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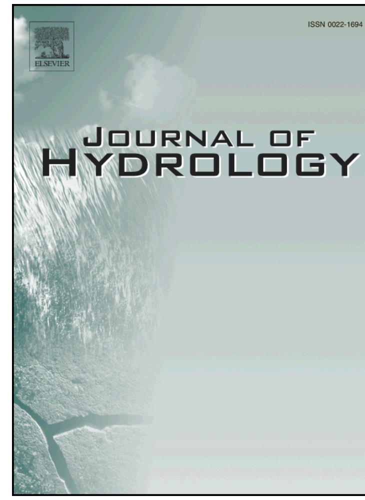

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The relevance of Philip theory to Haverkamp quasi-exact implicit analytical formulation and its uses to predict soil hydraulic properties

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ABSTRACT

The quasi-exact implicit (QEI) analytical formulation of Haverkamp equation, which might be also known as Parlange model, and its one- and two-term approximate expansions are among the mostly used equations for in situ determination of soil sorptivity, S, and saturated hydraulic conductivity, K_s, from unsaturated one-dimensional (1D) cumulative infiltration into soils. However, from practical point of view, the approximate expansions are only valid from short to intermediate times and the QEI analytical formulation has a complex resolution, which makes its use complicated in inverting procedures. Therefore, alternative functions are needed to compute cumulative infiltration for longer times and improve inverting procedures for easier, more robust and accurate predictions of S and K_s. In this regard, current work presents and evaluates a new three-term approximation of the QEI analytical formulation. As a first step, we checked the accuracy of the proposed three-term approximate expansion with respect to the QEI formulation for three different soils (sand, loam and silt) in order to define its time domain validity. Next, the accuracy of the three-term approximate expansion to estimate S, K_s and the β parameter was compared to those obtained with one- and two-term approximate expansions. Lastly, a large dataset of field experimental data was inverted using the proposed three approximate expansions and the QEI analytical formulation, and goodness of fits predicted and measured cumulative infiltration curves and accuracy of the parameters (S, K_s and the β) estimates were compared. The results revealed that the three-term approximate has a larger validity time interval compared to one- and two-term approximate expansions, which allows its use for larger infiltration times durations resulting in more accurate estimates of K_s. Its capability of estimating S is also improved. The accurate prediction of β parameter is not still guaranteed for none of the approximate expansions. Compared to the one- and two-term approximate expansions, the experimental infiltration data usage revealed that the three-term approximate expansion resulted in higher performance and better prediction of hydraulic parameters.

Keywords: Sorptivity, Hydraulic conductivity, Sensitivity analysis, Infiltration constant, Inverse procedure.

1 INTRODUCTION

Soil scientists and hydrologists usually are concerned to predict soil hydraulic properties more accurately. In this regard, several attempts have been made for in situ determination of soil hydraulic properties. Soil sorptivity (S) and hydraulic conductivity (K) are among the most concerned hydraulic properties (Youngs, 1964; Reynolds and Elrick, 1985; Zhang, 1997; Valiantzas, 2010; Latorre et al., 2015). The ability of a saturated soil to conduct water by gravity is considered as saturated hydraulic conductivity (K_s). The S is also considered as the capacity of a medium to absorb or desorb liquid by capillarity (Philip, 1957).

Hydraulic properties of a porous medium are usually characterized by the key governing equation of Richards (1931). Philip (1957) solved the Richards (1931) equation to introduce a specific solution for unsaturated one-dimensional (1D) water infiltration in soil surface. Although the Philip model is easy to solve and seems more user-friendly, assumptions made by Philip (1957) are rarely found at field scale (Maheshwari and Jayawardane, 1992). Nevertheless, several researchers have applied the Philip (1957) model to estimate soil hydraulic properties. For example, Fahad et al. (1982) comparing several infiltration models showed that Philip (1957) and Kostiakov (1932) models had better agreement with experimental data compared to others. Years after Philip, Parlange et al. (1982) also solved the Richards (1931) equation and proposed a quasi-exact implicit formulation for one-dimensional (1D) water infiltration into soils, involving three inputs: K and S and the α parameter. On this basis, Haverkamp et al. (1990) redefined the Parlange et al. (1982) model, where α was replaced by a new integral shape constant, β factor, that was defined as a constant at 0.6 but could be defined as function of the soil diffusivity, the hydraulic conductivity function, and the initial and final volumetric water contents (Haverkamp et al., 1994). According to Moret-Fernández and Latorre (2017) and Lassabatere et al. (2009), β is strongly related to the soil textural properties. Because of the complexity of the implicit model, Haverkamp et al. (1994) proposed one- and two-term approximate expansions valid only for short to intermediate times. The two-term approximate expansion, which is function of K, S, and β, is consistent with the approach proposed by Philip (1957). The one-term approximate expansion reduces to the Philip one-term model (Philip, 1957) that was developed to quantify water infiltration induced exclusively by capillarity (horizontal infiltration, no gravity).

The two-term approximate expansion proposed by Haverkamp et al. (1994) has been largely used to estimate K and S. For instance, Vandervaere et al. (2000) calculated S and K by applying a specific derivative linearization method to the two-term approximate expansion. Although this method gave reasonable results, this equation keeps the uncertainty about the optimal infiltration time that should be chosen. The "short to medium infiltration time" term defined for the two-term approximate expansion is ambiguous and makes the application of this method quite subjective. On the other hand, Latorre et al. (2015) demonstrated that this method was largely affected by experimental problems during the water infiltration experiment like bubbling and the influence of the contact sand layer commonly used under disc infiltrometer. As an alternative, these authors, demonstrated that the implementation of the QEI formulation in the inverse procedure allowed more robust estimates of K and S. However this alternative is time-consuming and requires a more complex computation and coding, due to the implicit nature of the QEI.

As described above, given the limitations of the two-term approximate expansion and the difficulties to compute the QEI formulation introduced by Haverkamp et al. (1994), alternative functions are needed to release the constraints on time and allow easier, more robust and accurate estimations of the soil hydraulic properties. This work proposes a new three-term infiltration model derived from the asymptotic development of the QEI formulation close to zero to get a model valid for larger times and to improve the estimates of K, S and β when fitting to experimental data. As a first step, we checked the accuracy of the three-term approximate expansion with respect to the QEI formulation and defined its validity time interval. Accuracy and validity time intervals were compared to those related to the regular one- and two-term approximate expansions. Finally, the three-term approximate expansion was implemented into an inverting procedure for the estimation of S, K and parameter β from fitting experimental double ring infiltrometer measurements (assuming 1D water infiltration). The estimates were compared to those obtained with the QEI formulation.

2 METHOD AND MATERIALS

2.1 Theoretical background

The governing equation for one-dimensional Darcian downward flow in variably saturated rigid porous media is given by the following form of the Richards (1931) equation:

$$\frac{\partial\theta}{\partial t}=\frac{\partial}{\partial z}\left[D(\theta)\frac{\partial\theta}{\partial z}\right]+\frac{\partial K(\theta)}{\partial z} \tag{1}$$

where θ ($L^3\ L^{-3}$) is the volumetric water content, t (T) is time, K the hydraulic conductivity (L T^{-1}), z is a vertical coordinate (L) positive downwards, and $D(\theta)$ ($L^2\ T^{-1}$) is the diffusivity defined by Klute (1952) as:

$$D(\theta)=K(\theta)\frac{dh}{d\theta} \tag{2}$$

where h is the matric component of soil water potential (L). The respective initial and boundary conditions for downward infiltration are:

$$\begin{aligned} &z=0, \quad t>0, \quad \theta=\theta_s \\ &z\geq 0, \quad t=0, \quad \theta=\theta_i \\ &z\to\infty, \ t>0, \quad \theta=\theta_i \end{aligned}$$

where θ_s and θ_i are the saturated and initial volumetric water contents, respectively. Several attempts have been made to solve Richards' equation in order to find solutions for water infiltration into soils. In this regard, Philip (1957) suggested the following infinite power series for cumulative infiltration:

$$I(t)=A_1t^{1/2}+A_2t+A_3t^{3/2}+A_4t^2+...+A_nt^{n/2} \tag{3}$$

where A_2 (L $T^{-1/2}$) to A_n (L $T^{-n/2}$) are coefficients. Philip (1957) showed that A_1 is equal to the S (L $T^{-1/2}$) and A_2 (L $T^{-3/2}$) is proportional to K_s (L T^{-1}). For horizontal or very short downward infiltration times, one can keep the first term only and simplifies the Eq. (3) as follows (Philip, 1957):

$$I(t)=S\ t^{1/2} \tag{4}$$

Philip (1957) also showed that in practical Eq. (3) can be approximated by two first terms as below:

$$I(t) = St^{1/2} + At \tag{5}$$

However, Kutílek and Krejča (1987) showed that if one removes terms three and more from this infinite power series in inverting procedures, estimates of parameter A lower than zero may be found. This is meaningless regarding the physics of water flow since it would imply a negative value to the effect of gravity on water infiltration (upward infiltration). Therefore, they suggested to apply the three-term Philip model rather than its one- or two-terms expansions:

$$I(t) = c_1 t^{1/2} + c_2 t + c_3 t^{3/2} \tag{6}$$

Besides, Haverkamp et al. (1994) also presented a quasi-exact analytical solution of the Richards (1931) equation for one dimensional (1D) cumulative infiltration, I_{1D}, which was firstly proposed by Parlange et al. (1982) and redifined by Haverkamp et al. (1990):

$$\frac{2\left(K_s - K_i\right)^2}{S^2} t = \frac{2}{1-\beta} \frac{\left(K_s - K_i\right)\left(I_{1D} - K_i t\right)}{S^2} - \frac{1}{1-\beta} \cdot \ln\left[\frac{1}{\beta} \exp\left(2\beta\left(K_s - K_i\right)\left(I_{1D} - K_i t\right)/S^2\right) + \frac{\beta - 1}{\beta}\right] \tag{7}$$

where, S (L T$^{-0.5}$) is the sorptivity for initial and final water contents θ_i and θ_s (L^3 L^{-3}), K_i and K_s (L T^{-1}) are the hydraulic conductivities corresponding to θ_i and θ_s, respectively, and β is an integral shape parameter that ranges between 0.3 and 1.7 for sand to silty soils (Lassabatere et al., 2009). Fuentes et al. (1992) proposed the following expression for parameter β:

$$\beta = 2 - 2\frac{\int_{\theta_i}^{\theta_s} \left(\frac{K(\theta) - K_i}{K_s - K_i}\right)\left(\frac{\theta_s - \theta_i}{\theta - \theta_i}\right) D(\theta)\, d\theta}{\int_{\theta_i}^{\theta_s} D(\theta)\, d\theta} \tag{8}$$

While for very short times Eq. (7) simplifies to Eq. (4) (Haverkamp et al., 1994), for short-intermediate infiltration times and K_i close to zero (initially dry soils), Eq. (7) can be simplified to the following two-term approximate expansion (Haverkamp et al., 1994):

$$I_{1D} = S\sqrt{t} + \frac{2-\beta}{3} K_s\, t \tag{9}$$

which is consistent with the two-term Philip (1957) model (Eq. 5).

In addition to one- and two-term approximate expansions of QEI, we also applied the Taylor series up to third order in powers of 0.5 to the QEI formulation to introduce its three-term approximate expansion as below:

$$I_{1D} = S \cdot t^{\frac{1}{2}} + \frac{2-\beta}{3} K_s \cdot t + \frac{1}{9}\left(\beta^2 - \beta + 1\right)\frac{K_s^2}{S} \cdot t^{\frac{3}{2}} \tag{10}$$

To do this, we considered that K_{i} is close to zero. A detailed description of the steps of the derivation of this approximation is reported in Appendix A. Equation 10 is identical to Eq. (6) providing that the following equalities are considered:

$$c_1 = S \tag{11}$$

$$c_2 = \frac{2-\beta}{3} K_s \tag{12}$$

$$c_3 = \frac{1}{9}\left(\beta^2 - \beta + 1\right)\frac{K_s^2}{S} \tag{13}$$

2.2 Synthetic data

Synthetic infiltration curves were generated numerically applying the QEI formulation of Eq. (7). Synthetic sand, loam, and silt soils defined by Carsel and Parrish (1988) were used for simulations (Table 1). In this case, van Genuchten (1980) model for water retention curve and Mualem (1976) model for hydraulic conductivity curve were considered. The free web page http://swi.csic.es/ was employed to compute Eq. (7) to generate the synthetic curves for synthetic soils (see Latorre et al., 2015 for the description of webpage). These synthetic data were computed for time datasets 1 to 10000 seconds (~2.75h) and considering very dry initial state and a saturated final state. These synthetic cumulative infiltrations can be considered as the response of the synthetic soils to water infiltration experiments and constitute the experimental data to be inverted with the different inverting procedure based on the use of the different approximate expansions.

For all the synthetic soils and cases, the S was computed using the equation of Parlange (1982):

$$S^2 = \int_{\theta_0}^{\theta_s} \left(\theta_s + \theta - 2\theta_0\right) D\left(\theta\right) d\theta \tag{14}$$

The diffusivity $D(\theta)$ can be easily computed from the soil hydraulic conductivity functions $K(\theta)$ and $\theta(h)$ using Eq. (2). More details on the use of the QEI formulation can be found in Lassabatere et al. (2009). We also computed β values for each soil using Eq. (8).

Then, we had synthetic experimental data obtained by direct model and then we used the models (QEI and approximations) for the inversion of the data and the estimation of S, K_s and β. The details about the inverting procedure is reported later.

<<Table 1 about here>>

2.3 Time validity domains for the approximate expansions

The time domain validity of the one-, two- and three-term equations, were determined considering the following time divergence analysis. In order to assess the accuracy of the approximate expansions, we computed their values for the same synthetic soils and time datasets. Then, the distance between the target QEI formulation and the approximate expansions were evaluated using the relative error. The following criterion was considered to define the threshold when approximate expansions start to diverge from the target implicit formulation:

$$t = t_{divergence} \quad if \quad \frac{\left| mean\left(I_t - \hat{I}_t\right) \right|}{\max(I_t)} \times 100 \geq 1 \tag{15}$$

where I_t is a vector of target values obtained with the QEI formulation at time t and $\hat{I}_t$ is the vector of values predicted by the one-, two-, or three-term approximate expansions. Implicitly, Eq. (15) is based on a maximum relative error of 1% and defines the corresponding divergence time, $t_{divergence}$. The interval [0, $t_{divergence}$] characterizes the validity intervals of the approximate expansions.

2.4 Inverting procedure and objective function

The synthetic cumulative infiltration values computed with the QEI formulation were considered as experimental data and inverted with different inverting procedures. Different inverting procedures were designed on the basis of the use of the three approximate expansions with the same aim of deriving the three optimum values for parameters S, K_s, and β. The main goal of this investigation involves the demonstration that the use of the three-term approximate expansion will be more accurate since this approximate expansion is closer to the target QEI formulation. For the other expansions, the discrepancy between their prediction and the QEI formulation may induce erroneous estimates for S, K_s, and β. For the fitting, we considered a quadratic objective function that calculates the squared difference between the experimental data (synthetic cumulative infiltration) and the model (approximate expansion used to fit the data). The optimized parameters S, K_s, β are the values that minimize the objective function.

In addition to these tests, we investigated the shape of the objective functions that determine the quality of the inverting procedures. To this end, the objective function, Q, which represents the difference between the models and the experimental data to be fitted was employed:

$$Q = \sum_{i=1}^{N} \left(\left(I_i - \hat{I}_i \right) \Delta t_i \right)^2 \tag{16}$$

where N is the number of experimental (I, t) and modelled values ($\hat{I}, t$) values, and Δt_i is the time interval between consecutive infiltration values. Since all the models used for the fits (i.e., all the approximate expansions) include three parameters, the 2-D S-K_s, S-β and K_s-β error maps, using in each plane the remaining target parameter, were plotted. The objective functions were determined for several infiltration times including 40, 1000 (~16 min), and 10000 s (~2.75h) in order to characterize the inversion for short, intermediate and long times. The minimum of the objective function defined the optimum S, K_s and β, then compared to the theoretical target set (Table 1).

2.5 Experimental data

To evaluate the efficiency of the approximate expansion with real data, field experimental data were supplied by SWIG database (Rahmati et al., 2018a; Rahmati et al., 2018b). In total, 828 individual soil infiltration curves were selected from SWIG (Rahmati et al., 2018a; Rahmati et al., 2018b) representing 1D cumulative infiltrations measured with double rings. For these data, we compared the inverting procedure based on the three-term approximate expansion and on the QEI formulation. Note that even if the implementation of the QEI formulation is possible, it slows down the computing procedure and may lead to undermined parameters in certain cases. The three-term approximate expansion (Eq. 10) was next to experimental data and estimated K and S were compared to those obtained with the implicit equation (Eq. 7). Note that, for this case, parameter β is considered as constant.

2.6 Statistical analysis

The root mean square error, *RMSE*, (Eq. 17), Error Ratio, *ER*, (Eq. 18), and Nash and Sutcliffe (1970), *E*, (Eq. 19) criterions along with determination coefficient (R^2) were applied to assess the quality of the fits as well as the accuracy of the predicted values of S, $\log(K_s)$, and β parameters:

$$RMSE = \sqrt{\frac{\sum (X_m - X_p)^2}{n}} \tag{17}$$

$$ER = \frac{RMSE}{\overline{X}_m} \times 100 \tag{18}$$

$$E = 1 - \frac{\sum (X_m - X_p)^2}{\sum (X_m - \overline{X}_m)^2} \tag{19}$$

where X_m and X_p, respectively, are the experimental and predicted cumulative infiltrations, or known (target) and estimated values of parameters, $\overline{X}_m$ is the mean of X_m values and n is the number of data sets for each evaluation. In the case of cumulative infiltration assessment, n is the length of time vectors [0:1:40], [0:1:59, 60:20:1000], and [0:1:59, 60:20:10000]) related to time ranges of 0 - 40, 0 - 1000, and 0 - 10000 s, respectively. When the criterions were used to assess the accuracy of the model estimates (S, $\log(K_s)$, or β), n equaled to number of examined soils. The *RMSE* and *ER* (standardized *RMSE* in percent) can be regarded as the computation of the objective function and values of *RMSE* and *ER* near zero denote a great accuracy. The value of *E* is comprised in [$-\infty$, 1] and the following considerations hold: $E = 1$ shows a perfect match, $E = 0$ means that the average of the observed values is as good as predicted values, $E < 0$, means that the model is worse for prediction than taking only the mean of the observed values, $E > 0.9$ shows that the slope between observed and predicted values is nearly unity.

3 RESULTS AND DISCUSSION

3.1 Quasi-exact implicit model *vs*. approximate expansions

We evaluated the accuracy of the one-, two-, and three-term approximate expansions (Eqs. 4, 9, and 10) with respect to the quasi-exact implicit formulation using the known values of S, K_s, and β parameters. Note that the sorptivity had to be computed previous the use of the QEI formulation. The results obtained applying the criterion defined by Eq. 16 are shown in Figure 1. From Figure 2 it is possible to deduce that the one-term expansion diverges from the implicit model for time ranges of 1, 600, and 2720 seconds respectively for the synthetic soils sand, loam and silt. Conversely, the two-term approximate expansion diverges from the implicit model for time ranges of 440, 3720, and >10000 seconds, respectively, for sand, loam and silt soils, while the three-term approximate expansion allows extending the validity time domain up to 9400, >10000, and >10000 seconds, respectively, for the three synthetic soils. Figure 1 shows that for the synthetic sand, the three-term approximate expansion drives to slightly overestimate cumulative infiltration in comparison to the QEI formulation for times larger than around 5000 seconds. Instead, the one- or two-term approximate expansions underestimate the cumulative infiltration for any times with a significant error for times larger than the validity time. In fact, these trends (underestimation *vs*. overestimation) apply to any type of soils and result from inherent mathematical properties of these equations (Lassabatere et al., 2009). However, for the other soils, the validity time domains seem much larger for all approximate expansions with more accuracy for the finer soils (Figure 1, loam and silt *vs*. sand). From these numerical results, we may conclude that the accuracy of the approximate expansions is dependent on the soil type and the time considered for the comparison between models. The approximate expansions should be used only within their related validity time intervals, and validity time dependency upon soil type should be accounted for, in particular for very permeable soils for which the validity times are quite short. Consequently, the miss-use of the approximate expansions out of their validity time intervals may spoil the accuracy of these expansions and thus the performance of inverting methods which are investigated below.

<<Figure 1 about here>>

<<Figure 2 about here>>

3.2 Inverting synthetic data with three-term approximate expansion

The sensitivity analysis of the three-term approximate expansion (Eq. (10)) performed for the synthetic sand, loam, and silt (Figure 3 to Figure 5) showed that the infiltration time has a drastic influence on the S-β, K_s-S and K_s-β error maps. In other words, the objective function of Eq. (16) strongly depends on the selected time datasets. The quasi-vertical contour lines observed in the S-K_s planes for very short infiltration times (i.e. 40 s) indicates that good estimations of S can be expected but no accurate estimations may be found for K_s. Indeed, the objective function defines large valleys parallel to the K_s-axis, preventing from identifying a clear minimum in the K_s-S plane (Figure 3 to Figure 5, plane K_s-S and t = 40 s). The accuracy of the K_s-S error map progressively improves with increasing infiltration time (i.e. 1000 and 10000 s), where the valley shape is transformed to a well. Such a change in shape is due to the fact that longer infiltration times allow more contribution of the terms related to K_s in approximate expansions (Figure 3 to Figure 5), thus increasing the possibility to estimate the parameter K_s. From a physical point of view, it corresponds to a large contribution of gravity to water infiltration.

A different behavior was observed in the related β planes (S-β and K_s-β). In these cases, the respective quasi-vertical and horizontal valleys shape observed for all soils and medium infiltration time indicates that many values of β give very similar values for the objective functions and thus that β cannot be estimated with the three-term equation. Although the accuracy of the β estimate tends to increase with increasing times (Figure 3 to Figure 5), only the S-β error map for sand at very long time changed to the shape of a well that is necessary to ensure a unique value for β estimate. However, these results may be questioned since the previous analysis has demonstrated the three-terms approximate expansions is no longer valid for times larger than 5000s (Fig. 1 and 2); thus questioning the validity of this approximate expansions at such large times. From these results, we can conclude the following statements: (i) short infiltration times may be enough to approach S, (ii) intermediate infiltration times are required for a proper estimation of K_s, and (iii) no infiltration time may be adequate for the optimization of β, with no reliable values, even at very long time. Similar results were obtained by Latorre et al. (2018).

<<Figure 3 about here>>

<<Figure 4 about here>>

<<Figure 5 about here>>

3.3 Constant β *vs*. variable β

Results from the previous section revealed that the estimation of the β cannot be properly done. Therefore, a new analysis was performed to confirm these results by evaluating the effect of constant versus soil dependent values of β on the accuracy of predicted K_s and S. To this end, K_s and S were estimated by applying the three-term expansion to the sand, loam, and silt synthetic infiltration curves for 10000 s under three different scenarios: 1) β predicted through curve fitting process, like the two other parameters (S and K_s), 2) β fixed at a constant value of 0.6 as suggested by Haverkamp et al. (1994), and 3) β fixed at a constant value of 1.1 as suggested by Latorre et al. (2018). The results (Figure 6) revealed that the curve fitting process provided β values of 0.3, 1.41, and 1.53 respectively for sand, loam, and silt soil versus target values of 0.63, 1.27, and 1.53. Figure 6 shows that the accuracy of predicted S and K_s are nearly independent to applied scenario for β with slight overestimation of S for sand. Therefore, we may suggest to use a constant value of β to predict S and K_s parameters rather than using soil dependent β. The results are supported by those obtained by Latorre et al. (2018) and indicate that β values obtained from the three-term expansion cannot be considered as a consistent value, and S and K_s can be estimated with a fixed value of β.

<<Figure 6 about here>>

3.4 Compartitive of K_s and S estimates depending on the approximate expansion

Since the three-term expansion does not give robust estimations of β (see section 3.2 and 3.3), this section only will focus on the comparison of the accuracy of the three approximate expansions regarding K_s and S optimization. In the following, we consider that parameter β is fixed at a given value, *i.e.*, 0.6, as suggested by Haverkamp et al. (1994). To this end, the one-, two-, and three-term approximate expansions were fitted to the experimental cumulative infiltrations for the same time datasets (40 to 10000 seconds) and synthetic soils. The results showed that the coefficients of determination R^2 were close to unity for all the expansions and the time datasets (Table 2), but the accuracy and reliability of the predicted parameters were totally dependent on the selected approximate expansion and time datasets (short, intermediate or long) (Figure 7 and Figure 8). Clearly, the three-term approximate expansion performed better. As shown in Figure 7, the one-term approximate expansion is only valid for very short infiltration times (time < 40 s). When this approximate expansion is applied to longer infiltration times, it overestimates S and the magnitude of this overestimation increases with the infiltration time (Figure 7). These results are in line with the results obtained by Minasny and McBratney (2000). The two-term expansion has a larger validity time interval, which increases the accuracy of the inverting procedure in estimating S for broader time intervals (with an increase up to a value between 2000 and 5000 s from sandy to silty soils). However, S remained underestimated for longer infiltration times (Figure 7). Lastly, the three-term approximate expansion showed the most accurate and reliable predictions of parameter S regardless of applied infiltration times (Figure 7). Its estimates remained accurate for even very large times.

In addition to S, the two-term approximate expansion overestimates K_s for all applied infiltration times (Figure 8). This contradicts the conclusions of Vandervaere et al. (2000) reporting that the two-term model is valid for short infiltration times. Contrary to the two-term approximate expansion, the three-term approximate expansion predicted K_s more accurately and reliably for the whole range (40 to 10000 s) of applied infiltration times. However, it seems that the accuracy of predicted K_s using the three-term approximate expansion was dependent on soil permeability and e.g. longer infiltration measurement was needed to predict K_s more accurately for loam and silt soils rather than sandy soil. The need for longer infiltration times for much accurate predictions of K_s was already reported by Latorre et al. (2018).

<<Table 2 about here>>

<<Figure 7 about here>>

<<Figure 8 about here>>

3.5 Performance evaluation against experimental data

In order to verify the results obtained with the synthetic soils and cumulative infiltrations, we used field experimental data. We compared the K_s and S estimates fitting the one-, two-, and three-term expansions and also the QEI formulation to a large dataset of 1D infiltration curves measured with double ring infiltrometer. The estimates obtained with the QEI formulation are regarded as the most accurate estimates and then constitute the benchmark. The experimental infiltration curves were selected from SWIG database (Rahmati et al., 2018a; Rahmati et al., 2018b) comprising infiltration times of 65 to 718800 seconds with an average infiltration time of 26740 seconds indicating nearly long infiltration times for most cases. The results (Table 3) revealed that although no considerable differences are seen between models with respect to R^2, the E criterion shows that the three-term expansion (with E = 0.924 and 0.995 respectively for S and K_s) showed higher accuracy than the one- (with E = 0.-067 for S) and two-term (with E = 0.764 and 0.123 respectively for S and K_s) expansions in comparison with the implicit formulation. Figure 9 and Figure 10 also compare the predicted values of S and K_s from the one-, two-, and three-term expansions to those obtained with the implicit formulation. The Figure 9 revealed that the one-term expansion results in the overestimation of S while the two-term expansion underestimates S in most cases. These discrepancies could be related to the long infiltration durations of some experimental curves. Therefore, these long infiltration durations make the one- and two-term expansions to be used out of their validity period and consequently to spoil the estimates of the related inverting procedures. In contrast to these expansions, the three-term expansion, which allows exploring longer infiltration while remaining valid, provides much accurate prediction of S since all plotted values are close to 1:1 line (Figure 9). The Figure 10 also depicts that the two-term expansion, valid only for short infiltration times, provides overestimation of K_s and less accurate fits with higher values of *RMSE* and lower values of E (Table 3). In contrast, the three-term expansion, that overcomes the limitations of the other expansions, showed a much better accuracy for K_s since plotted values are very close to 1:1 line (Figure 9), along with nice fits, very small values of RMSE and higher values of E in Table 3.

<<Figure 9 about here>>

<<Figure 10 about here>>

<<Table 3 about here>>

4 Conclusions

In this study, we introduce the three-term approximate expansion for modelling 1D water infiltration into soils, which was compared to the analytical implicit equation and the one- and two-term approximate expansions in synthetic sand, loam and silt soil and experimental measurements. The three-term expansion is much closer to the implicit formulation and its validity time interval is enlarged significantly, in comparison to the one- and two-term expansions. However, it may be noted that for very large times, out of the validity time intervals, the proposed approximate expansion exhibits an increase in its slope and the prediction of values higher than the implicit formulation. In this case, the three-term approximate expansion is no longer valid and should not be considered. The three-term approximate expansion improves the quality of fits and estimates for S and K_s.

Results come from the following considerations: The three-term approximate expansion applied to the experimental data gave results similar to those obtained with the implicit formulation. Fits were of the same quality, as well. Instead, the two other expansions (one- and two-term) provided different estimates, with underestimation of S and over-estimation of K_s. The use of the three-term approximate expansion proves quite interesting. Compared to the other approximate expansions, it offers a significant increase in the quality of predictions and estimates with no additional difficulty regarding the computation. Instead, the use of the implicit QEI formulation may be complicated to be used. This. This approximate expansion could be used beneficially for direct and inverse modelling water infiltration into soils and methods of soil hydraulic characterization. For instance, most soil hydraulic characterization methods use either the one- or two-term approximate expansions, e.g., the so-called BEST methods (Lassabatere et al., 2006;

Yilmaz et al., 2010; Bagarello et al., 2014). The accuracy of these methods could be improved by considering the three-term instead.

However, further developments and investigations may be advised, regarding the mathematical properties of this approximate expansion. Indeed, when used out of its time validity intervals, this approximate expansion produces a change in its slope and crosses the implicit formulation (e.g. case of the sand in this study). Conversely, the two other approximate expansions remain below the reference implicit formulation and keep a regular concave shape. The impact of such change in shape and the fact that the proposed approximate expansion may exceed the reference implicit formulation should be questioned in next research investigations.

Similar to two other approximate expansions, the three-term expansion also does not give robust estimations of β. However, a sensitivity analysis showed that the three-term expansion has less sensitivity on β parameter. We also showed that the accuracy of the predicted S and K_s has very less dependency to β parameter showing that a fixed value of β = 0.6 (Haverkamp et al. 1994) or β = 1.1 (Latorre et al. 2018) resulted in enough accuracy in S and K_s parameters predictions. Therefore, by fixing β parameter, the three-term expansion will have two parameters (S and K_s) only and one could use Data Solver application in Excel to minimize the objective function between measured and predicted cumulative infiltration values at given times and then to predict the S and K_s parameters.

5 Author contributions

The main idea of article was put forward by David Moret-Fernández and Mehdi Rahmati. The data analysis was conceived, designed, and performed by Mehdi Rahmati, Borja Latorre, and David Moret-Fernández. The article was written by Mehdi Rahmati, Borja Latorre, Laurent Lasabatere, Rafael Angulo-Jaramillo, and David Moret-Fernández.

6 Competing interests

The authors declare no competing interests.

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Table 1- Values of initial (θ_i), saturated (θ_s) and residual (θ_r) water contents, saturated hydraulic conductivity (K_s), α and n shape parameters of the Van Genuchten (1980) water retention curve and Mualem (1976) hydraulic conductivity curve, sorptivity (S) and shape factor (β) calculated from the soil hydraulic properties (Moret-Fernández and Latorre, 2017).

Soil	θ_i	θ_s	θ_r	α	n	K_s	S	β
	$cm^3 cm^{-3}$			cm^{-1}		mm s^{-1}	mm $s^{-0.5}$	
Sand	0.045	0.43	0.045	0.145	2.68	8.25×10^{-2}	1.521	0.63
Loam	0.078	0.43	0.078	0.036	1.56	2.88×10^{-3}	0.367	1.27
Silt	0.034	0.46	0.034	0.016	1.37	6.93×10^{-4}	0.238	1.50

Table 2- The goodness of fit (R^2) of one-, two-, and three-term approximate expansions to the implicit formulation of Haverkamp et al. (1994)

Model	Soil	Mean	Max	Min	STD
	Sand	0.970	0.998	0.934	2.1e-2
1-term Eq.	Loam	0.999	1.000	0.994	2.0e-3
	Silt	1.000	1.000	1.000	1.0e-4
	Sand	1.000	1.000	0.9998	1.0e-4
2-terms Eq.	Loam	1.000	1.000	0.9999	2.0e-5
	Silt	1.000	1.000	1.000	3.0e-6
	Sand	1.000	1.000	0.9998	1.0e-4
3-terms Eq.	Loam	1.000	1.000	1.000	3.0e-6
	Silt	1.000	1.000	1.000	2.0e-6

Table 3- Accuracy and statistics of estimated parameters of one-, two, and three- term approximate expansions compared to the implicit formulation of Haverkamp et al. (1994) using SWIG database (Rahmati et al., 2018a; Rahmati et al., 2018b) (n = 753)

Approximate expansions	Parameter	E	R^2	RMSE	ER (%)	Mean ± STD
One-term	S (cm $h^{-0.5}$)	-0.067	0.85	13.77	107.81	19.18 ± 23.35
	K_s (cm h^{-1})	-	-	-	-	-
Two-term	S (cm $h^{-0.5}$)	0.764	0.805	6.47	50.68	10.18 ± 11.3
	K_s (cm h^{-1})	0.123	0.995	83.2	423.42	36.42 ± 169.9
Three-term	S (cm $h^{-0.5}$)	0.924	0.932	3.68	28.82	11.61 ± 12.47
	K_s (cm h^{-1})	0.995	0.998	6.3	32.04	21.3 ± 93.02

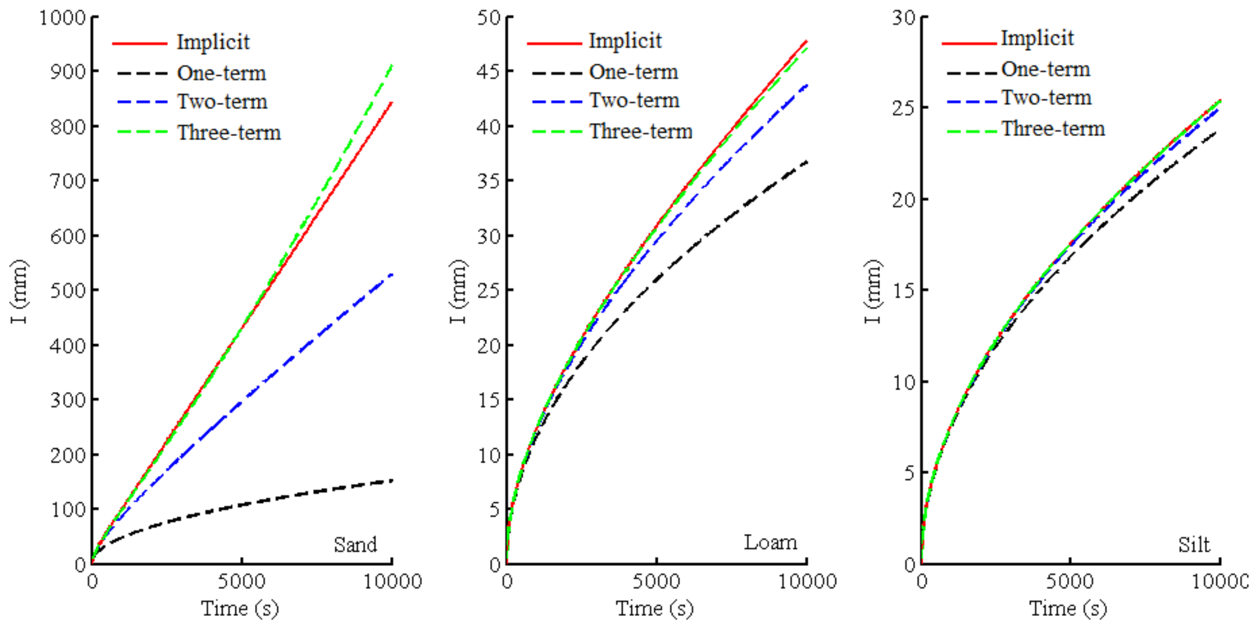

Figure 1- Accuracy evaluation of one-, two-, and three-term approximate expansions with respect to the implicit formulation of Haverkamp et al. (1994) using the synthetic soils

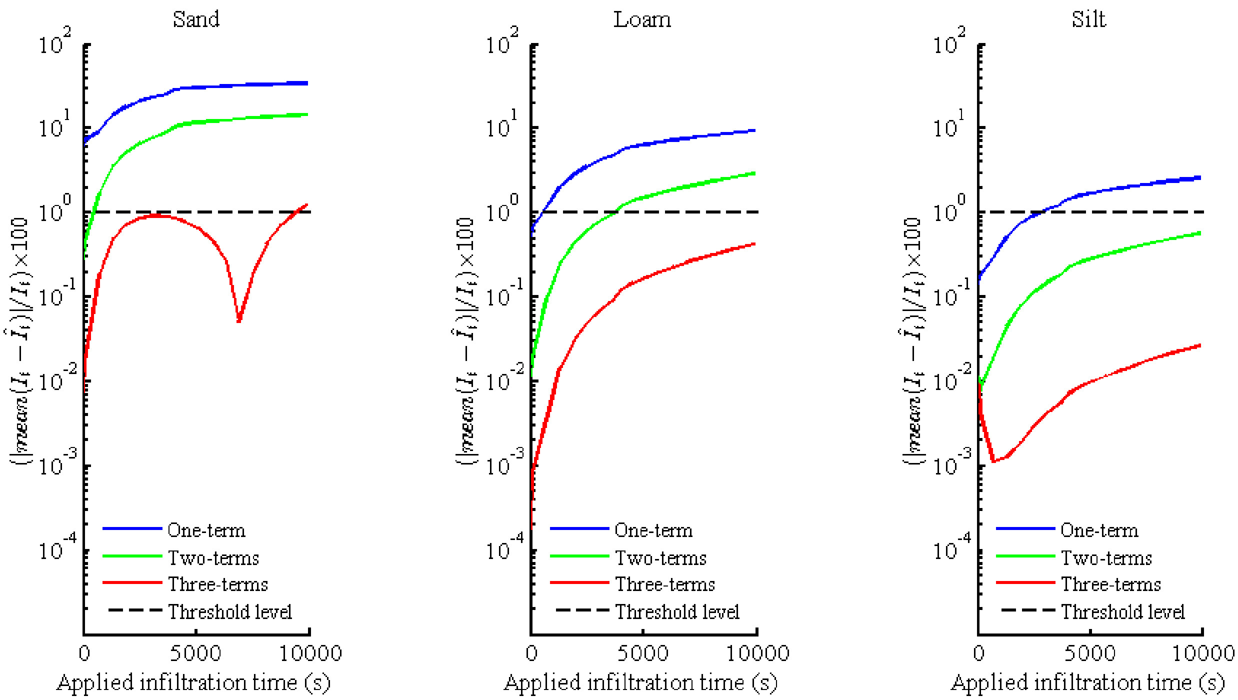

Figure 2- Evaluating the time dependent divergence of one-, two-, and three-term approximate expansions with respect to the implicit formulation of Haverkamp et al. (1994) using the synthetic soils

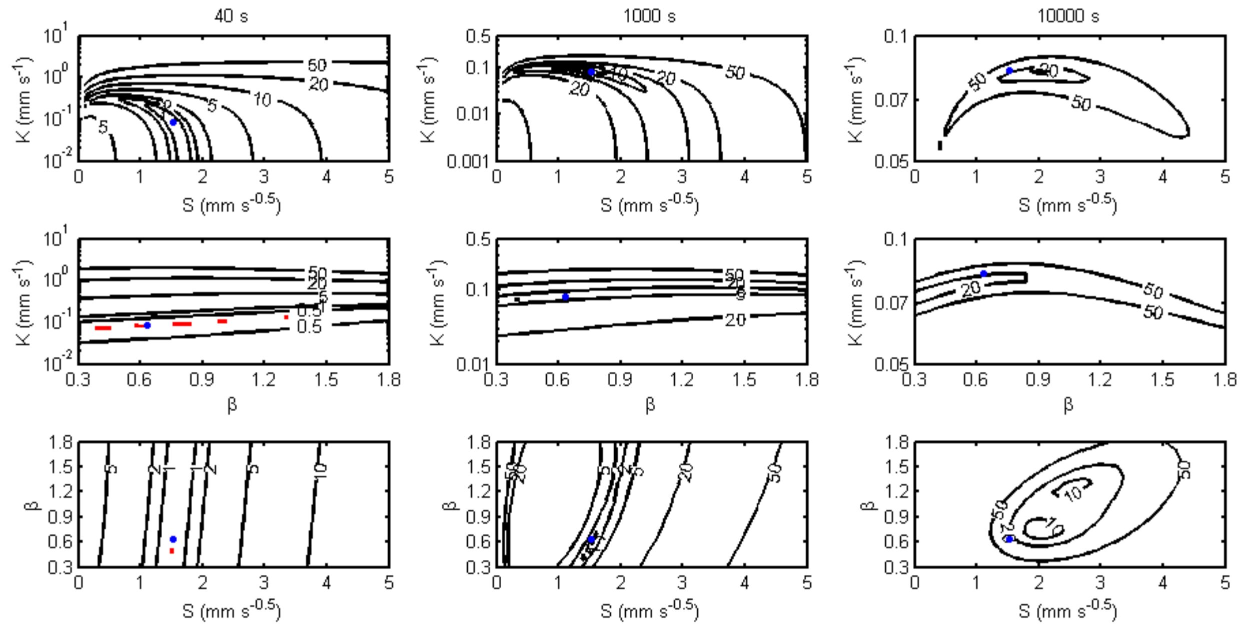

Figure 3- Objective function of the three-term approximate expansion (Eq. (10)) for the sandy soil using three different infiltration times (40, 1000, and 10000 s). Contour lines correspond to the RMSE (cm), the red lines delineate the zone where the objective is lower than a given uncertainty (RMSE = 0.03 mm) and where the estimates should be located, and the blue dots corresponds to known values of parameters.

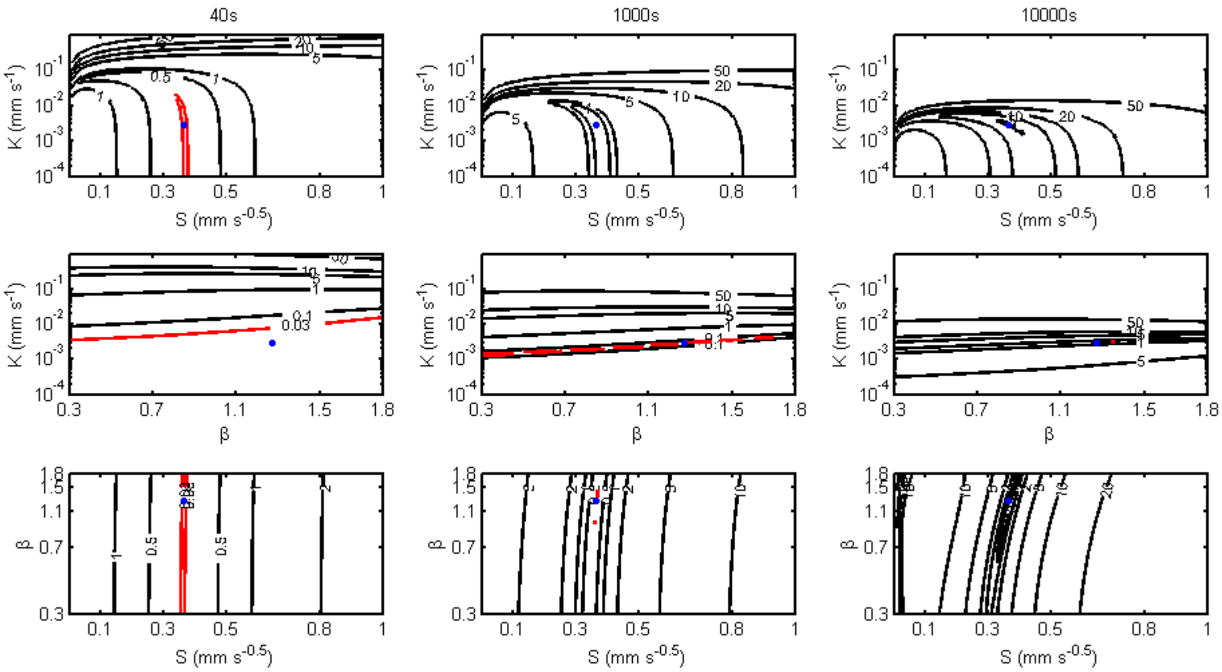

Figure 4- Objective function of the three-term approximate expansion (Eq. (10)) for loam soil using three different infiltration times (40, 1000, and 10000 seconds). Contour lines correspond to the RMSE (cm), the red lines delineate the zone where the objective is lower than a given uncertainty (RMSE = 0.03 mm) and where the estimates should be located, and the blue dots corresponds to known values of parameters

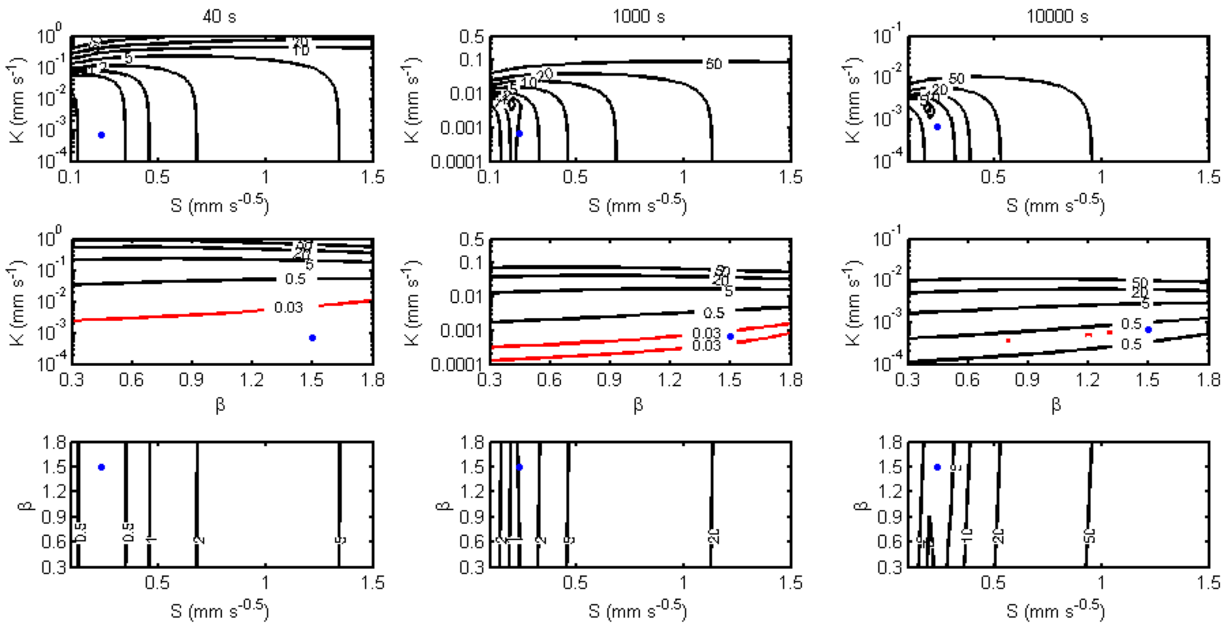

Figure 5- Objective function of the three-term approximate expansion (Eq. (10)) for silt soil using three different infiltration times (40, 1000, and 10000 seconds). Contour lines correspond to the RMSE (mm), the red lines delineate the zone where the objective is lower than a given uncertainty (RMSE = 0.03 mm) and where the estimates should be located, and the blue dots corresponds to known values of parameters

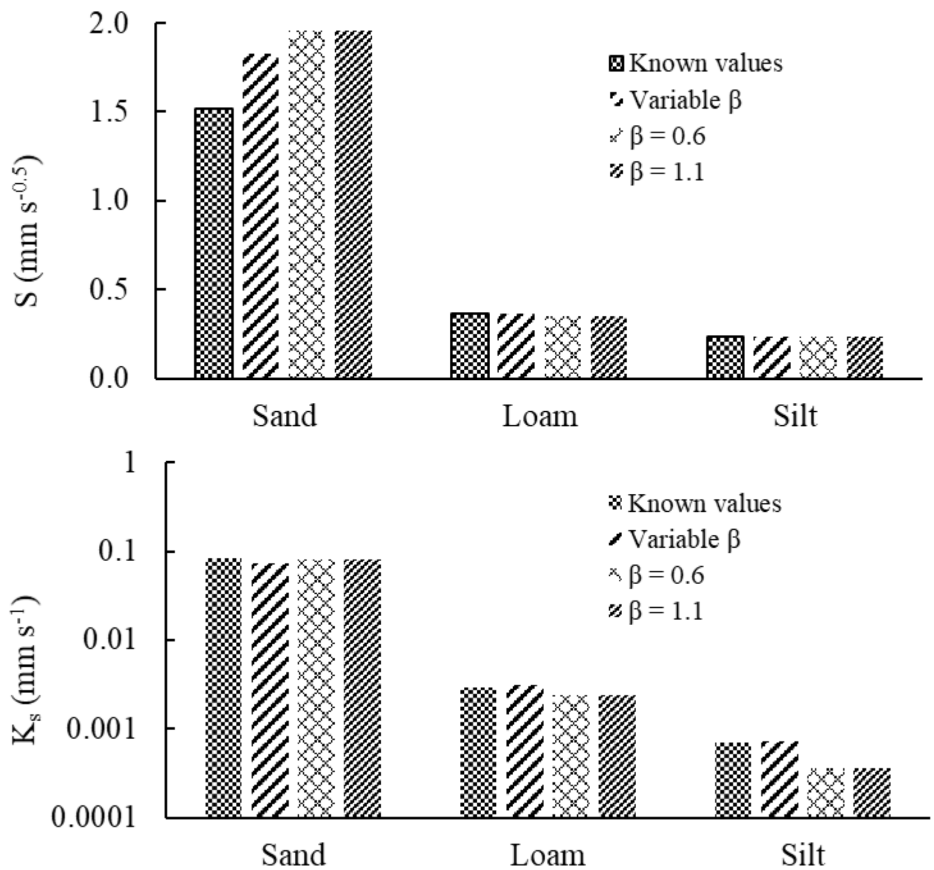

Figure 6- Comparison of S and K_s estimates with the target values using different scenarios for β in the three-term expansion

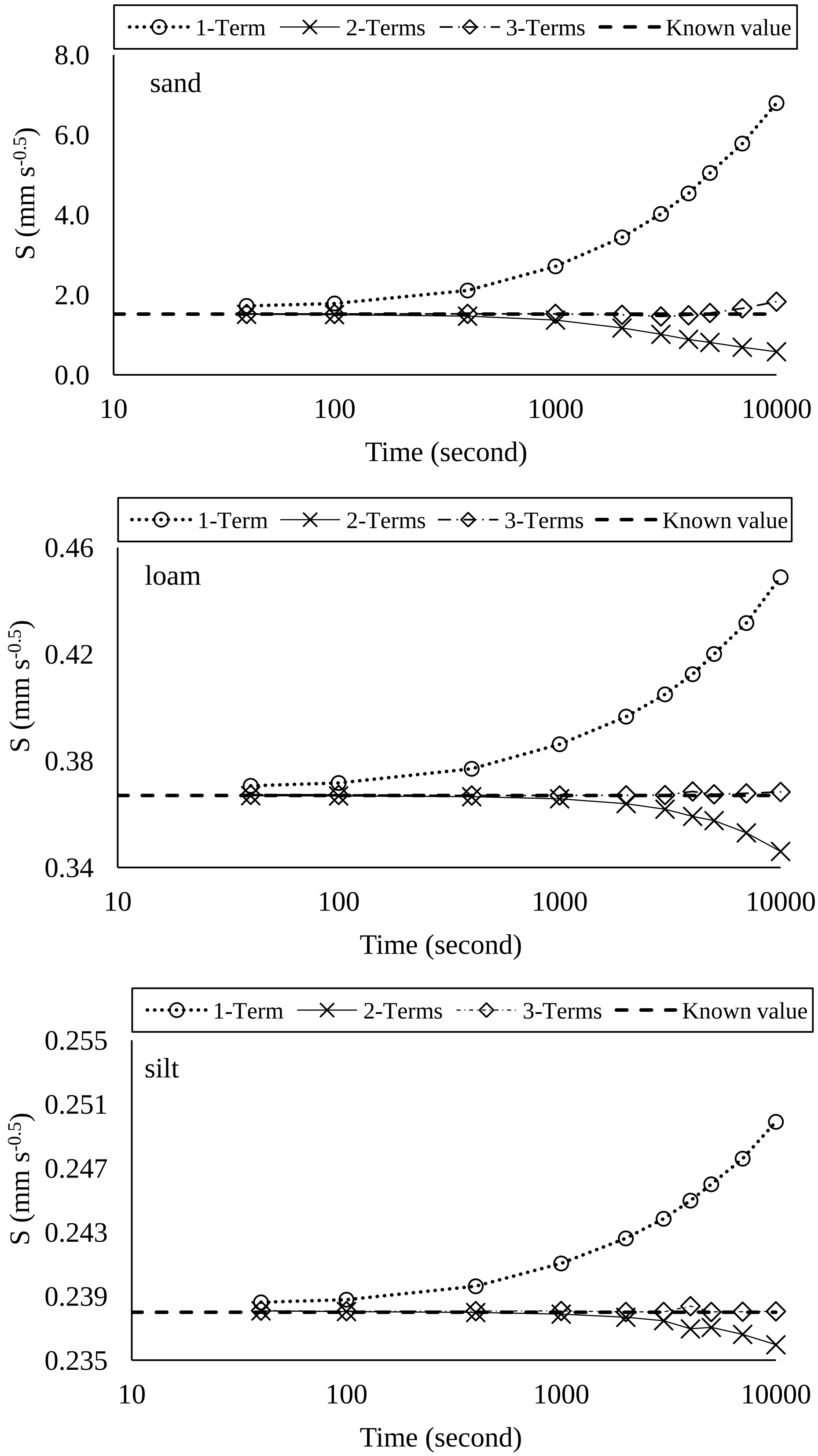

Figure 7 –Accuracy and reliability of predicted *S* for one-, two-, and three-term approximate expansions

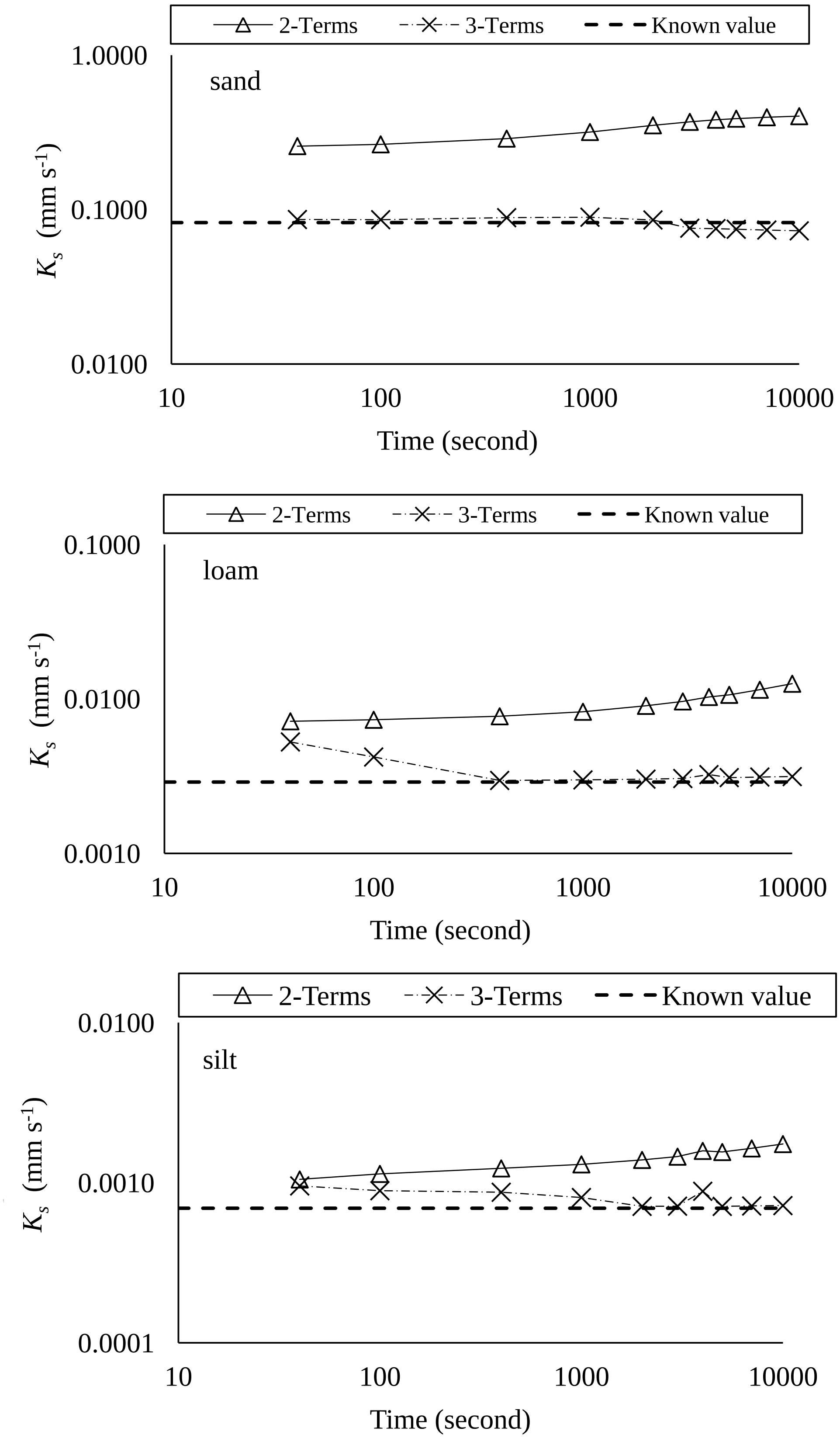

Figure 8 –Accuracy and reliability of predicted K_s for the two- and three-term approximate expansions

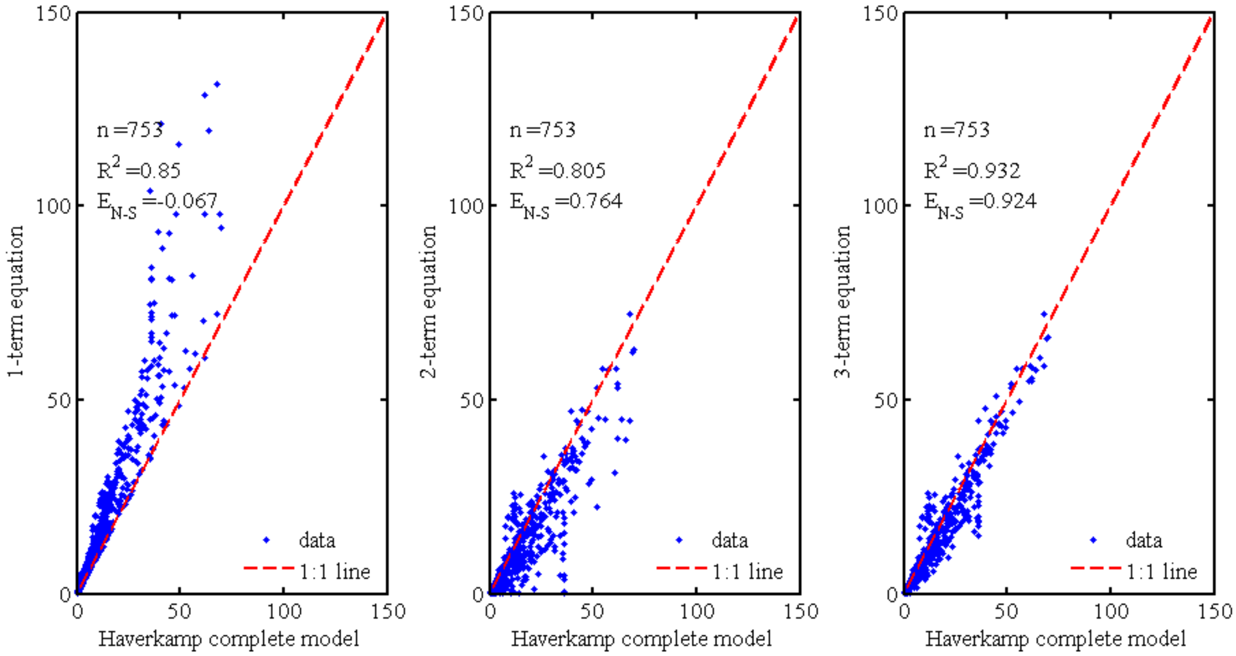

Figure 9- Predicted *S* (mm $s^{-0.5}$) values by the one-, two- and three-term expansions compared to those obtained with the implicit formulation of Haverkamp et al. (1994)

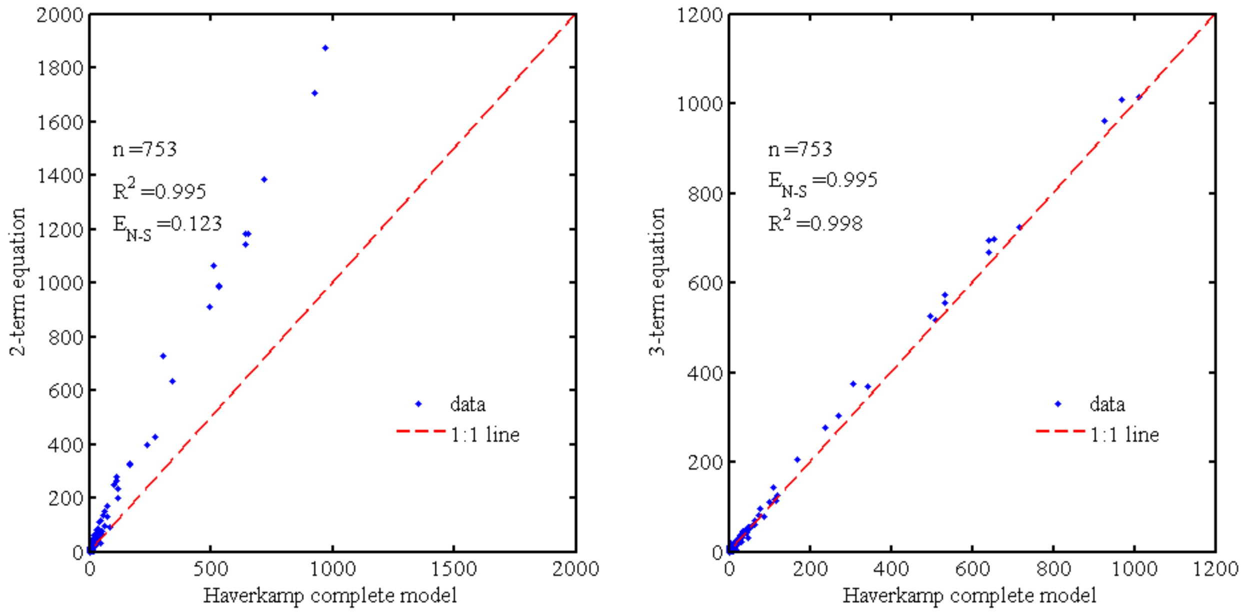

Figure 10- Predicted K_s (mm s^{-1}) values by the two- and three-term expansions compared to those obtained with the implicit formulation of Haverkamp et al. (1994)

Appendix A

In order to drive the three-term approximation of the QEI formulation, first we simplified the model assuming that K_i is close to zero:

$$\frac{2K_s^{\,2}}{S^2}t = \frac{2}{1-\beta}\frac{K_s}{S^2}I_{1D} - \frac{1}{1-\beta}\cdot\ln\left\{1+\frac{1}{\beta}\left[\exp\left(\beta\frac{2K_s}{S^2}I_{1D}\right)-1\right]\right\} \quad \text{(A-1)}$$

Then, a change of variables is made to simplify terms and explicit dependence with $t^{0.5}$:

$$(1-\beta)y^2 = x - \ln\left\{1+\frac{1}{\beta}\left[\exp\left(\beta x\right)-1\right]\right\} \quad \text{(A-2)}$$

where y and x are defined as below:

$$y^2 = \frac{2K_s^{\,2}}{S^2}t \quad \text{(A-3)}$$

$$x = \frac{2K_s}{S^2}I_{1D} \quad \text{(A-4)}$$

Two Taylor series as below are used assuming that z<<1 in both cases:

$$\exp(z)-1 = z+\frac{z^2}{2}+\frac{z^3}{6}+\frac{z^4}{24}+\ldots \quad \text{(A-5)}$$

$$\log(1+z) = z-\frac{z^2}{2}+\frac{z^3}{6}-\frac{z^4}{24}+\ldots \quad \text{(A-6)}$$

Substituting these series sequentially in Eq. A-2 results in:

$$y^2 = \frac{x^2}{2}+\frac{(be-2)x^3}{6}+\frac{(be^2-6be+6)x^4}{24}+\ldots \quad \text{(A-7)}$$

Then, a time expansion of x is proposed in the form of powers of $t^{0.5}$:

$$x = ay+by^2+cy^3+\ldots \quad \text{(A-8)}$$

Substituting Eq. A-7 in Eq. A-8 leads to the coefficient values:

$$a = \sqrt{2}$$
$$b = \frac{2-\beta}{3} \qquad \text{(A-9)}$$
$$c = \frac{\beta^2 - \beta + 1}{9\sqrt{2}}$$

These coefficients are substituted and Eq. A-8 is finally expressed in the original variables:

$$I_{1D} = S \cdot t^{\frac{1}{2}} + \frac{2-\beta}{3} K_s \cdot t + \frac{1}{9}\left(\beta^2 - \beta + 1\right)\frac{K_s^2}{S} \cdot t^{\frac{3}{2}} + \ldots \qquad \text{(A-10)}$$

The first three terms are selected to approximate the QEI formulation which is reported in equation 10 of main text.

The relevance of Philip theory to Haverkamp equation and its uses to predict soil hydraulic properties

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Highlights

- This work approximates the quasi-exact implicit formulation of Haverkamp model up to three-term expansion.
- The accuracy of the proposed approximation is evaluated based on the quasi-exact implicit model.
- A sensitivity analysis is conducted to define the infiltration time for which this expansion remains valid.
- The proposed approximation is compared with Philip (1957) one- and two-term approximate expansions.