = \min \mathcal{I}$ when $\mathcal{I} \neq \varnothing$. If $n_I^W = n^*$, the first-best is feasible and (NDE) does not bind ($\lambda = 0$). If $n = n_I^W > n^*$, there exists a unique $0 \leq \rho^W < 1$ such that $\rho^W A_{n-1,1} + (1-\rho^W)A_{n,1} = (\rho^W S^N_{n-1} + (1-\rho^W)S^N_n)C$. Letting $\rho^W = 0$ when $n_I^W = n^*$, it is then optimal to produce with probability $\rho(v) = \rho^W$ when $n(v) = n_I^W - 1$ and to produce with probability 1 when $n(v) \geq n_I^W$. That is, the production pivot is determined by a lottery between n_I^W and $n_I^W - 1$. The Lagrange multiplier λ is given by $\tilde{n}_I(\lambda) = n_I^W - 1$, and satisfies $0 \leq \lambda < \bar{\lambda}$.

Case II: $\lambda > \bar{\lambda}$. Here, instead $p_i(v_L, v_{-i}) = 0$, excluding L-types. (That is, $\phi = 0$.) Clearly, it is then optimal to set $\rho(v) = 1$ whenever $n(v) \geq n_E$.[35]

Case III: $\lambda = \bar{\lambda}$. Now $p_i(v_L, v_{-i})$ does not affect $\mathcal{L}$ but it does affect expected transfers, so it is chosen to relax (NDE). As in case II, $\rho(v) = 1$ is optimal whenever $n(v) \geq n_E$, implying that the production pivot is n_E. Including L-types at a given $n(v) \geq n_E$, raises welfare but lowers expected revenues. In particular, a marginal increase in the probability $p_i(v_L, v_{-i})$ then raises welfare by $v_L q(v)$ and lowers transfers by $-w_L q(v)$, so expected transfers can be transformed into additional welfare at the rate $|v_L/w_L|$.

[35]Since $\lambda > 0$, the welfare optimum requires (NDE) to bind. This is achieved by leaving some rent to H-types without affecting total welfare.

Since this is independent of v, we simply let ϕ denote the probability that an L-type receives the good when there is production, that is, $p_i(v_L, v_{-i}) = \phi$ for all v. The optimal inclusion probability, denoted ϕ^W, is defined in (13). Note that $A_{n_E,\phi}$ is linear and strictly decreasing in ϕ. From exclusive profit-maximization we have $A_{n_E,0} > S^N_{n_E}C$. Note further that, by definition of $\mathcal{I}$, $A_{n_E,1} < S^N_{n_E}C$ if and only if $\mathcal{I} = \varnothing$. Hence, $0 < \phi^W < 1$ if and only if $\mathcal{I} = \varnothing$.

Case III clearly yields higher welfare than case II and when $\mathcal{I} \neq \varnothing$, case I yields higher welfare than case III. This concludes the proof. ■

Proof of Corollary 1. Recall that full inclusion is always feasible for $q \leq \hat{q}$. $\mathcal{I}$ then contains the profit-maximizing pivot n_I, which never exceeds n_E. So the welfare-maximizer is fully inclusive. In addition, she sets $n^W_I \leq n_I$ so L-types get the good with a higher probability than under profit-maximization. For larger q, the profit-maximizer is fully exclusive, whereas a welfare-maximizer never fully excludes: she is fully inclusive if $\mathcal{I} \neq \varnothing$ and partially inclusive even when $\mathcal{I} = \varnothing$ since $\phi^W > 0$. ■

Proof of Proposition 3. Case (ii) is obvious. In case (i), $\phi = 1$ so that $p = v_L$ and $b_H = b_n \equiv h_n v_H + (1 - h_n)v_L$ with $n = n_I$. In general, bidding above $p = v_L$ can be attractive only if it increases the probability of production. In a candidate equilibrium where L-types bid $p = v_L$, H-types bid b_n and threshold $T_n = n\delta_n + Nv_L$ where $\delta_n \equiv b_n - v_L$, an individual buyer bidding $b \geq p$ generates project success rate S^{N-1}_{ℓ} where $\ell = \lceil \frac{b-p}{\delta_n} \rceil$. Bidding above p reduces by ℓ the number of the other $N-1$ buyers who need to be H-type for the project to succeed. Bid increments that do not raise ℓ are weakly dominated, so we need only consider bids of the form $b = v_L + \ell\delta_n$ for integer values of ℓ. We need to check that H-types are willing to set $\ell = 1$. Deviating to $\ell = 0$ is not a problem by incentive compatibility in the general mechanism solution. It remains to verify that deviating to a bid $b = v_L + \ell\delta_n$ is weakly inferior for integer values of $\ell \geq 2$ in the case of $n = n_I$, but it is as simple to prove it for all n so we do.

In the putative equilibrium with $p = v_L$ and $T = Nv_L + n\delta_n$, if the two types continue to make respective bids, $b_H = v_L + \delta_n$ and $b_L = v_L$, then the production probability is S^N_n. From the perspective of a single buyer of the H-type playing the equilibrium strategy, this probability is higher at S^{N-1}_{n-1}, and falls to S^{N-1}_n if he deviates to bid v_L, but rises to $S^{N-1}_{n-\ell}$ if he deviates to the proposed bid with some $\ell \geq 2$. The first two options give this buyer the same expected utility because inequality (IC_H) binds; this payoff is $(v_H - v_L)\, S^{N-1}_n$. The deviation option gives,

$$(v_H - v_L - \ell\delta_n)\, S^{N-1}_{n-\ell}$$

So, substituting for $\delta_n = h_n\,(v_H - v_L)$ and dividing by $(v_H - v_L)\, S^{N-1}_{n-\ell}$, we seek to show that,

$$(1 - \ell h_n) \leq S^{N-1}_n / S^{N-1}_{n-\ell}, \qquad \forall \ell \geq 2$$

Now the right-hand side can be written as the product of $(1 - h_n)\,(1 - h_{n-1}) \,....\, (1 - h_{n-\ell})$, but h_n is increasing in n, so this expression weakly exceeds $(1-h_n)^\ell$. Now $h_n \in [0,1]$ so defining $a = 1 - h_n$, we have $a \in [0,1]$, so for any $\ell \geq 1$,

$$1 - a^\ell = (1-a)\left(1 + ... + a^{\ell-1}\right) \leq (1-a)\ell$$

Rearranging terms and substituting back for a, this gives $1 - \ell h_n \leq (1-h_n)^\ell$, concluding the proof of implementation.

This optimal outcome is still uniquely implemented in pure strategy Pareto undominated equilibrium. The only candidates for alternative Pareto undominated equilibria are where $n' \neq n$ H-types are needed who all bid $b'_{n'} = (T_n - (N-n')v_L)/n'$. It is readily verified that this breaks H's IC when $n' < n$ and when $n' > n$, it is an equilibrium but is Pareto dominated as the entrepreneur and H-types are worse-off: an H-type buyer expects to obtain $(v_H - b'_{n'})S^{N-1}_{n'-1} < (v_H - v_L)S^{N-1}_{n'-1}$ (as $b'_{n'} > v_L$) while in the optimal equilibrium he obtains $(v_H - b_n)S^{N-1}_{n-1} = (v_H - v_L)S^{N-1}_n \geq (v_H - v_L)S^{N-1}_{n'-1}$. ■

Proof of Proposition 4. The proof of case (i) is similar to that of Proposition 3(i) and therefore omitted. As for case (ii), note that $b^W = v_H - \phi^W(v_H - v_L)(1 - h_{n_E})$ and that $T^W = n_E b^W + (N - n_E)b_L$ where $b_L = \phi^W v_L$. L-types are just willing to pay b_L while H-types are exactly indifferent between bidding b^W for the reward and bidding b_L for the lottery. Proposition 2 implies that any other BNE gives lower total and consumer welfare, and is thus Pareto dominated. ■

Proof of Corollary 2. For $q > \hat{q}$, where exclusion is optimal, the intuitive result that profits are decreasing in C and increasing in q is easily verified from the profit expression:

$$\pi^E_{n_E} = \sum_{k=n_E}^{N} f^N_k(kv_H - C) = \mathbb{E}_k[\max\{0, kv_H - C\}]$$

where $\mathbb{E}_k$ denotes the expectation operator. Since an increase in q induces a first-order stochastic dominating distribution of k, and the expectation is taken over an increasing (utility) function, the expectation is increasing in q. The impact of C is more immediate: profits fall at the rate $S^N_{n_E}$.

For $q \leq \hat{q}$, note that $\pi^I_n = (Nv_L - C)S^N_n + (v_H - v_L)qNf^{N-1}_{n-1}$, which is clearly strictly decreasing in C. Taking derivatives with respect to q yields,

$$
\begin{aligned}
\frac{\partial \pi_n^I}{\partial q} &= (Nv_L - C)(\frac{\partial S_n^N}{\partial q}) + N(v_H - v_L)\left(f_{n-1}^{N-1} + \frac{\partial f_{n-1}^{N-1}}{\partial q} q\right) \\
&= (Nv_L - C)Nf_{n-1}^{N-1} + N(v_H - v_L)f_{n-1}^{N-1}\left(1 + q\frac{n-1-(N-1)q}{q(1-q)}\right) \\
&\qquad\qquad\qquad\qquad \text{(by Lemmas A.1(vi) and (v))} \\
&= (Nv_L - C)Nf_{n-1}^{N-1} + N(v_H - v_L)f_{n-1}^{N-1}\left(\frac{n-Nq}{1-q}\right) \\
&= \frac{Nf_{n-1}^{N-1}}{1-q}\Big((Nv_L - C)(1-q) + (v_H - v_L)(n - Nq)\Big) \\
&= \frac{N(v_H - v_L)f_{n-1}^{N-1}}{1-q}\Big(n - \tilde{n}_I\Big)
\end{aligned}
$$

Recall that $\tilde{n}_I = \frac{C - Nv_L + q(Nv_H - C)}{v_h - v_L}$ and $n_I = \lceil \tilde{n}_I \rceil$. So $n_I > \tilde{n}_I$ except at critical values of q at which $n_I = \tilde{n}_I$. These exceptional values have measure zero; they occur on the boundary between strategy types. It follows that the maximal profit $\pi_{n_I}^I$ is strictly increasing in q. ■

Proof of Proposition 6. We prove that $n_I' \leq n_I$. The statements about profits, consumer and total welfare follow.

We define for each n,

$$
C_n(q) = \frac{N(v_L - qv_H) + n(v_H - v_L)}{1-q} \tag{22}
$$

From (9) it follows that the entrepreneur sets $n_I = n$ in the region between curves $C = C_{n-1}(q)$ and $C = C_n(q)$, picking the more efficient, lower n_I on the boundaries. Hence, $n_I = \arg\min_n \{C \leq C_n(q)\}$. Feasibility of the n-type strategy requires $C \leq T_n$. In particular, feasibility is guaranteed for all $n \geq 1$ when $C = Nv_L$. We show first that the n_I-type strategy is feasible by demonstrating that $C_n(q) < T_n$ for all $n < N$:

$$
\begin{aligned}
& \frac{N(v_L - qv_H) + n(v_H - v_L)}{1-q} < Nv_L + nh_n(v_H - v_L) \\
\Leftrightarrow\ & N(v_L - qv_H) + n(v_H - v_L) < N(1-q)v_L + nh_n(1-q)(v_H - v_L) \\
\Leftrightarrow\ & n(v_H - v_L)(1 - h_n(1-q)) < qN(v_H - v_L)
\end{aligned}
$$

The result follows from Lemma A.1(ix).

Next we show that there exist unique values $0 < q_1' < \cdots < q_{N-1}'$ so that the entrepreneur is indifferent between strategies of type n and $n+1$ (as long as both are feasible) when $q = q_n'$, independently of C. Note that $\pi_n' = (qNS_{n-1}^{N-1} - nS_n^N)\delta_n'$ where $\delta_n' = h_n(v_H - p_n') = h_n(Nv_H - C)/(N - nh_n)$. Hence, $\left[\pi_{n+1}' - \pi_n'\right]/(Nv_H - C)$ is independent of C. There must exist a q_n' where the entrepreneur is indifferent, because the difference is strictly negative when $q > 0$ is very small while it is strictly positive when $q < 1$ is close to one. Straightforward calculations show that a marginal increase in q

above q_n' increases the difference $\pi_{n+1}' - \pi_n'$, and the uniqueness result follows.

Similar steps show that at $q_n = n/N$, $\pi_n'(q_n) > \pi_{n+1}'(q_n)$, which implies that $q_n' > q_n$. It then follows that the optimal inclusive strategy is of type n_I' where n_I' is the smallest n such that both $q \leq q_n'$ and $C \leq T_n$. ■

Proof of Proposition 9. An inclusive strategy with pivot n gives expected revenue,

$$\begin{aligned} R_n^I(q) &= S_n^N(Nv_L + \mathbb{E}[k|k \geq n]h_n(v_H - v_L)) \\ &= S_n^N\left(\frac{(1-h_n)Nv_L + h_n(qNv_H)}{1-(1-q)h_n}\right) \\ &= NS_{n-1}^{N-1}\left((1-h_n)v_L + h_n(qv_H)\right) \end{aligned} \tag{23}$$

using $b_n = h_n v_H + (1-h_n)v_L$ and Lemma A.1(iii) then (ii). S_{n-1}^{N-1} falls with n and so does the term in parentheses for $q < \hat{q}$ since then $v_L > qv_H$; recall that h_n rises with n. $R_n^I(q)$ is then strictly decreasing in n. In particular, when $n_I^W < n_I$ the platform makes strictly higher expected profits from not-for-profits, proving claim b(i). (This also holds on $q = \hat{q}$ except that $R_0^I(\hat{q}) = R_1^I(\hat{q})$.)

An exclusive strategy with pivot n gives expected revenue,

$$R_n^E(q) = S_n^N \mathbb{E}[k|k \geq n]v_H = NS_{n-1}^{N-1}qv_H \tag{24}$$

Clearly, $R_n^E(q)$ is strictly decreasing on $n \geq 1$ and increasing in q. Recall that entrepreneurs only use exclusive strategies on $q > \hat{q}$ and on here $qv_H > v_L$ so $R_n^E(q) > R_n^I(q)$. In particular, the platform makes strictly lower profit from welfare-maximizing entrepreneurs if they adopt inclusive strategies with $n_I^W = n_E$, proving claim b(ii). If instead $n_I^W < n_E$, the platform may prefer either not-for-profits for their higher success probability or profit-maximizers for their higher conditional expected revenue.

When $C \leq Nv_L$, $n_I^W = 0 < n_E$. For $C \leq v_H$, $n_E = 1$ and $Nv_L = R_0^I(\hat{q}) = R_1^E(\hat{q})$ and $R_0^I(q) < R_1^E(q)$ for all $q > \hat{q}$. Hence, we define $\bar{q}(C) = \hat{q}$ in this region. For $C > v_H$, $n_E > 1$ and $R_0^I(\hat{q}) > R_{n_E}^E(\hat{q})$. $R_{n_E}^E(q)$ is strictly increasing in q and $R_{n_E}^E(1) = Nv_H > Nv_L = R_0^I(q)$. Continuity implies uniqueness and existence of $\bar{q}(C)$ defined by $Nv_L = R_{n_E}^E(\bar{q}(C))$. Finally, define $\underline{q}(C) = (Nv_L - C)/(Nv_H - C)$. Then for $q \leq \underline{q}(C)$, $n_I = 0$, because $\underline{q}(C) = C_0^{-1}(q)$ where C_0 was defined in the proof of Proposition 6. This proves claims a(i),(ii),(iii). ■

Proof of Proposition 10. Using Lemma A.1 (iii) and (ii), we can rewrite the profit as,

$$\pi(p) = -S_n^N C + mpqNS_{n-1}^{N-1} = \big(mpqN - C\big)S_n^N + mpqN(1-q)f_{n-1}^{N-1}$$

Using Lemma A.1 (vi) and (v), the optimal p must satisfy,

$$
\begin{aligned}
0 = \frac{\partial \pi}{\partial p} \quad &= \quad \big(mpqN - C\big)Nf_{n-1}^{N-1}q' + (mqN + mpq'N)S_n^N \\
&\quad + mpqN(1-q)f_{n-1}^{N-1}\frac{n-1-(N-1)q}{q(1-q)}q' \\
&\quad + \big(mqN(1-q) + mpN(1-2q)q'\big)f_{n-1}^{N-1}
\end{aligned}
$$

Using again Lemma A.1 (ii), and defining $c = C/Nm$, this is equivalent to,

$$
\begin{aligned}
0 \quad &= \quad \big(pqN - cN\big)f_{n-1}^{N-1}q' + (q + pq')\big(S_{n-1}^{N-1} - (1-q)f_{n-1}^{N-1}\big) \\
&\quad + pf_{n-1}^{N-1}\big(n-1-(N-1)q\big)q' \\
&\quad + \big(q(1-q) + p(1-2q)q'\big)f_{n-1}^{N-1}
\end{aligned}
$$

Taking out a factor S_{n-1}^{N-1} and rearranging yields,

$$0 = q + pq' + \big((n-1)p - cN\big)h_n q'. \blacksquare$$