Chebyshev interpolation of phase space mapping for efficient optical characterization

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Abstract.

The phase space mapping, between an object and image plane, characterizes an optical system within geometrical optics framework. We propose a novel idea to characterize the phase mapping in axially symmetric optical systems for arbitrary object locations not restricted to a specific object plane. The idea is based on decomposing the phase mapping into a set of bivariate equations corresponding to different values of the radial coordinate at a specific object surface (most likely the entrance pupil). These equations are then approximated through bivariate Chebyshev interpolation at Chebyshev nodes, which guarantees uniform convergence. Additionally we propose the use of a new concept (*effective object phase space*), defined as the set of points of the phase space at the first optical element (typically the entrance pupil) that are effectively mapped onto a set of points included at the image surface. The *effective object phase space* provides a way to avoid tracing rays that do not reach the image surface, by means of an inclusion test.

Keywords: Phase space optics, geometrical optics, ray tracing.

1. Introduction

Ray tracing routines are at the core of optical systems design and characterization. At first sight, one may think that the computational cost of these routines is not a relevant issue due to current computer capabilities. However, there are, at least, two good reasons to avoid tracing many rays. First, although not many rays are typically required to geometrically characterize the optical performance of an optical system for a fixed punctual source, this is not true if the operation has to be repeated many times, because there are extended sources or an evaluation for different object locations is required (e.g field of view or depth of focus analysis). A second reason is that in many applications the rays emerging from an object point must be selected in order to fill the aperture-stop surface. This indirectly implies avoiding tracing those rays not reaching the final image surface (vignetting). Optical design programs (e.g. Zemax or Code V) solve this requirement through costly iterative ray tracing algorithms, called *ray-aiming*. Hanika et al.[1] proposed an algorithm, in somewhat similar philosophy, within computers graphic community.

The natural strategy to avoid tracing too many rays is to characterize the phase mapping equations with some type of functional relations. For axially symmetric optical systems, the classical approach has been to approximate rays coordinates at image space through Taylor series expansions with respect to three independent object-pupil coordinates (a recent example can be found in Zheng et al. [2]). Nowadays, Computer Graphics community appears to be specially interested in such approach [1, 3, 4], coining the keyword *polynomial optics*, and applying it even in the absence of axial symmetry. In that field, tracing large number of rays is typically required; hence there is a strong motivation to reduce the associated computation cost. The proposed methodology is based on two steps: 1) To compute a polynomial expansion of the phase mapping at a reduced, randomly sampled, set of real traced rays [1]; 2) To evaluate the polynomial coefficients solving the associated set of linear equations, through, for e.g., using non-linear least squares fitting [1]. If many high order terms are required, close to zero coefficients could appear, then making techniques to handle sparsity required [4].

However, it is well known that the approximation accuracy in Taylor expansion decreases when the object ray coordinates drift apart from the base (aberration-free) one [5]. Additionally, uniform convergence (error reduction when increasing the number of monomials) in Taylor expansion is not even guaranteed for typical regions of interest, as pointed out by Forbes three decades ago [5, 6]. Another serious drawback of fitting (versus interpolation) is that it introduces an error in the phase mapping equations even in those real traced data where there is no error. A final inconvenience is that computing the coefficients of the series expansion (for a specific system) implies elaborated algebraic operations.

An alternative to series expansion is polynomial interpolation. After tracing a small set of rays, the final ray coordinates [7] or the characteristic function [8] can be interpolated, making use of the fact that the optical mapping is a symplectic one, and,

consequently, smooth (class C^∞) [9]. We (as others e.g. [1]) followed this approach applying piecewise polynomial interpolation (splines) in [10].

However, it is problematic to ensure uniform convergence and, at the same time, to keep a physical interpretation when doing interpolation with three independent variables instead of two. We opt now for a novel strategy (seminal idea presented at [11]) which, while remaining faithful to the interpolation approach, tries to overcome those problems. The basic idea is to decompose the phase mapping into a set equations in two variables corresponding to different values of the radial coordinate of a ray at an specific object surface. These equations are then approximated through bivariate Chebyshev interpolation, at Chebyshev nodes, which guarantees uniform convergence.

We note that Gagnon et al. [12] already proposed using univariate Chebyshev interpolation for meridional ray tracing in rotationally symmetric systems. Later on, Jesus [13] extended the concept to bivariate Chebyshev interpolation of both surface and ray-surface intersection data. Our approach differs from theirs in that we directly interpolate phase mapping equations, instead of single-step optical transformations (refraction, surface description).

2. Phase space characterization

A ray crossing an optical system is fully characterized by four coordinates: two of them describing the ray coordinates at a plane orthogonal to the optical axis and the other two determining the momentum (direction vector of the ray multiplied by the refractive index). The fundamental problem of geometrical optics is to determine the relations between these coordinates at the image plane with respect to their counterparts at the object space. The full set of rays configure a vector space called phase space. Then a *phase space mapping* is established between the object and image space [9].

In an axially (also called rotationally) symmetric optical system the phase space mapping is reduced to three independent variables that are invariant under rotations of the optical system. Let us denote the ray location and momentum (normalized) vectors by $\vec{q}$ and $\vec{p}$, respectively. Typically, the three reduced coordinates (e.g. [14] p. 36) are : $|\mathbf{q}|^2$, $|\mathbf{p}|^2$ and $\mathbf{p} \cdot \mathbf{q}$. This set of coordinates can be directly applied to the problem of computing the propagation of light in a gradually varying refractive index medium, considering that, at the end, the fourth coordinate of the complete phase space can be recovered ([9] pp. 123-124). However, when dealing with an optical system comprising interfaces that separate media of different refractive indices, these coordinates can not be directly applied; then, a discrete ray tracing is required. An alternative is to use a new set of coordinates which are function of those, and, consequently, are themselves rotationally invariant.

When cylindrical coordinates are used, $\vec{q}$ is characterized by the distance from the ray location to the axis of rotation (ρ) and the angle (ϕ), formed between a fixed meridional plane and another one containing the ray location. We set a local coordinate system: $(\vec{e}_\rho, \vec{e}_\phi)$ at each point along a ray. In this local coordinate system the ray

momentum $\vec{p}$ is described by two coordinates: 1) the radial momentum component (p_ρ), obtained as the dot product of the momentum $\vec{p}$ with $\vec{q}$; 2) the angular (skew) momentum component (p_ϕ), given by the modulus of the cross product: $p_\phi = |\vec{q} \times \vec{p}|$ [15].

A geometrical interpretation of these momentum coordinates is as follows. Let us set X-Z as the meridional plane of the optical system. A point O of a specific ray has a direction vector described by two angles [15]: 1) α_ϕ : angle between the ray and the meridional plane; 2) α_ρ : angle between the ray and the orthogonal plane to X-Z containing the optical axis (Y-Z). Then: $p_\rho = n \sin \alpha_\rho$ (n : refractive index) is the radial and $p_\phi = n \sin \alpha_\phi$ the angular momentum components respectively.

It is convenient to relate α_ρ and α_ϕ with the natural angles ($\alpha_x, \alpha_y, \alpha_z$) involved in the cosine directions that are used for ray tracing routines. Figure 1 shows the geometrical relationship between these angles.

Figure 1. Relationship between $(\alpha_\rho, \alpha_\phi)$ and $(\alpha_x, \alpha_y, \alpha_z)$

At image space we used cartesian coordinates $\vec{q}'(x', y')$. Using the three independent coordinates (ρ, p_ρ, p_ϕ) the phase space mapping equations can be written:

$$\begin{aligned} x' &= f_1(\rho, p_\rho, p_\phi) & y' &= f_2(\rho, p_\rho, p_\phi) \\ p'_\rho &= f_3(\rho, p_\rho, p_\phi) & p'_\phi &= f_4(\rho, p_\rho, p_\phi) \end{aligned} \quad (1)$$

where primes implies coordinates at image space. We note that we prefer to define ray position coordinates at image space in cartesian coordinates in order to characterize both coordinates (and corresponding functions f_1 and f_2) with the same units. Axial rotationally symmetry also imposes that the fourth of Eqs. 1 is calculated straightforward just applying the *skew invariant* ([15] p. 35): $p'_\phi = \frac{\rho p'_\phi}{\rho'}$.

3. The *effective object phase space*

The core philosophy of this work is to explode the potentials of Eqs. 1 to fully characterize at once the geometrical properties of the optical system. To visualize these properties, an object and an image planes are usually fixed. Hence this type of representation restricts the characterization of the optical system to the correspondence between these two planes.

We propose a new graphical representation that intends to provide extra information of the optical mapping associated to different object planes into a single 2-D graph. The idea is that determining the set of rays passing through the optical system and impinging at a given image surface is equivalent to obtain the complete set of points of the reduced phase space (ρ, p_ρ, p_ϕ) defined at the first optical element, which are mapped onto a set (non-null) of points at the image surface. From now on, we shall call it the *effective object phase space*. The first optical element may represent the entrance pupil or the first optical surface of the optical system (depending on the application). The entrance pupil should be used when it is easily and accurately computable, and the first optical surface otherwise.

Real rays traced through the entrance pupil of an optical system may not reach the final image plane because of several possible factors: 1) total internal reflection at an interface, 2) vignetting, i.e rays blocked by the aperture stop; 3) rays missing a ray-surface intersection. The *effective object phase space* provides a compact visualization tool to describe all possible light (independent of object location) captured by an optical system.

An useful feature of the *effective object phase space* is that it provides a way to replace ray aiming algorithms by a simpler inclusion test: starting from the fan of rays emerging from an object target, the inclusion test allows us to avoid tracing those rays not belonging to the *effective object phase space* because, by definition, they do not reach the involved image surface.

We have implemented such an inclusion test following this sequence:

- (i) To compute phase mapping Eqs. 1 through ray tracing with a dense sampling of (ρ, p_ρ, p_ϕ) .
- (ii) To obtain the set of rays $(\rho(u), p_\rho(u), p_\phi(u))$, $u = 1 \dots n$ from the previous set that reach the final image plane.
- (iii) To compute the smallest convex set (*convex hull*) containing $(\rho(u), p_\rho(u), p_\phi(u))$, $u = 1 \dots n$. (We implemented this step with the help of *DelaunayTri* and *convexHull*

MATLAB[®] functions.)

- (iv) Finally a ray is determined to be included inside the *effective object phase space* if its coordinates belong to the convex hull calculated in the previous step.

As an example, Figure 2 shows the *effective object phase space* computed for a triplet optical system described in section 7. This example shows that when the radial coordinate ρ increases, the range of the radial momentum coordinates p_ρ of the *effective object phase space* is shifted and stretched. This occurs because the rays with positive and large radial momentum (rays directed outwards of the optical system) are lost when the incident ray location is moving towards the peripheral part of the pupil.

Figure 2. *Effective object phase space* of a triplet lens system.

4. Interpolation of the phase mapping

As mentioned in the Introduction our proposal is to decompose Eqs. 1 into a set of equations in two variables:

$$x' = g_1^j(p_\rho, p_\phi) \quad j = 1..m$$

$$\begin{aligned} y' &= g_2^j(p_\rho, p_\phi) & j &= 1..m \\ p'_\rho &= g_3^j(p_\rho, p_\phi) & j &= 1..m \end{aligned} \quad (2)$$

for j being the index for the set of $\rho(j)$ coordinates sampling the effective object space volume (sampling density is chosen as dense as required) and g_1^j , g_2^j and g_3^j denoting different functions. Then, these functions are approximated by Chebyshev polynomials.

In univariate polynomial interpolation, the interpolation error depends on both the sampling points and the shape of the function to be interpolated. Using Chebyshev interpolation points guarantees two important properties: 1) the error term exclusively depending on the sampling is minimized ([16] p. 32); 2) uniform convergence (uniform norm of the interpolation error tends to zero with increasing number of monomials [16] p. 117) of the interpolation, something that does not occur with equally spaced points. If L2 norm is used, the criteria is relaxed and a sufficient condition for uniform convergence is that the interpolation points are the zeros of a system of orthogonal polynomials [17]. Additionally, the Chebyshev polynomial (using Chebyshev interpolations points) is the best uniform approximation to any continuous function in a bounded interval ([16], p. 119). These properties of univariate Chebyshev interpolation can be extrapolated to bivariate interpolation if partial derivatives exist and are bounded [17]. Drawing on the fact that the optical mapping is symplectic, the partial derivatives of Eqs. 2 exist and are bounded, hence ensuring an uniformly convergent polynomial interpolation provided that Chebyshev interpolation points are used.

Considering all the aforementioned, it is fully justified to write as identities:

$$\begin{aligned} g_1^j(p_\rho, p_\phi) &= \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} a_{kl}^j T_k(p_\rho) T_l(p_\phi) & j &= 1..m \\ g_2^j(p_\rho, p_\phi) &= \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} b_{kl}^j T_k(p_\rho) T_l(p_\phi) & j &= 1..m \\ g_3^j(p_\rho, p_\phi) &= \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} c_{kl}^j T_k(p_\rho) T_l(p_\phi) & j &= 1..m \end{aligned} \quad (3)$$

where $T_n(x)$ is the Chebyshev polynomial of the first kind and a_{kl}^j , b_{kl}^j and c_{kl}^j are the coefficients of the different expansions.

In practice, the Chebyshev expansion is truncated, so, instead of an identity, we get an approximation:

$$\begin{aligned} g_1^j(p_\rho, p_\phi) &\approx \sum_{k=0}^K \sum_{l=0}^L a_{kl}^j T_k(p_\rho) T_l(p_\phi) & j &= 1..m \\ g_2^j(p_\rho, p_\phi) &\approx \sum_{k=0}^K \sum_{l=0}^L b_{kl}^j T_k(p_\rho) T_l(p_\phi) & j &= 1..m \\ g_3^j(p_\rho, p_\phi) &\approx \sum_{k=0}^K \sum_{l=0}^L c_{kl}^j T_k(p_\rho) T_l(p_\phi) & j &= 1..m \end{aligned} \quad (4)$$

K and L in Eqs. 4 are called the ranks of the Chebyshev expansions in the two independent variables. The truncation of the Chebyshev expansion introduces an avoidable interpolation error. However, as we will illustrate with an example in section 7, the interpolation error rapidly goes to negligible values even with moderate ranks expansions, showing the potentials of our approach.

In this work, we used the public domain MATLAB[®] *chebfun toolbox* [18] for all operations related to the Chebyshev interpolation. Throughout this work, for simplicity, we set $K = L$, i.e. we assume that the same expansion rank in p_ρ and p_ϕ provides similar results. This is not necessary the most opportune choice, because in axially symmetric systems, phase mappings typically exhibit a larger variation with p_ρ than with p_ϕ . However, selecting the optimal ratio between K and L is only attainable through empirical analysis for each particularly case.

5. Sampling the *effective object phase space*

As said before, the optimal interpolation of Eqs. 3 is attained using Chebyshev interpolation points. For this purpose, we have to evaluate Eqs. 2 at these points. This is done through ray tracing, which is assumed to provide exact results.

Fixing $\rho(j)$, $j = 1..m$, the *effective object phase space* volume is reduced to an area which we call *effective phase space area*. For $\rho = 0$, due to the symmetry, the effective phase space area is a circle. However, for any other value of $\rho(j)$ the effective phase space area is shifted and stretched showing complicate forms. Figure 3 shows (ray tracing evaluation of Eqs. 2) these phase space areas for different values of ρ : $\rho = 0$ mm, $\rho = 6.5$ mm, $\rho = 13$ mm and $\rho = 19.5$ mm; corresponding to the optical system example (Cooke triplet) described in section 7. Therefore, considering that Chebyshev interpolation properties described in section 4 are only valid for rectangle domains, for each $\rho(j)$ we are forced to restrict the interpolation to the largest area axis-parallel rectangle inscribed inside the effective phase space area. Using rays out of this effective areas would create null control points, which eventually would significantly increase the interpolation error.

If the effective phase space area is approximated with a convex polygon, the problem of finding this largest area inscribed rectangle is computationally treatable, although it is far from simple [19]. For this purpose, we developed a customized algorithm, based on a convolution iterative process, comprising the following sequence:

- (i) Define our interest area as the smallest axis-aligned rectangle containing the convex polygon (frontier of the phase space area), by simply taking its minimal and maximal coordinate values. We call this rectangle G .
- (ii) Discretize G (taking as step, e.g., 1) to obtain a binary 2D array from the frontier dots, where bit 1 and 0 correspond to a location being contained or not in the convex polygon, respectively.
- (iii) Sweep in a double loop in (n_x, n_y) , the dimensions of a smaller all-ones rectangular array S .

- (iv) For each (n_x, n_y) pair, test if S fits in the polygon. This is done by convolving the binary mask with S, and detecting the locations (if any) where the convolution yields the area of S ($n_x \times n_y$). If $(n_x \times n_y)$ is larger than the previously saved value (initialized to 0), then the dimensions and center location of S are saved.
- (v) Final largest-area S dimensions and location in G are converted back to moment coordinates.

The enclosing rectangles are defined by $[p_{\rho min}^j, p_{\rho max}^j] \times [p_{\phi min}^j, p_{\phi max}^j]$. Therefore, for interpolation within them we applied shifted/scaled Chebyshev polynomials: $T^*(x/c)$ [17]. At each of these rectangles we sampled a set of Chebyshev interpolation points using a fixed rank of interpolation.

Again using the example described in section 7, Fig. 3 shows the Chebyshev interpolation points (blue crosses) corresponding to the phase space areas associated to each value of $\rho(j)$. The plots show that the rectangular Chebyshev domains loose some regions of the effective phase space areas. At this point, a key assumption is that, although Eqs. 4 are strictly only valid inside the enclosing rectangles, it still can provide a good approximation for rays located in those lost effective phase space areas. We will check this hypothesis empirically with the application example of section 7.

We point out that optically axial symmetric systems are invariant under reflections with respect to planes containing the axis of symmetry (p. 113 [9]). This implies that we only need to perform real ray tracing coordinates at half-of momentum enclosing rectangles: $[p_{\rho min}^j, p_{\rho max}^j] \times [0, p_{\phi max}^j]$.

6. Spot diagrams from phase space mapping interpolation

The Chebyshev expansion provided by Eqs. 4 can be applied to compute spot diagrams associated with a punctual source without tracing any new ray following the sequence:

- (i) To select the position of the object points.
- (ii) To choose a set of points sampling the entrance pupil plane: $(\rho(u), \phi(u))$, $u = 1..n$. with the condition that $\rho(u)$ are contained in the set $\rho(j)$, $j = 1..m$ used for the phase mapping interpolation.
- (iii) To compute the corresponding momentum coordinates associated to each ray defined by the line joining the punctual source and pupil points: $(p_\rho(u), p_\phi(u))$, $u = 1..n$.
- (iv) Having previously obtained the effective object phase space as a convex hull (see section 3), to eliminate those rays of the set $(p_\rho(u), p_\phi(u))$, $u = 1..n$ located outside each effective phase space area. We implemented this step as a conventional point-in-polygon computation (with the help of *inpolygon MATLAB*[®] function).
- (v) To obtain ray coordinates at the last refractive surface given by Eqs. 4.
- (vi) To compute the ray intersections with the desired image surface through propagation of ray trajectories at an homogeneous image space.

In case of an object point located at a finite distance, Figure 4 shows the relation between the angles $(\alpha_\rho, \alpha_\phi)$, and the pupil coordinates (x_p, y_p) , that are needed to compute the momentum coordinates: $p_\rho = n \sin \alpha_\rho$ and $p_\phi = n \sin \alpha_\phi$. If the punctual object is located at infinity, the angles $(\alpha_\rho, \alpha_\phi)$ directly provided the momentum.

We note that Eqs. 4 provide the image space coordinates $(x', y', p'_\rho, p'_\phi)$ at a coordinate system relative to a fixed meridional plane. To obtain the coordinates with respect to an absolute cartesian coordinate system, we need to apply a rotation:

$$\begin{pmatrix} x'_{abs} \\ y'_{abs} \end{pmatrix} = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix} \begin{pmatrix} x' \\ y' \end{pmatrix} \quad (5)$$

where θ is the angle forming the pupil position vector (x_p, y_p) with the fixed meridional plane ($Y=0$ in Fig. 4) .

Figure 3. Coordinates of p_ρ (abscissas) and p_ϕ (ordinates) of Chebyshev interpolation points (blue crosses) for effective phase space areas corresponding with: (a) $\rho = 0$ mm; (b) $\rho = 6.5$ mm; (c) $\rho = 13$ mm; (d) $\rho = 19.5$ mm. Close red lines represent the edges of the effective phase space areas.

Figure 4. $(\alpha_\rho, \alpha_\phi)$ angles from pupil coordinates (x_p, y_p) for an object point located at a finite distance. $Y=0$ is set as the meridional plane.

7. Examples of application

To illustrate the potentials of our procedure, we used an optical system comprising spherical refracting surfaces, which allows an exact ray tracing. Particularly we used a Cooke triplet described elsewhere [20]. The maximum field of angle is 12° .

The Cooke triplet is a good example for our purposes because is the starting point of many optical systems design and it has a considerable good chromatic aberration correction (even in the secondary spectrum). So far we have not not considered the chromatic response of the optical system.

For each j we evaluated the interpolation error of Eqs. 4 with increasing value of the rank in the Chebyshev expansion. We computed the average of the normalized (to the RMS of the spot diagram) root-mean square error (NRMSE) for $\rho(j)$ sampled in nine different locations. Results are show in table 1.

Table 1 shows that $rank = 15$ already provides a good approximation of the phase mapping.

After it, we computed the error in the estimation of spot diagrams for different

Table 1. Normalized root-mean square error (NRMSE) as function of Chebyshev rank interpolation for a Cooke triplet.

Chebyshev rank	5	7	9	11	13	15	17	19	21
NRMSE (%)	0.2	0.03	0.008	0.002	7.6E-4	2.3E-4	7E-5	2E-5	6.E-6

locations of the punctual object. The exact ray tracing ensures a rigorous computation of the interpolation error.

For an object point at infinity, Figure 5 shows the spot diagrams at the image plane calculated through exact ray tracing and estimated with Chebyshev interpolation. NRMSE for spot diagrams estimation were 2.8E-5 %, 5.3E-6 %, 1.7E-5 % and 2E-2 %, for angle of incidences: 0°, 4°, 8° and 12° respectively. These results show that the error in the spot diagram estimation is close to the order of the interpolation error of Eqs. 4 with the exception of the extreme angle of incidence 12° (maximum field of angle), where the error is significantly increased. Anyhow, the 2E-2 % for this case is still acceptable for most of optical applications.

Figure 5. Exact (red crosses) and estimated (blue circles) ray impacts (μm) (Cooke triplet) for different field angles: (a) 0°; (b) 4°; (c) 8°; (d) 12°.

Figure 6 shows the interpolation error for different finite location of the punctual object. For illustrative purposes, instead of the NRMSE we used decibel units ($dB =$

$10\log_{10}NRMSE$). The ordinate represents the distance of the punctual object from the optical axis in mm. The abscissa represents the axial distance of the punctual object to the entrance pupil normalized to the axial location providing the maximum field of angle: $z_n = \frac{z * Pupil\ Radius}{\tan(Maximum\ field\ of\ angle)}$.

Figure 6. Interpolation error (dB) as function of finite object location: normalized axial coordinate (abscissa), distance to optical axis in mm (ordinate)

Just as Fig. 5, Fig. 6 shows the error to estimate spot diagrams is very low for any region that it is not very close to regions where momentum takes values close to that associated to the maximum field of angle (abscissa close to 1). Although, again, as for the infinity case, the error in such regions could be acceptable for most of optical applications.

8. Discussion

We have proposed a new way to fully characterize, for arbitrary object locations, axially symmetric optical systems. Obviously such characterization it is confined within geometrical optics framework.

As explained before, the methodology decomposes the three-variate phase mapping equations into a set of bivariate ones, defined in a domain that we called object phase space areas. To correctly apply bivariate Chebyshev interpolation, we had to limit

the interpolation domain to the largest area axis-parallel rectangle inscribed inside the effective phase space area. This poses an intrinsic limitation to our method (because some data is ignored for the interpolation), that could be avoid if some recent advances in interpolation techniques for irregular domains would be applied, such as the theory of *Weakly Admissible Meshes* (WAM) [21]. However such approach would imply high demanding computation cost and mathematical complexity that is out of the scope of this work. Furthermore, as we showed with our application example, the extrapolation error introduced when applying Chebyshev expansion to rays, within the phase space areas but out of the rectangle interpolation domains, is not critical.

Conversely we found that the interpolation errors become problematic at the edges. Typical spot diagrams exhibit a core region where the point density is highest. However, optical designers are accustomed to observe that around this core some ray impacts are scattered toward the periphery of spot diagrams (elsewhere called *flares* [22]). These flares are related to, either the presence of large aberrations, or the contribution of edge regions on the field of view or pupil. The spot diagram associated to the maximum field of angle of the cooke triplet, used in this work, shows these flares (Fig. 5.(d)) . At these situations, the estimation error of the spot diagrams is increased although, at least in this case, the error is still acceptable.

We note that, for our methodology, the restriction that the pupil entrance radial coordinates are fixed is not strictly required. It could be avoid if the Chebyshev coefficients (a_{kl}^j , b_{kl}^j and c_{kl}^j in Eqs. 4) are themselves interpolated for a new set of ρ coordinates instead of those fixed at $j = 1..m$. This should be used if, for instance, the spot diagram must be computed with a cartesian (instead of polar) sampling of the pupil entrance.

Although we used only one application example, it contains enough ingredients to confirm the hypothesis of our work. Anyhow, we plan to extend the analysis to other type of optical systems such as wide field of view objective lenses.

9. Acknowledgments

This work was supported by grant FIS2016-75891-P from the Spanish Ministerio de Economía y Competitividad.

10. References

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