Maximality in finite-valued Łukasiewicz logics defined by order filters

Marcelo Coniglio[1], Francesc Esteva[2], Joan Gispert[3], and Lluís Godo[4]

[1] Centre for Logic Epistemology and the History of Science, University of Campinas, Brazil.
coniglio@cle.unicamp.br
[2] Artificial Intelligence Research Institute (IIIA), CSIC, Barcelona, Spain.
esteva@iiia.csic.es
[3] Facultat de Matemàtiques i Informàtica, Universitat de Barcelona, Barcelona, Spain.
jgispertb@ub.edu
[4] Artificial Intelligence Research Institute (IIIA), CSIC, Barcelona, Spain.
godo@iiia.csic.es

1 Preliminaries and first results

In this talk we consider the logics L^i_n obtained from the $(n+1)$-valued Łukasiewicz logics $Ł_{n+1}$ by taking the order filter generated by i/n as the set of designated elements. The $(n+1)$-valued Łukasiewicz logic can be semantically defined as the matrix logic

$$Ł_{n+1} = \langle \mathbf{LV_{n+1}}, \{1\}\rangle,$$

where $\mathbf{LV_{n+1}} = (LV_{n+1}, \neg, \to)$ with $LV_{n+1} = \left\{0, \frac{1}{n}, \dots, \frac{n-1}{n}, 1\right\}$, and the operations are defined as follows: for every $x, y \in LV_{n+1}$, $\neg x = 1 - x$ and $x \to y = \min\{1, 1 - x + y\}$.

Observe that $Ł_2$ is the usual presentation of classical propositional logic CPL as a matrix logic over the two-element Boolean algebra $\mathbf{B}_2$ with domain $\{0,1\}$ and signature $\{\neg, \to\}$. The logics $Ł_n$ can also be presented as Hilbert calculi that are axiomatic extensions of the infinite-valued Łukasiewicz logic $Ł_\infty$.

The following operations can be defined in every algebra $\mathbf{LV_{n+1}}$: $x \otimes y = \neg(x \to \neg y) = \max\{0, x + y - 1\}$ and $x \oplus y = \neg x \to y = \min\{1, x + y\}$. For every $n > 1$, $x^n = x \otimes \cdots \otimes x$ (n-times) and $nx = x \oplus \cdots \oplus x$ (n-times).

For $1 \le i \le n$ let $F_{i/n} = \{x \in LV_{n+1} \ : \ x \ge i/n\} = \left\{\frac{i}{n}, \dots, \frac{n-1}{n}, 1\right\}$ be the order filter generated by i/n, and let

$$\mathsf{L}^i_n = \langle \mathbf{LV_{n+1}}, F_{i/n}\rangle$$

be the corresponding matrix logic. From now on, the consequence relation of L^i_n is denoted by $\models_{\mathsf{L}^i_n}$. Observe that $Ł_{n+1} = \mathsf{L}^n_n$ for every n. In particular, CPL is L^1_1 (that is, $Ł_2$). If $1 \le i, m \le n$, we can also consider the following matrix logic: $\mathsf{L}^{i/n}_m = \langle \mathbf{LV_{m+1}}, F_{i/n} \cap LV_{m+1}\rangle$.

The logic $\mathsf{L}^1_2 = \langle LV_3, \{1, 1/2\}\rangle$ was already known as the 3-valued paraconsistent logic J_3, introduced by da Costa and D'Ottaviano see [4] in order to obtain an example of a paraconsistent logic maximal w.r.t. CPL.

Definition 1. Let L_1 and L_2 be two standard propositional logics defined over the same signature Θ such that L_1 is a proper sublogic of L_2. Then, L_1 is *maximal* w.r.t. L_2 if, for every formula φ over Θ, if $\vdash_{L_2} \varphi$ but $\nvdash_{L_1} \varphi$, then the logic L_1^+ obtained from L_1 by adding φ as a theorem, coincides with L_2.

In order to study maximality among finite-valued Łukasiewicz logics defined by order filters we obtain the following sufficient condition:

Theorem 1. *Let $L_1 = \langle \mathbf{A}_1, F_1 \rangle$ and $L_2 = \langle \mathbf{A}_2, F_2 \rangle$ be two distinct finite matrix logics over a same signature Θ such that $\mathbf{A}_2$ is a subalgebra of $\mathbf{A}_1$ and $F_2 = F_1 \cap A_2$. Assume the following:*

1. *$A_1 = \{0, 1, a_1, \ldots, a_k, a_{k+1}, \ldots, a_n\}$ and $A_2 = \{0, 1, a_1, \ldots, a_k\}$ are finite such that $0 \notin F_1$, $1 \in F_2$ and $\{0, 1\}$ is a subalgebra of $\mathbf{A}_2$.*
2. *There are formulas $\top(p)$ and $\bot(p)$ in $\mathcal{L}(\Theta)$ depending at most on one variable p such that $e(\top(p)) = 1$ and $e(\bot(p)) = 0$, for every evaluation e for L_1.*
3. *For every $k + 1 \leq i \leq n$ and $1 \leq j \leq n$ (with $i \neq j$) there exists a formula $\alpha^i_j(p)$ in $\mathcal{L}(\Theta)$ depending at most on one variable p such that, for every evaluation e, $e(\alpha^i_j(p)) = a_j$ if $e(p) = a_i$.*

Then, L_1 is maximal w.r.t. L_2.

We use this result to prove that

Theorem 2. *Let $1 \leq i, m \leq n$. Then L^i_n is maximal w.r.t. $\mathsf{L}^{i/n}_m$ if the following condition holds: there is some prime number p and $k \geq 1$ such that $n = p^k$, and $m = p^{k-1}$.*

Corollary 1. *Let $1 \leq i \leq p$. For every prime number p, L^i_p is maximal w.r.t.* CPL

Notice that the above corollary generalizes the well known result: *L_{p+1} is maximal w.r.t.* CPL *for every prime number p.*

Definition 2. Let L_1 and L_2 be two standard propositional logics defined over the same signature Θ such that L_1 is a proper sublogic of L_2. Then, L_1 is *strongly maximal* w.r.t. L_2 if, for every finitary rule $\varphi_1, \ldots, \varphi_n/\psi$ over Θ, if $\varphi_1, \ldots, \varphi_n \vdash_{L_2} \psi$ but $\varphi_1, \ldots, \varphi_n \nvdash_{L_1} \psi$, then the logic L^*_1 obtained from L_1 by adding $\varphi_1, \ldots, \varphi_n/\psi$ as structural rule, coincides with L_2.

Let i be a strictly positive integer, the *i-explosion rule* is the rule (exp_i) $\dfrac{i(\varphi \wedge \neg\varphi)}{\bot}$.

Lemma 1. *For every $1 \leq i \leq n$, the rule (exp_i) is not valid in L^i_n.*

Corollary 2. *Let $1 \leq i \leq p$. For every prime number p, L^i_p is not strongly maximal w.r.t.* CPL

2 Equivalent systems

Blok and Pigozzi introduce in [3] the notion of equivalent deductive systems in the following sense: Two propositional deductive systems S_1 and S_2 in the same language $\mathcal{L}$ are equivalent iff there are two translations τ_1, τ_2 (finite subsets of $\mathcal{L}$-propositional formulas in one variable) such that:

- $\Gamma \vdash_{S_1} \varphi$ iff $\tau_1(\Gamma) \vdash_{S_2} \tau_1(\varphi)$,
- $\Delta \vdash_{S_2} \psi$ iff $\tau_2(\Delta) \vdash_{S_1} \tau_2(\psi)$,
- $\varphi \dashv\vdash_{S_1} \tau_2(\tau_1(\varphi))$,
- $\psi \dashv\vdash_{S_2} \tau_1(\tau_2(\psi))$.

Theorem 3. *For every $n \geq 2$ and every $1 \leq i \leq n$, L^i_n and L^{n+1} are equivalent deductive systems.*

From the equivalence among L_n^i and $Ł_{n+1}$, we can obtain, by translating the axiomatization of the finite valued Łukasiewicz logic $Ł_{n+1}$, a calculus sound and complete with respect L_n^i that we denote by H_n^i.

Since $Ł_\infty$ is algebraizable and the class MV of all MV-algebras is its equivalent quasivariety semantics, finitary extensions of $Ł_\infty$ are in 1 to 1 correspondence with quasivarieties of MV-algebras. Actually, there is a dual isomorphism from the lattice of all finitary extensions of $Ł_\infty$ and the lattice of all quasivarieties of MV. Moreover, if we restrict this correspondence to varieties of MV we get the dual isomorphism from the lattice of all varieties of MV and the lattice of all axiomatic extensions of $Ł_\infty$. Since $Ł_{n+1} = \mathsf{L}_n^n$ is an axiomatic extension of $Ł_\infty$, $Ł_{n+1}$ is an algebraizable logic with the class $MV_n = \mathcal{Q}(Ł\mathbf{V}_{n+1})$, the quasivariety generated by $Ł\mathbf{V}_{n+1}$, as its equivalent variety semantics. It follows from the previous theorem that L_n^i, for every $1 \leq i \leq n$, is also algebraizable with the same class of MV_n-algebras as its equivalent variety semantics. Thus, the lattices of all finitary extensions of L_n^i are isomorphic, and in fact, dually isomorphic to the lattice of all subquasivarieties of MV_n, for all $0 < i < n$.

Therefore maximality conditions in the lattice of finitary (axiomatic) extensions correspond to minimality conditions in the lattice of subquasivarieties (subvarieties). Thus, given two finitary extensions L_1 and L_2 of a given logic L_n^i, where K_{L_1} and K_{L_2} are its associated MV_n-quasivarieties, L_1 is strongly maximal with respect L_2 iff K_{L_1} is a minimal subquasivariety of MV_n among those MV_n-quasivarieties properly containing K_{L_2}. Moreover, if L_1 and L_2 are axiomatic extensions of L_n^i, then K_{L_1} and K_{L_2} are indeed MV_n-varieties. In that case, L_1 is maximal with respect L_2 iff K_{L_1} is a minimal subvariety of MV_n among those MV_n-varieties properly containing K_{L_2}.

The lattice of all axiomatic extensions $Ł_\infty$ is fully described also by Komori in [7], thus from the equivalence of Theorem 3, we can obtain the following maximality conditions for all axiomatic extensions of L_n^i.

Theorem 4. *Let $0 < i, m \leq n$ be natural numbers such that $m|n$. If L is an axiomatic extension of L_n^i, then L is maximal with respect to $\mathsf{L}_m^{i/n}$ iff $L = \mathsf{L}_m^{i/n} \cap \mathsf{L}_{p^{k+1}}^{i/n}$ for some prime number p with $p|n$ and a natural $k \geq 0$ such that $p^k|m$ and $p^{k+1} \not| m$.*

As a corollary we obtain that the sufficient condition of Theorem 2 is also necessary.

Corollary 3. *Let $1 \leq i, m \leq n$. Then L_n^i is maximal w.r.t. $\mathsf{L}_m^{i/n}$ if and only if there is some prime number p and $k \geq 1$ such that $n = p^k$, and $m = p^{k-1}$.*

To obtain results on strong maximality we need to study finitary extensions of $Ł_\infty$. The task of fully describing the lattice of all all finitary extensions of $Ł_\infty$, isomorphic to the lattice of all subquasivarieties of MV, turns to be an heroic task since the class of all MV-algebras is Q-universal [1]. For the finite valued case it is much simpler, since MV_n is a locally finite discriminator variety. Any locally finite quasivariety is generated by its critical algebras [5]. Critical MV-algebras were fully described in [6] and using this description we can obtain some results on strong maximality.

First we need to introduce the following matrix logics: For every $1 \leq i, m \leq n$,

$$\bar{\mathsf{L}}_n^i = \langle \mathbf{LV_{n+1}} \times \mathbf{LV_2}, F_{i/n} \times \{1\} \rangle \qquad \bar{\mathsf{L}}_m^{i/n} = \langle \mathbf{LV_{m+1}} \times \mathbf{LV_2}, (F_{i/n} \cap LV_{m+1}) \times \{1\} \rangle$$

Theorem 5. *Let $0 < i \leq n$ be natural numbers, let p be a prime number and let $r = \max\{j \in \mathbb{N} : p^j|n\}$. Then we have: For every j such that $(i-1)p < j \leq ip$, $\mathsf{L}_n^i \cap \bar{\mathsf{L}}_{p^{r+1}}^{j/np}$ is strongly maximal with respect to L_n^i. Moreover, every finitary extension of some L_k^j is strongly maximal with respect L_n^i iff it is one of the preceding types.*

As a particular case we can also prove the following result.

Theorem 6. *Let p be a prime number. Then, for every j such that* $0 < j \leq p$*:*

- $\bar{\mathsf{L}}_p^j$ *is strongly maximal with respect to* CPL *and it is axiomatized by* H_p^j *plus the* j*-explosion rule* (exp_j) $j(\varphi \wedge \neg\varphi)/\bot$.
- L_p^j *is strongly maximal w.r.t.* $\bar{\mathsf{L}}_p^j$.

3 Ideal paraconsistent logics

Arieli, Avron and Zamansy introduced in [2] the concept of *ideal paraconsistent logics.*

Definition 3. Let L be a propositional logic defined over a signature Θ (with consecuence relation $\vdash_L$) containing at least a unary connective $\neg$ and a binary connective $\to$ such that:

(i) L is paraconsistent w.r.t. $\neg$, i.e. there are formulas $\varphi, \psi \in \mathcal{L}(\Theta)$ such that $\varphi, \neg\varphi \nvdash_L \psi$; and $\to$ is a deductive implication, i.e. $\Gamma \cup \{\varphi\} \vdash_L \psi$ iff $\Gamma \vdash_L \varphi \to \psi$,.

(ii) There is a presentation of CPL as a matrix logic $L' = \langle \mathbf{A}, \{1\}\rangle$ over the signature Θ such that the domain of $\mathbf{A}$ is $\{0, 1\}$, and $\neg$ and $\to$ are interpreted as the usual 2-valued negation and implication of CPL, respectively, such that L is a sublogic of CPL.

Then, L is said to be an *ideal paraconsistent logic* if it is maximal w.r.t. CPL, and every proper extension of L over Θ is not $\neg$-paraconsistent.

Lemma 2. *Let* $0 < i \leq n$. L_n^i *is paraconsistent w.r.t.* $\neg$ *iff* $\frac{i}{n} \leq \frac{1}{2}$

Since for every $0 < i \leq n$, there is a term definable implication $\Rightarrow_n^i$ which is deductive implication next result follows from Theorem 6

Theorem 7. *Let p be a prime number, and let* $1 \leq i < p$ *such that* $i/p \leq 1/2$*. Then,* L_p^i *is a* $(p+1)$*-valued ideal paraconsistent logic.*[1]

References

[1] Michael E. Adams and Wiesjlaw Dziobiak, *Q-universal quasivarieties of algebras*, ***Proceedings of the American Mathematical Society***, 120 (1994), pp.1053–1059.

[2] Ofer Arieli, Arnon Avron, and Anna Zamansky, *Ideal paraconsistent logics*, ***Studia Logica***, vol. 99 (2011), no. 1-3, pp.31–60.

[3] Willem J. Blok and Don L. Pigozzi, *Local deduction theorems in algebraic logic*, ***Algebraic logic (Budapest, 1988), volume 54 of Colloq. Math. Soc. János Bolyai***, pp. 75–109. North-Holland, Amsterdam, 1991.

[4] Itala M. L. D'Ottaviano and Newton C. A. da Costa, *Sur un problème de Jaśkowski (in French)*, ***Comptes Rendus de l'Académie de Sciences de Paris (A-B)***, vol. 270 (1970), pp.1349–1353.

[5] Wiesjlaw Dziobiak, *On subquasivariety lattices of semi-primal varieties*, ***Algebra Universalis***, vol. 20 (1985), pp.127–129.

[6] Joan Gispert and Antoni Torrens, *Locally finite quasivarieties of MV-algebras*, ***ArXiv***, (2014) pp.1–14. Online DOI: http://arxiv.org/abs/1405.7504.

[7] Yuichi Komori, *Super-Lukasiewicz propositional logics*, ***Nagoya Mathematical Journal***, vol. 84 (1981), pp.119–133.

[1]Strictly speaking, in this claim we implicitly assume that the signature of L_p^i has been changed by adding the definable implication $\Rightarrow_p^i$ as a primitive connective.