Master’s Thesis:

Financial contagion in the interbank market

Supervisor: Pere Colet  
Student: Patricio C. Sánchez
"There are $10^{11}$ stars in the galaxy. That used to be a huge number. But it’s only a hundred billion. It’s less than the national deficit! We used to call them astronomical numbers. Now we should call them economical numbers."

RICHARD FEYNMAN
Abstract

Financial networks have been extensively studied in view of their importance to the world economy. In particular, the stability of the interbank market is the cornerstone for the prevention of financial crises, such as the one in 2007 when the bankruptcy of an initially small sector ended up collapsing the entire system. In this paper we will analyze the stability of interbank networks in terms of their connectivity, using the Gai and Kapadia model in which all banks are of the same size [1]. The objective is to perform an in-depth study, analyzing the dependence on the contagion probability and the contagion extend on the model parameters as well as on the network topology. Furthermore we extend the model to incorporate the diversity present in realistic financial networks and to account for external bank investments in private companies.
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to the IFISC staff, with whom to work is as fruitful as joyful,
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For all of you, this thesis will always be in part yours.

THANK YOU, GRACIAS, GRACIES, GRAZIE
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1 INTRODUCTION

In 2007, a global economic crisis hit the most developed countries in the world. This tragic period, known as the Great Recession [2], has its origin in the United States due to a combination of several factors, including product overvaluation, the high number of bank crimes and critical failures in economic regulation. This set of errors, coupled with globalization and the high connectivity among financial institutions, not only facilitated the emergence of the crisis in the first instance, but provided the ideal scenario for spreading at a dizzying pace around the world.

The Great Recession surprised economic experts both by its rapid spread and because of the extent to which it spread, affecting the entire world. One of the most determining factors for this titanic contagion was the interconnected structure of the interbank market, whereby the fall of a few banks caused a cascade effect that culminated with the collapse of the entire system. Until then, it was widely believed that the diversity of interbank connections mitigated exposure to a critical impact, thereby reducing the likelihood of contagion. However, reality turned out to be diametrically opposed to what was expected, and after the initial commotion, researchers soon came to work to understand the origin of the failure in their theories.

To understand the reason for these dramatic discrepancies, we must first understand the interconnected structure of the interbank market (see for example [3] for an extensive study of the Austrian market). To begin with, we must distinguish between two kinds of interbank interactions depending on the time scale in which they occur. On the one hand, we have purchases and long-term loans, which generate profits thanks to the upward trend of the economy. These agreements tend to have a low risk and a collateral\(^1\), but the gains derived from them are obtained late. The counterpart to these loans comes from the overnight interbank market, whose main function is to provide banks with the settlement of accounts at the end of the day by efficiently replacing liquidity excess in the system, thus enabling them to meet the requirements imposed by regulatory institutions \(^2\).

The overnight market is arguably the most important of all interbank markets to understand the rapid contagion of the crisis of 2007 [11], since it plays a fundamental role in the monetary and payment system of countries, at the same time as it provides the banks with an essential safety valve for its correct operation. Networking relationships create interdependencies with potential for contagion, in particular in the absence of a collateral as in the overnight market, thus triggering the system to be much more susceptible to a large scale breakdown of financial intermediation due to domino effects of insolvency [12], [13], [14], [15], [16].

\(^1\)In economics, a collateral is a property or an asset that a borrower offers to the lender as a way to secure the loan. In case the borrower cannot meet its duties with the loan, the lender can seize the collateral as an exchange for its losses.

The first classical bank-run model proposed by Diamond and Dybvig (1983) [17], and later extended by Allen and Gale (2000) [18], shows that in normal times the connections between banks lead to a better allocation of liquidity and a more effective distribution of risk among financial institutions, a phenomenon commonly known as “too interconnected to fail”. However, shortly afterwards Gai and Kapadia (2008) [1] studied just the opposite effect, that is, the amplification of shocks due to those same banking interconnections. Through their network model, they showed that the probability that a random initial shock would end up affecting a certain fraction of the total banks (i.e. systemic failure) is reduced by increasing the connectivity of the market, while the price to pay is that the total amount of banks that default in case of a systemic failure is dramatically enlarged.

This double-edged property, known as “robust yet fragile”, has been extensively studied (eg in [19]), and in particular related to the importance of the network topology [20]. The main finding of these works is that the more clustered networks are, the more robust to the contagion, but more fragile in the fall of one of the big banks. This fact helps to improve the results of regulatory policies, since the control of the most important banks leads to a clear improvement in the spread of risk.

In recent years, and following the philosophy of studying interbank networks, several new models have appeared with the aim of adding new ingredients to the original formula in a race to realism. To cite a few examples, there are a number of empirical studies related to interbank network topology (Boss et al for the Austrian market [13], Alves et al for the European market [21], Iori et al in Italy [22] and Soramäki for the US fedwire\(^3\) system [23]), indicating the scale-free nature of the network and more precisely, the existence of a core-periphery structure in which a fully connected central core is linked to smaller banks by a few links. On the other hand, Georg (2011) [24] presented a model in which banks were allowed to optimize their portfolios, taking into account the dynamic component of the system and the decision making. Later, Ladley (2013) [25] developed a model composed by three ingredients: households which can deposit their funds in banks, loans for the private investments of households, and banks that allocate their resources following a genetic algorithm to maximize their expected returns. HaÅflaj and Kok (2013) [26] similarly introduced an agent-based model in which the optimization of bank’s expected returns is done in order to study the emergence of network structures when varying certain prudential regulatory parameters. With respect to information considerations, Montagna and Kok (2016) [27] presented a multilayer model to take into account the different types of transactions between banks, considering that each of them has partial information of these layers and concluding that the contagion in the multilayer system is much greater than the simple sum of each of them, due to nonlinearities arising from the coupling between layers. Finally, recent studies from Battiston et al (2016) [28] point in the direction of distress propagation as a main part of financial contagion, (and not just direct default), in particular focusing on the importance of cycles when amplifying a financial shock.

\(^3\)Federal Reserve Wire Network
Thus, it is clear that the implementation of graph theory in the interbank market is a deeply studied field and still has a lot of predictive potential to be squeezed. In addition, it has not only proven to be extremely useful when understanding the origin of the rapid spread of the 2007 crisis, but also provides us with an invaluable useful tool for analyzing the effects of regulatory policies on the system, and knowing in advance their possible consequences.

1.1 Outline

Our objective will be to study the fundamental properties of the overnight interbank system from a network model, and to include some fundamental extensions to make it more realistic. However, we believe that simplicity is crucial for understanding the underlying mechanisms behind any problem, so we will often focus on analyzing the different parameters in order to extract the basic ideas behind them.

The starting point will be the Gai and Kapadia model [1], which will provide the essential parts of the system and it will allow us to understand the fundamentals of the interbank market, and from which we will study the importance of network connectivity in the risk of contagion (the so-called "too interconnected to fail" property).

The next logical step will be to take into account the heterogeneities of the graph, first studying the degree distribution, considering the scale-free nature of the connections, and later on by including banks with different sizes, thus analyzing the famous "too big to fail" property. We will then introduce correlations between the connectivity and the size of the banks. To end the study of heterogeneities, without leaving loose threads, we will conclude with the construction of the interbank network through certain rules that are governed only by the nature of the banks.

Finally, we will analyze how exogenous effects reinforce the direct contagion mechanism, by considering a private sector to which the interbank market is coupled.
2 THE GAI-KAPADIA MODEL FOR FINANCIAL CONTAGION

In this section we will present a network model in order to reproduce the most important features of the overnight interbank market. Nonetheless, the dynamical day by day evolution won’t be considered, since our study will focus on the impact of risk contagion.

Among the huge number of bank models in the literature, Gai and Kapadia’s model [1] (GK model from now) stands out as one of the most popular given its simplicity and predictability. Initially conceived as a simple way of introducing interdependencies between banks through a complex network, it has proved to be a source of inspiration in a number of papers [20] [29] [30] where the original idea is modified by extending it to different topologies, coupling different networks or studying the impact of regulatory policies. Following the same philosophy, we will first introduce the basic GK model, starting from the the bank’s balance sheet and finally defining the interbank relationships, to later add different ingredients, paying special attention to the different parameters that emerge to the surface as we expand the model.

2.1 Balance sheet

Before introducing the complex network into our model, we must first know how banks operate and which are their working regimes, ie the time scale in which they work with each of their financial operations. Although each bank has its own peculiarities and invests in different assets, there is a general framework in which all of them can be addressed and that will serve as a reference for our study.

A bank’s balance sheet is composed of a series of elements of different nature and origin, with different maturities and whose amount can vary greatly. However, they all belong to two different groups:

1. **Assets**, composed of anything that can be sold for a certain value.

2. **Liabilities**, those obligations that must eventually be paid, ie the claim of an asset.

The excess between assets and liabilities is known as equity, and is closely related to the bank’s solvency and its ability to cope with short-term liabilities.

Depending on the duration of the maturity, assets and liabilities can be long or short term. The first of them consist of long-term investments, mortagages and loans, and generate large dividends to the bank in exchange for keeping the money frozen for a considerable period of time (illiquid). On the other hand, short-term transactions tend to have two origins; either they are value fluctuations, for example due to customer deposits, or overnight interbank loans, whereby banks with excess liquidity provide to those who do not have it in a way that they can carry out the transformation of the maturity of long-term risk assets and short-term volatile liabilities.
The difference between assets and liabilities (commonly known as equity) of all banks must be positive in order to not be in bankruptcy. The equity of bank \( i \) is defined as:

\[
E_i = q A_i^{ext} + (1 - \phi_{i,t}) A_i^{IB} - L_i^{ext} - L_i^{IB} - D_i
\]

where \( q A_i^{ext} \) are the (illiquid) non-interbank assets, where \( q \) is a multiplier to consider firesales of the illiquid assets\(^4\). The term \( (1 - \phi_{i,t}) A_i^{IB} \) refers to interbank assets that will be returned, where \( \phi_i \) is the fraction of banks that owe money of bank \( i \) at time \( t \) and are in default. From the liabilities side, \( L_i^{IB} \) are liabilities from the interbank market, and \( D_i \) are the deposits of customers\(^5\). The condition to be in default is therefore \( E_i \leq 0 \), and to prevent banks from reaching this extreme case due to economic fluctuations, regulatory institutions impose a minimum capital buffer \( K \), typically established as \( 4 - 5\% \) of the total assets of the bank, \( E_i > K = 0.04 A_{\text{total}} \). In what follows we will assume that at the initial time all banks have the minimum capital, \( E_i = K = 0.04 A_{\text{total}} \).

As for the quantitative aspects of the model, the exact fractions of the elements of the balance sheet need to be fixed. Following the values mentioned in the literature\(^3\), we will consider that interbank assets represent a 16% of the total assets, with 4% of capital buffer imposed by regulatory institutions and 80% of illiquid assets. Interbank liabilities are automatically given by the interbank assets (network), while deposits are chosen so that previously values are met. For the moment we do not consider firesales \( (q = 1) \).

\(^4\)A bank that goes into default may try to sell illiquid assets to improve its balance sheet. These assets suffer a depreciation, which may eventually trigger a depreciation of the illiquid assets in the entire market. A detailed study of these effects can be found in section 3.4.

\(^5\)These deposits are taken such that the balance sheet is fullfilled up to capital requirements.
2.2 Network model

Banks that do not have enough cash at the end of the day go to the overnight market to obtain this money, either through loans with other banks or with the central bank. These banking interconnections are implemented in the model through a complex network. Given a bank $i$ borrowing a certain amount of liquidity to another bank $j$, we will have a directional edge with $i$ as source and $j$ as target. Therefore, ingoing links reflect interbank assets for the source node, while outgoing links are related to the corresponding liabilities.

Figure 2: Simple scheme of an interbank market. Bank A has liabilities with banks B and C, so that its failure would lead to the lost of the loans for the lenders. Simultaneously, bank B also owes money to bank A. Finally, bank B has a second liability with bank C.

Initializing the system is as simple as considering a directed network (see figures 2 and 3) with a certain topology, and then define the interbank assets of each bank evenly distributed among all its lenders, automatically setting all interbank liabilities. Then, we attack the network through an initial failure (random default) under zero recovery assumption\(^6\), that spreads over the system through the interbank exposures.

\(^6\)The recovery rate $R$ is a parameter to control the fraction of a loan that is returned to the lender in case of default of the borrower. For simplicity, in this work we will consider $R = 0$. 

6
The implementation of the interbank network model can be split in two main parts: the initialization of the networks and the banks’ properties, and the dynamical evolution to spread the initial default. For a pseudocode of the GK model see Appendix A: pseudocodes.

<table>
<thead>
<tr>
<th>NETWORK GENERATION</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. Generate a directed graph.</td>
</tr>
<tr>
<td>2. Assign a size to each node ( A^{total} ), and define the fraction of interbank assets ( A^{IB} ) and initial equity ( E ).</td>
</tr>
<tr>
<td>3. Distribute ( A^{IB} ) uniformly among all in-neighbors.</td>
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</tbody>
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<tr>
<th>CASCADE OF DEFAULTS</th>
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<tr>
<td>4. Select a random bank ( i ) to default ( E_i = 0 ).</td>
</tr>
<tr>
<td>5. All out-neighbors of bank ( i ) loose their loans, ( E_j \rightarrow E_j - e_{ij} ).</td>
</tr>
<tr>
<td>6. Compute new defaults and repeat Step 5.</td>
</tr>
</tbody>
</table>

2.3 The GK model in Erdős-Rényi networks and banks of the same size

The results of the GK model\(^8\) for an Erdős-Rényi network, where all banks have the same balance sheet size, are presented in Figure 4. Both the probability and the extent of contagion\(^9\) are represented as a function of the average network connectivity \( \langle k \rangle \). In order

\(^8\) The results are in complete agreement with previous works [1], [20]. However, we find them from a balance sheet composed by a 16\% of interbank assets, not a 20\% as mentioned in Caccioli’s paper.

\(^9\) That is, the fraction of banks that have entered bankruptcy at the end of the simulation.
to neglect smaller shocks, the likelihood of contagion has been computed as the fraction of runs in which at least 5% of the total banks go bankrupt. Correspondingly, the extent of the shock only refers to these systemic cases. In addition, from now on unless otherwise stated, results refer to a system with $N = 10^3$ banks and $10^3$ independent simulations.

Figure 4: Probability and extent of contagion of the GK model in an Erdős-Rényi graph with 1000 vertex. The probability of contagion continuously grows, reaching a maximum around $<k> = 3$, and then progressively decays until it vanishes for higher connectivities. The contagion rapidly spreads through the whole network. Statistics are obtained from $m = 1000$ independent simulations, where errorbars are so small that are not shown.

The first finding to catch our attention in Figure 4 is the existence of a finite window of connectivities in which the probability of contagion is non-zero. A priori, one could have assumed that the more connected the network the less susceptible the banks would be, given the diversification of risk through interbank lending. Hence, the evolution of the likelihood of contagion would follow a monotonously decreasing behavior.

However, the maximum likelihood of contagion is in a particular range of intermediate connectivities, and only for large values of the mean degree one find the diversification behavior that was expected. Thus, there is a concrete value of network connectivity for which the banking system is especially susceptible to financial contagion in crisis situations.

With regard to the extent of contagion, for low values of the mean degree we see how the behavior is approximately equal to the probability of contagion. This result is not particularly surprising, given that in this region financial spreading occurs as soon as a bank of the giant vulnerable cluster, or contiguous to it, is attacked and goes bankrupt.

\[ \text{Probability to a systemic failure } (\sum_i \delta(E_i) \geq 0.05N) \text{ can be understood as a binomial process, so that the error associated to each point is given by } \sqrt{p(1-p)/m}. \]

\[ \text{In fact, according to a large majority of economists, a high risk diversification ensures stability against financial contagion. Following the recent crisis of 2007, this argument was completely discredited and is now considered as false by most researchers in financial networks [32].} \]
thus extending the shock to the entire cluster (which roughly corresponds to the entire connected component of the network). As we increase the number of connections between banks, these two behaviors change, since banks begin to develop resilience to contagion. Nevertheless, the system presents the so called robust-yet-fragile\textsuperscript{12} property: since the extension of the contagion grows monotonously as a function of the connectivity, there is a double-edge effect in the form of a regime of connectivities for which the probability of contagion is low, but whose extension affects the whole system.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure5.png}
\caption{Comparison between random failure and targeted attack (highly/poorly connected with outgoing links). Selective attacks lead to the same results than random failure.}
\end{figure}

For completeness, the results of a targeted attack are presented in figure 5, where the cases of default of the bank with more out-degree and a bank with a single outgoing link are compared to random failure. On one hand, we observe that the bankruptcy of a poorly connected bank gives exactly the same behavior than in the random case: this is not surprising at all, since the default of the initial bank directly spreads through her neighbor, thus being equivalent to a random failure. On the other hand, when a highly connected bank falls, the situation is similar but with more links to spread. However, since the contagion resilience of banks is given by their in-degree, which is determined by the average degree of the graph, the evolution is the same than in the random case. Thus, selective attacks in Erdős-Rényi graphs are equivalent to random failures.

\textsuperscript{12}Robust-yet-fragile is a well-known concept in network theory, where it refers to the resistance to contagion of a network under random or targeted attacks. Economists borrowed the concept [33], giving it an alternate meaning to refer to networks in which the probability of contagion is relatively low, but the potential consequences of such contagion result in the collapse of the whole system (the extent of the contagion reaches all banks). It is this latter meaning that we will use in the paper, unless otherwise stated.
2.3.1 Capital buffer effects

At this point, one might ask which are the implications when varying the only free parameter of the system, i.e. the ratio between the number of interbank loans and the capital buffer that each bank maintains. Since this capital buffer is imposed by regulatory institutions, it acts as a measure to study the capacity to control financial contagion.

Figure 6 shows how more restrictive policies, maintaining a capital buffer of 5% in reserve, prevents the susceptibility of the system to a systemic contagion. Not only the window of connectivities for which contagion appears is reduced considerably with this small change, but the probability decreases and its maximum moves towards smaller values of the mean degree. On the contrary, a more flexible policy results in a much greater likelihood of contagion, and in fact the plateaux of intermediate connectivities for which the likelihood of contagion is maximal also increases.

Notice that once the contagion of a considerable number of banks occurs, the size of this shock only depends on the network topology, which in these three cases is an Erdős-Rényi graph. Hence, the results about the extent of the contagion are the same in all cases.

![Figure 6: Probability of contagion for different capital buffers. The larger the capital buffer, the smaller the probability of contagion and the window of susceptibility.](image)

Thus, despite having made only a first contact through a random network with simple rules, we have already found interesting results regarding the systemic risk of the system, and in particular about the importance of the connectivity of the network. However, the model is far from reflecting the complexity of the true interbank market, and many of the elements that are being overlooked can be decisive in the behavior of the system.

Our objective in the next sections will be to add wealth to the model initially devised by Gai and Kapadia, and increase in realism progressively without losing sight of simplicity in modeling, while studying the parameters that will inevitably be added to our work.
3 GAI-KAPADIA MODEL WITH HETEROGENEITIES

Despite the fact that the original GK model has proved to be very revealing when it comes to unveiling the importance of network connectivity, and although it has been an excellent starting point for the modeling of the interbank market given its simplicity, it is far from being a realistic model that includes the richness of real-world systems. In this direction, the first logical step towards the inclusion of this realism, using the empirical evidence observed in the interbank market, is to consider different sources of heterogeneities in the system.

First, our assumption about the network as a completely random graph misses the diversity of connections in the interbank market. Larger banks typically have many and varied counterparties \(^{13}\), which allows them to find financing when they need it at an optimal cost. By contrast, less important banks are limited to negotiating loans with a reduced amount of neighbors. Thus, the inclusion of this topological component in the model will be of vital importance.

On the other hand, having considered all banks as of equal size we have lost all the implications related to non-uniform exposure to financial risk. Unsurprisingly, banks not only move very different amounts of money according to their size, but the financial loans of the interbank market have different amounts. At the end of the day this translates into a completely heterogeneous loan distribution, diametrically opposed to the original approach of uniform loans.

3.1 Scale-free networks

The empirical evidence found in the literature \[3\] \[34\] \[35\] suggests that the interbank network is far from being purely random, as it would be in an Erdős-Rényi network. On the contrary, one finds that the number of connections between banks varies considerably, with a few banks highly connected to the rest of the network (serving as hubs) and many others with a reduced number of connections.

To understand how this property affects financial stability, we will analyze scale-free networks with power-law degree distributions, \(P(k) = k^{-\gamma}\). To explain these degree distributions, in 1999 Albert and Barabasi \[36\] presented a generative mechanism known as preferential attachment, with which one can create a scale-free network whose degree distribution follows a power-law with a fat tail. This directly translates into the existence of nodes with any number of connections, just as observed in the interbank market. As we shall see, this fact will have great consequences in terms of the properties of the system.

\[^{13}\]In economy, the term counterparty refers to the other part in a financial transaction. In particular, every buyer has a seller and vice versa.
For our model we will not consider an Albert-Barabasi network, but an equivalent stochastic method known as Chung and Lu model [37], which allows to independently set the in/out-degree exponents$^{14}$. As in the previous case with an Erdős-Rényi network, and from now on for the rest of this work (unless explicitly stated otherwise), we maintain the same fractions in the balance sheet and only the cases in which at least a 5% of the system defaults are considered. The only difference lies now in the network topology, which in fact is enough to greatly affect the behavior of the system, as we can see in figure 7.

![Figure 7: Probability and extent of contagion for an Erdős-Rényi and a scale-free graphs. The scale-free topology seems to reduce the probability to a systemic contagion, but widens the window of susceptible connectivities. The extent is roughly the same in both cases.](image)

The scale-free topology not only reduces the window of infection to almost half, thus enhancing the stability of the system, but it slightly affects the extent of the contagion, so that the robust-yet-fragile property of the network is preserved.

The effects of risk diversification are greatly mitigated in the case of a scale-free network, where the probability of contagion is reduced to almost a half. The reason for that is that highly connected banks distribute their interbank assets through a large number of neighbors, thus increasing their robustness to contagion (a property known as too interconnected to fail in economic terms).

Surprisingly, the decayment of the probability follows a long and progressive tail, which finally disappears around $\langle k \rangle = 10$. The reason for this behavior is that by increasing the connectivity of a scale-free network, the nodes that increase their mean degree more are those belonging to hubs, that is, the already highly connected ones. Therefore, poorly connected banks remain being susceptible since their degree does not increase, and con-

$^{14}$See Appendix B for a detailed study of this method.
sequently they are also the firsts to suffer contagion effects. On the contrary, banks with larger in-degree become still more robust to the bankruptcy of their neighbors as the average connectivity of the network increases. Hence, ”big” banks behave as a fire-wall that decreases the probability of contagion. Notwithstanding, as we increase the network connectivity, eventually we reach a point at which the average degree is so high that all banks lose their vulnerability to contagion.

The results found in this section change dramatically when the initial default is not random. The selective attack on certain banks may entail major changes in the evolution of the system, as it’s shown in Figure 8.

![Figure 8: Scale-free network under random and selective attack. In green, the attack to nodes with only one liability, which reduces the probability of contagion. In red, the targeted attack to banks with higher out-degree: for lower mean degrees this type of attack seems catastrophic for the system.](image)

As we can observe, attacking a bank with a single outgoing link reduces considerably the probability of contagion. The reason underlying this fact is simple: poorly connected nodes are linked with high probability to highly connected banks, and since assets are uniformly distributed, the shock to a ”big” bank is very small. On the contrary, a random attack can lead to the failure of any bank, so that, on average, its contagion probability is higher. The case of a targeted attack to the bank with more outgoing links is more subtle. For low values of the average degree the system can not distribute risk optimally, so that attacking a well connected bank leads to a fragile situation in which the probability of a systemic contagion is very high\textsuperscript{15}. As soon as the connectivity is large enough, banks become more and more resilient to contagion, so that the probability rapidly decreases.

\textsuperscript{15}We can say that the model for a scale-free network is ‘robust-yet-fragile’ in the network theory sense, given that the system is robust against a random attack but fragile under a selective one.
3.1.1 Different in and out exponents

To finalize this section, in figures 9 and 10 we find the results for scale-free networks with different in-degree and out-degree exponents. The motivation behind this analysis is based on the fact that although wealth distributions follow a power-law, banking networks from different parts of the world may follow slightly different exponents, and moreover they could reflect different distributions for lenders and borrowers. Given our special interest in the study of the different parameters of the model, we believe it is necessary to know the effects of different exponents for the two types of degree.

![Figure 9: Scale-free network with $\gamma_{\text{in}} = 3$ and different out-exponents. The lower the out-exponent, the smaller the probability to contagion.](image)

To change the out-exponent of the network translates directly into the distribution of liabilities, that are endogenously determined by the system through the interbank assets. Decreasing this exponent leads to a system where outgoing links are more heterogeneously distributed, so that one could expect that the initial default is more dangerous since it spreads through more neighbors, while the amount of these loans are independent from the out-degree. However, since the in-exponent is fixed, the risk diversification of each bank due to interbank assets is fixed as well. That means that highly connected banks are the most probable targets for loans, which in turn are the most robust banks against contagion. That is why reducing the out-exponent increases the robustness to contagion. On the contrary, varying the in-exponent affects the risk diversification in the system. Reducing the value of this exponent we obtain a system with ingoing links more distributed through banks, so that financial entities that would have received previously a big shock from a single link, now can share their interbank assets among several neighbors. On the other hand, "big" banks lose partially their robustness, so that some of them can now bankrupt under certain attacks. This fact not only reduces the probability of contagion and largely widens the contagion window, but it induces a shift into the maximum probability of contagion (since now there are no "small" banks to default in the first stages).
Figure 10: Scale-free network with $\gamma_{out} = 3$ and different $in$-exponents. The maximum of the contagion suffers a shift to higher $\langle k \rangle$, and is reduced as the exponent decreases.

3.2 Heterogeneous bank’s sizes

As in the case of heterogeneous degree distributions, empirical studies of the interbank market also point to a large disparity in the size of the accounts of financial institutions [38] [39]. These differences in size are not only reflected in the amount of the loans that each bank manages, but also affect how they are distributed, since decisions on how to distribute such loans are clearly influenced by the economic capacity of each entity.

Different economic studies [40] [41] [42] highlight that the distribution of wealth follows a power-law with an exponent between 2 and 3, so that to implement this empirical fact in our model we will consider a distribution of assets such that $P(A) \sim A^{2.5}$. Nonetheless, this heterogeneity not only affects the total amount of the loans, but also plays a key role in their allocation. Given a certain financial entity, the loans provided to the system are not evenly distributed among all counterparties, but the distribution of these is given by the size of each of the banks with which it maintains contact. Thus, one can consider that the amount of a loan is given by the interbank assets owned by bank $i$, multiplied by the relative size of the borrowing bank $j$ among the rest of the banks requiring liquidity,

$$e_{j \rightarrow i} = A_{iB}^{iB} \frac{A_{j}^{total}}{\sum_{<ik>} A_{k}^{total}}$$

where $< ik >$ sums for all banks with liabilities to bank $i$ (loans from $k$ to $i$).
The results for heterogeneous banks’ sizes are shown in Figure 11, for an Erdős-Rényi network. As we can observe, the maximum probability of contagion is lower in the heterogeneous case. This is not surprising at all, since now there are big banks that can absorb shocks from their neighbors due to their huge amount of capital, a feature known as ”too big to fail”. Thus, these banks mitigate the contagion spreading.

Nonetheless, the consequences of the redistribution of loans are worrisome: banks that would have survived to the default of a neighbor in the homogeneous case, now get bankruptcy due to the non-uniform exposures when the wrong neighbor falls. For this reason, a broad window of connectivities that previously did not show systemic contagion now does. Hence, we conclude that a heterogeneous distribution of loans reduce the effectiveness of risk diversification for intermedium and large connectivities.

With this, we are now in disposition to compare the differences between the famous ”too interconnected to fail” and ”too big to fail” properties. A simple procedure to do it is to perform targeted attacks to the most connected and the richest bank, and then analyze the regimes in which each of these properties are dominant.

As can be seen in Figure 12, for lower connectivities highly connected banks play a keyrole in the system, and attacking them leads to extremely large contagion probabilities. After reaching its maximum, the probability quickly decreases. For $\langle k \rangle \geq 3$, attacking these banks is less effective than a random default. Hence, for highly connected networks

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\[\text{Figure 11: Simulations of an Erdős-Rényi network for homogeneous (blue) and heterogeneous (orange) distribution of assets and interbank loans. Heterogeneities in bank’s size leads to a long tail of connectivities that are susceptible to contagion.}\]
this type of attack enhances the resilience of the system to contagion.

On the other hand, to start the contagion with the default of the richest bank produces a monotonous increasing of the contagion probability. Around \( \langle k \rangle = 2 \) the curve intersects the evolution corresponding to the attack of the richest bank, and from this moment keeps the higher probability of the three cases. It is easy to understand why attacking big banks is critical: the default of the richest bank involves a large amount of money, and hence a huge shock that eventually spreads to the whole system. As far as the connectivity of the network is high enough to propagate, the systemic contagion is ensured. Thus, we conclude that the *too big to fail* property is dominant except for very low connectivities.

![Figure 12: Results for a selective attack of the highly connected bank and the richest one in a scale-free network with heterogeneous assets. At lower connectivities the *too interconnected to fail* feature is essential, while for the rest of regimes are governed by "*too big to fail* banks".](image)

### 3.3 Correlations

So far we have studied the distribution of loans and the banks’ size independently in order to analyze them in detail without additional effects. However, to consider that these sources of heterogeneity are no related might seem incoherent.

One expects that banks with larger financial volume would also have a very large amount of loans given their greater economic capacity. This intuition stems from the fact that larger bank have more financial elements to balance, so that at the end of the day it is easier for stochastic variations to avoid a perfect fit. Thus, our topic of study in this section will be the correlations between the degree distribution of the network and the size of the banks that compose the system.
Since we are interested in preserving the scale-free nature of the network, but we also want to maintain the power-law of the distribution of wealth, the correlation of both properties must be done carefully. To overcome this problem we propose a relatively simple procedure:

- Make a list of $N$ assets’ sizes distributed according to a power-law.
- Generate a scale-free network with $N$ nodes.
- Assign the largest size of the list to the node with highest degree, and continue this way until no nodes remain. In case of tie, choose at random.

![Figure 13: Comparison between two scale-free networks, where the assets’ distribution is random (blue) or correlated with the degree (orange). As we observe, correlated networks are more resilient to contagion.](image)

The results for a correlated network are reported in Figure 13. The behavior is very similar to the case of a scale-free network with non-correlated heterogeneous distribution of assets. However, the effects of correlations seems to reduce the probability of contagion, thus slightly enhancing the stability of the system.

Remind that the spreading capacity of the contagion relies in the connectivity of the graph, and that scale-free networks have a slight bias to form a core of highly connected banks (hubs) linked to a periphery. Hence, since in correlated networks hubs are in fact also the richest banks, they behave as a firewall over contagion. Only very big shocks can lead to the default of one of these big banks, so that the probability of contagion is reduced.
3.4 Financial networks generated by means of economic rules

To finalize this section, we will study how networks with the properties and features of the interbank system can be generated, by means of rules that are justified in economic terms and not simply by the use of artificial networks.

We will present two alternatives to generate networks from assets’ distribution, and a third method with potential to introduce coevolution in the model, based on the negotiation of loans (due to lack/excess of liquidity). Although these methods may not accurately represent real-life decision-making, we believe that they represent a good starting point for understanding where the network structure arises from purely economic arguments.

For the static scale-free networks case, we will define a probability function that determines how links are added to the network [43]. The method is based on the following procedure:

**GENERATION OF FINANCIAL NETWORKS**

1. Generate a collection of $N$ assets distributed according to a power-law, $P(A) \propto A^{-\alpha}$.
2. Add a link between nodes $i$ and $j$, with probability:
   \[
   p_{i,j} = b(A_i + A_j)
   \]
   being $b$ a normalization constant, $b^{-1} = \sum_{i,j} (A_i + A_j)$.
3. Repeat step 2 until the $m$ links of the network are set.

Other probabilities of connection can be used in step 2, in particular we have also considered:

\[
\tilde{p}_{i,j} = c A_i^{\beta_{\text{out}}} A_j^{\beta_{\text{in}}}
\]

where the normalization constant is given by $c^{-1} = \sum_{i,j} A_i^{\beta_{\text{out}}} A_j^{\beta_{\text{in}}}$, and $\beta_{\text{in}}$, $\beta_{\text{out}}$ determine the in/out exponents of the degree distribution.

The results for the two proposed probability functions are shown in figures 14 and 15. As can be seen, both degree distributions have long tails and follow more or less a straight line in a logarithmic plot.

A notable property of these networks, besides their scale-free nature, is that they have correlation between the size of the nodes (assets) and their connectivity, since the networks are generated from the distribution of wealth of each bank. Therefore, by this procedure it is not necessary to correlate the size and degree artificially, as we did in the previous section.
Our third proposal to generate a scale-free network aims to introduce the possibility of coevolution in the network, that is, that the network evolves over time without losing its statistical properties. As in the previous case, we start from a power-law type distribution of wealth, and then we assign an equity value to each bank such that:

\[ E_i = A_i \cdot u(K + \delta, K/2) \]  \hspace{1cm} (5)

where \( u \) is a Gaussian random number with mean \( K + \delta \) and standard deviation \( K/2 \). The parameter \( K \) is the capital buffer imposed by the regulators (in our model \( K = 0.04A^{total} \)) and \( \delta \) is a small bias so that, on average, the system has excess of liquidity.

Then, for each edge of the system, we select the source node from the set of banks that have less equity than capital buffer \( (E < K) \), and analogously the target node from those banks.
with excess of liquidity \((E > K)\). The probability to select each node as a source/target depends on the amount of missing/surplus capital, so that banks with an equity farther from their buffer will have, on average, more interbank loans. The reason for this is that banks with excess liquidity tend to diversify their loans, so that the bankruptcy of one of their borrowers does not lead to a critical impact.

Finally, once all edges have been introduced, their weight is determined according to the excess of liquidity of the targeted node and the amount required for the source nodes,

\[
e_{j \rightarrow i} = (E_i - K_i) \frac{|E_j - K_j|}{\sum_{<ik>} |E_k - K_k|}
\]

where \(<ik>\) sums for all loans of bank \(i\) (with excess of liquidity, \(E_i - K_i > 0\)) to other banks \(k\) (with lack of liquidity \(E_k - K_k < 0\)).

This method is far away from the actual decision-making that banks make when agreeing on interbank lending, but it does capture some of the most essential reasoning in this process, that will suffice for the purpose that concerns us in this paper.

As can be observed in Figure 16, the decision-making procedure provides a very sharp degree distribution. From the logarithmic plot, one could say that it follows a power-law, despite there is a lot of dispersion in the data. In general, the conclusion we draw is that the method is not good enough to generate the interbank networks. However, since the steps applied in our method can be applied to the network as the loans mature, we believe that it is a great step towards the inclusion of coevolution in the network, so that the negotiations are reflected in the model.
4 DYNAMICAL EFFECTS OF THE COUPLING WITH AN EXTERNAL MARKET

The interconnected structure of the interbank market has proved to be less robust to contagion than it originally seemed. Despite loans are necessary for the optimal functioning of banks, our study shows that they also act as exposures to financial shocks when liabilities can not be returned, thus representing a channel for direct contagion. However, this mechanism is not the only one acting in financial systems, but it can be exogenously reinforced by additional effects.

In this section we will study what is probably one of the most important external factors for the stability of the interbank market: the influence of investments in private companies. When a bank can not meet its obligations and enters bankruptcy, a common procedure is to sell part of its illiquid assets for a reduced amount of the original value. This procedure, commonly known as firesales, provides a mechanism to return a fraction of the bank’s liabilities, but it involves a certain cost for the system. Since part of the illiquid assets are in the form of investments in private firms, the sale at a reduced price of these investments may entail a depreciation of the company’s value. Consequently, all banks related to the company suffer a lost of capital due to depreciation, thus enhancing instability.

We will begin by studying the limiting case of a single company, in which all banks have invested a certain part of their assets. The devaluation of this firm will reduce the equity of all banks, which after defaulting, will trigger a new devaluation of the company. Therefore, this initial study introduces a mechanism that amplifies financial shocks.

Finally, we will add a private sector composed by several firms in which banks can invest, and we will study the effects of overlapping portfolios. As in the case of the topology and the banks’ size, an important part of our study will be to analyze the different parameters that naturally appear in this generalization.

4.1 Single firm exposure: firesales

As a first approach to modelize firesales, we will consider a single private company in which all banks have invested a fraction $c$ of their total assets. We will follow a similar approach than in the studies of Caccioli et al [44] [29]: each time a bank defaults the firm’s value decreases by a depreciation factor $\phi$, related to its liquidity capacity and solvency. This leads to a lost in the banks’ assets proportional to the amount they have invested ($c$) and the depreciation factor of the company ($\phi$), so that assets of bank $i$ change as,

$$A_i \rightarrow A_i (1 - c\phi) \quad (7)$$

This devaluation of banks’ equity can eventually lead to more defaults, so that the value of the company falls again. Hence, the coupling between the interbank market and the private sector acts as an amplifier for financial shocks. It is worth clarifying that the bank initially in default (the one that makes the firesale), does not increase its equity at all.
Since in its balance sheet the investment in the firm is considered an asset, the sale of this part to obtain liquidity does not affect the equity, but only helps the repayment of the outstanding loans. For simplicity, we will consider that this liquidity is used to reimburse external agents and costumers, and therefore, that interbank loans are not returned (the recovery rate is still zero, $R = 0$).

The results for a company with depreciation factor $\phi = 0.3$, in which all banks have deposited a fraction $c = 0.02$ of their total assets, are shown in Figure 17. For simplicity, and with the aim of studying the mechanism of firesales isolatedly, we have considered an Erdős-Rényi network with homogeneously distributed assets.

As we can see, the addition of a secondary contagion channel has catastrophic consequences for the stability of the system: both the maximum probability and the likelihood of contagion increase, and moreover the probability curve presents a long tailed evolution instead of a sharp decayment. On the contrary, the extent of the contagion remains the same than without private investments.

To understand these effects, we need to analyze how the devaluation mechanism affects the banking system. On one hand, each default induces a financial shock to the other banks’ capital that reduce their resilience to contagion. Due to the company devaluation, bank’s have a balance sheet so unhealthy that now small shocks to which they would have survived can trigger its default.

On the other hand, when the number of bankruptcies reach a critical value, bank’s automatically default due to the cumulative depreciation, independently of their loans.

Figure 17: Contagion and extent of contagion for an Erdős-Rényi graph with a firm or without it. The parameters in the company case are $c = 0.02$ and $\phi = 0.3$. 

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This case applies only when the devaluation mechanism is stronger than direct contagion, that is, for large values of the $c\phi$ product. Under this condition, all banks become insolvent at the same time since they all have the same capital and investment. Thus, a scenario in which firesales govern the dynamics of the system can be detected by checking if the fraction of defaulted banks is 1 for any value of the connectivity.

As a summary, the inclusion of a company cause the following dramatical changes in the system:

1. There is an increasing of the maximum probability that comes from the fact that financial shocks are reinforced by an exogenous term, so that more banks default in average and hence a systemic failure is more probable.

2. The robustness to contagion is largely reduced, and more interconnections are required to counter firesales effects.

3. Finally, the average degree at which banks were able to diversify loans to prevent contagion disappears, since the capital depreciation is independent from the number of links in the network. This translates into a long tail of contagion probabilities until disappearing.

A systematic study of the two new parameters introduced in the system ($c$ and $\phi$) is shown in Figure 18. In order to highlight the differences with the case without companies, we have fixed the average degree to $\langle k \rangle \approx 7.5$, since the probability of contagion without the company is 0. Then, we vary one of the parameters while maintaining the other constant.

![Figure 18](image-url)

**Figure 18:** (Left) Contagion probability for $\langle k \rangle \approx 7.5$ when varying the liquidity factor $\phi$. (Right) Contagion probability for $\langle k \rangle \approx 7.5$ when varying the inverted assets $cA$. 


One would expect that the larger the investments in the company ($c$) and its depreciation ratio ($\phi$), the more probable a systemic contagion would be. This intuition is confirmed by the monotonously increasing of the contagion probability as a function of $c$ and $\phi$, which follows a sigmoid-like function. However, the sigmoid-like overall shape is interrupted by some plateaus, which are more apparent for small $\phi$ on the left panel or for small $c$ on the right panel. In principle increasing either $c$ or $\phi$ should increase for all banks its susceptibility to default, since they lose a larger part of the external assets ($A^{\text{ext}}$). Instead, we observe that in certain regimes the probability of a systemic failure is constant, in spite of increasing the value of the losses suffered after each depreciation. A possible explanation for this phenomenon is that, since all banks have the same size and uniformly distribute their interbank assets ($A^{IB}$), each time the depreciation value equals a certain multiple of the fraction of interbank assets (ie the number of exposures), the shock coming from the company causes the default of all the banks with a certain number of counterparties.

It is also surprising that both parameters present such a similar behavior, which suggests that they are not completely independent one from the other. In fact, since we are in a homogeneous case in which there is only one company and all banks have invested a common fraction $c$ of their assets, all agents lose an amount $c\phi A$. That is why we can parametrize the system by the product $c\phi$.

![Figure 19: Firesales effects for different values of the parameter $c\phi$. The larger the parameter, the wider the contagion probability and its maximum.](image)

The effects for different $c\phi$ values are shown in figure 19. As we previously deduced, the value $c\phi$ is proportional to the contagion window and its maximum probability. Moreover, the extreme case in which devaluation of assets crashes the whole system can be observed, and as we pointed out before, the extent of contagion is 1 independently of the connectivity. Hence, figure 19 makes the summary of our findings and deductions about the inclusion of a private company into the banking system, at least at this description level.
4.2 Heterogeneous private sector

To this point, we have dealt with the inclusion of a single company, into which all banks are forced to invest. The analysis of this extremely simple case has been convenient in order to understand the fundamentals of firesales and to study how the different parameters affect the system. However, our initial goal was to describe the coupling between the interbank system and the whole private sector in form of firesales. Hence, due to the richness in number and heterogeneities of companies, we need to do a step forward in our model and consider a more general scenario.

Consider a banking system composed by $N$ banks, and a private sector with $M$ companies, each with its own liquidity factor $\phi$. Then, each bank $i$ with probability $p_{\text{invest}}$ invests in company $j$ a certain fraction of its assets $c_{ji}$, while with probability $1 - p_{\text{invest}}$ it does not invest in this company ($c_{ji} = 0$). This procedure is then repeated for each bank, with all coompanies, so that the total number of investments from banks into firms is given by $NMp_{\text{invest}}$. At the end of the day, this leads to an interbank system heterogeneously coupled to the private sector through a series of investments, as shown in Figure 20.

![Figure 20](image)

**Figure 20**: Schematic representation of the interbank system coupled to private companies. Banks invest a certain amount of their assets into companies, while keeping their interbank liabilities. At the same time, companies have their own liquidity rates, that are directly proportional to their depreciation under investments withdrawals.

The dynamics under these conditions is analogous to the case with a single company. Once a bank $i$ gets bankrupt and can not meet its duties, there’s an attempt to sell part of the companies’ investments in form of illiquid assets by a reduced value (firesales). The liquidity won by this mechanism is injected into the bank’s balance sheet, so that if this provides a positive equity $E_i > 0$ the bank is no more in default.
These firesales are reflected in the form of a depreciation of the companies’ values into which bank \( k \) has invested. As in the single company case, we will consider that the depreciation only depends on the liquidity factor \( \phi \), and not into the amount invested by bank \( i \) \((c_{jk})\). Hence, the value of a company \( j \) is reduced as, \( F_j \rightarrow F_j(1 - \phi_j) \), and all banks that had any investment into this company suffer a financial shock due to this depreciation according to\(^{18}\),

\[
A_i \rightarrow A_i(1 - c_{ji}\phi_j)
\] (8)

To compare independently the effects of the heterogeneity in the \( c \) and \( \phi \) values, and the multiplicity of firms, we will first analyze a system where all companies have the same \( c\phi \) value. Again, the network is Erdős-Rényi and assets’ are homogeneously distributed. Moreover, there are \( N = 1000 \) banks and \( M = 100 \) companies. The probability that a bank invests into a certain company \((p_{\text{invest}})\), is related to the number of firms in which a bank has invested a certain fraction of its external assets \((A_{\text{ext}})\). Hence, it plays the analogous role to the diversification of interbank risk, but for the depreciation of investments. We will study two different values of this parameter, \( p_{\text{invest}} = 0.3 \) (on average, each bank invests in 30 firms) and \( p_{\text{invest}} = 0.8 \) (80 investments in firms per bank).

![Figure 21](image)

**Figure 21:** Comparison between a private sector composed by a single or several companies. When more than one firm are present in the system, the contagion probability decreases proportionally to the number of different investments (related to \( p_{\text{invest}} \)).

Figure 21 shows a comparison between a single company exposure and the case where investments are done in several firms, and in both cases, all companies have the same depreciation ratio \( \phi = 0.3 \). In order to compare the different scenarios, the exposure to

\(^{18}\)This shock applies to all companies with investments from bank \( k \).
the private sector is the same. This means that the fraction of external assets invested in
the private sector is constant (with $\sum_j c_{ji} = 0.02$) regardless of whether the investment
are made in a single company or in more than one.

As can be see in Figure 21, including several companies (with identical $\phi$) to the system
reinforces stability. This result is not particularly surprising if we take into account that
the exposure to depreciation to which each bank is subject is lower for several firms. That
is, when a certain bank goes into default, all companies in which it has invested see reduced
its value. As a result, banks that had investments in these companies also lose part of
their equity in the form of depreciation of securities\textsuperscript{19}, but only the proportional part of
their private investments in those companies. In contrast, when there is only one firm
in which all banks invest, each bank failure leads to a depreciation of the company, and
therefore, all banks in the system suffer a financial shock of 100% of their investments.

The heterogeneity in the companies’ properties will presumably have minor effects into
the system, when compared to adding several firms to the market. Distributing the values
of the investments ($c$) and the liquidity factors ($\phi$) introduces the possibility of shock
with different size, but the general behavior is expected to be similar. This intuition
is confirmed by figure 22, where we compare the results for several companies all with
$c\phi = 0.003$ and the case in which $c$ and $\phi$ values are distributed according to gaussian
distributions, $f_D(c) = Gauss(0,0.01)$ and $f_D(\phi) = Gauss(0.3,0.1)$.

\textbf{Figure 22:} Coupling between several companies and the interbank companies, for a fixed
c$\phi = 0.03$ (blue), and $c\phi$ generated from a gaussian distribution (orange). The distributed
values give roughly the same behavior, with a slightly higher probability of contagion.

\textsuperscript{19}A security is a fungible, negotiable financial instrument that holds some type of monetary value. In
this case, itt represents an ownership position in a publicly-traded corporation (via stock).
The only effect in the curve is a little increasing in the probability of contagion for lower and higher connectivities, but not in intermedium. Of course, in both cases there is a complete spreading of the contagion due to multiple depreciation.

To complete the study of firesales effects, it could be useful to analyze how the probability of contagion evolves with the fraction of external assets invested in the external market. This is studied by varying the number of investments of each bank $p_{invest}^{20}$, and fixing the amount of each investment at a certain value (in our case, $c = 0.01$). Hence, the larger $p_{invest}$, the larger the fraction of external assets invested in the private sector (with a cap given by the fraction of external assets, $A^{ext} = 0.76A^{total}$).

This dependence is shown in figure 23, for an Erdős-Rényi graph with homogeneous assets and an average connectivity of $\langle k \rangle \approx 7.5$, in which the probability of contagion should be zero.

The results show a monotonous increasing as a function of $p_{invest}$, following more or less a sigmoid function. This is not surprising, since the higher the probability to invest, the more companies are related with each bank (and hence, the more shocks they receive from depreciations coming from a defaults). As we expected, the contagion always expands through the whole network, characteristic of the coupling with several companies.

![Figure 23](image)

**Figure 23:** Erdős-Rényi graph with homogeneous assets and an average connectivity of $\langle k \rangle \approx 7.5$. The larger the probability to invest in each company (and then the number of investments), the larger the probability of contagion.

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20 Notice that $p_{invest}$ is related to the interconnectivity between the interbank and the external markets.

21 This setup of the model is different for the one in Figure 21, in which we set the amount of total investments ($\sum_j c_{ji} = C_1$), not that of each individual investment $c_{ji} = C_2$. 

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5 CONCLUSIONS AND OUTLOOK

Throughout this work we have studied the importance of connectivity in financial networks for their robustness against systemic contagion. Starting from the original Gai-Kapadia model, we have discovered that risk diversification does not always increase the system’s stability against financial contagion, as is commonly believed in economic circles.

After analyzing the ‘robust-yet-fragile’ property of the system, we proceeded to extend the initial model to introduce heterogeneities, both in the network topology and in the financial entities themselves. Our results suggest that depending on the exact parameters related to such heterogeneities, the system may be much more unstable than we initially believed. In addition, after attacking the system through selective defaults of certain banks, we have determined that a high wealth is more effective to avoid contagion than a high connectivity, which seems to be of particular interest for the implementation of regulatory policies.

Finally, we have determined that the coupling between the interbank market and private sector investments serves as an amplification mechanism for financial shocks. And not only this but, due to overlapping portfolios, the whole system is susceptible to go bankrupt when there are fluctuations of values too high. In the case of an external market composed of several firms, we no longer see a progressive increase in the number of banks that go bankrupt as connectivity increases, but all banks may fall in the case of too high devaluations.

All these observations turn out to be highly dependent on the different parameters of the model, and until we have made a systematic study of them we have not been able to understand the true mechanisms behind financial contagion.

However, in this paper we have only analyzed the so-called ‘first-round effects’, that is, the direct contagion of a financial shock and how it is absorbed by the rest of banks. Recent studies point to the vital importance of second and third round effects, related to how banks take into account the solvency of other banks and adjust their accounts accordingly (also known as accounting), a fact that spreads throughout the system although no bank has defaulted.

In addition, the importance of the (partial) information that each bank knows about the rest, and how it affects renegotiations of loans, has also been analyzed.

All these factors, along with those that can be discovered in the next few years, must be thoroughly analyzed in order to understand exactly how they affect the financial system. In terms of understanding the real behavior of financial markets, it is of little use to begin to blindly add new elements to our economic models, or to develop complicated numerical methods for making short-term forecasts.

It will not be, however, through the in-depth study of the mechanisms, independently analyzed, that we will be able to understand the true behavior of financial contagion in order to develop prevention tools and new regulatory policies that will ensure a prosperous and sustainable future of the economy.
6 APPENDIX

6.1 Appendix A. Pseudocodes

Algorithm 1 Original GK model
1: Generate the Graph and set the parameters
2: for i in banks do
3: \( E_i = \text{capital requirement} \)
4: for all ingoing edges do
5: \( e_{ij} = \frac{A_i^{IB}}{k_i^{in}} \)
6: new_defaults = random (banks)
7: for i in new_defaults do
8: for all out-neighbors(i) do
9: \( E_j = E_j - e_{ij} \)
10: if \( E_j = 0 \) then
11: new_defaults.add(j)
12: Repeat from Step 7 until new_defaults is empty.

Algorithm 2 Coupling between the interbank market and one firm
1: Generate the Graph and set the parameters
2: for i in banks do
3: \( E_i = \text{capital requirement} \)
4: for all ingoing edges do
5: \( e_{ij} = \frac{A_i^{IB}}{k_i^{in}} \)
6: new_defaults = random (banks)
7: for i in new_defaults do
8: for k in banks do
9: \( E_k = E_k - c_k \phi A_k^{ext} \)
10: for i in new_defaults do
11: for all out-neighbors(i) do
12: \( E_j = E_j - e_{ij} \)
13: if \( E_j = 0 \) then
14: new_defaults.add(j)
15: if new_defaults not empty then
16: Go to Step 7
6.2 Appendix B. Generating scale-free networks: Chung-Lu model

With the aim to generate artificial scale-free networks, where both the in-degree and the out-degree can be set, we have applied the so called Chung-Lu procedure, a stochastic model similar to the ER and the static model. The procedure is as follows:

1. Begin with \( N \) indexed vertices, \( i = 1, \ldots, N \).
2. Assign two weights to each vertex, \( p_i = i^{-\alpha_{in}} \) and \( q_i = i^{-\alpha_{out}} \), corresponding to the in and out degrees respectively. The parameters \( \alpha_{in} \) and \( \alpha_{out} \) belong to \([0, 1)\).
3. Select a pair of nodes \((i, j)\) with probabilities \( p_i / \sum_k p_k \) and \( q_j / \sum_k q_k \) respectively, so that a directed edge from \( i \) to \( j \) is created.
4. Repeat the previous step until \( pN \) different links are made in the system.

It is easy to see that the mean degree of the generated network is \( 2p \). Moreover, since the probability that a link starts from a certain node is proportional to the outgoing weight of this node \( (q_i / \sum_k q_k) \), the out-degree of that vertex is given by,

\[
\frac{k_{out}^i}{\sum_j k_{out}^j} \approx \frac{(1 - \alpha_{out})}{(N/2)^{1-\alpha_{out}}}
\]

where \( \sum_j k_j = pN \). Then, the degree distribution for outgoing links follows a power-law \( p_{out}(k_{out}) \sim k^{-\gamma_{out}} \), where the exponent \( \gamma_{out} \) is,

\[
\gamma_{out} = \frac{1 + \alpha_{out}}{\alpha_{out}}
\]

Analogously, for the ingoing links,

\[
\gamma_{in} = \frac{1 + \alpha_{in}}{\alpha_{in}}
\]

Hence, the Chung-Lu model provides us with a mechanism to generate scale-free networks where both the in-degree and the out-degree can be fixed (between \( 2 < \gamma < \infty \)), just by adjusting the control parameters \( \alpha_{in} \) and \( \alpha_{out} \). Moreover, it is worth mentioning that since the method is based on the index of each vertex, both degree distributions are correlated, so that a node with high in-degree has also a high out-degree.
7 REFERENCES


