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Fundamental gaps in numerical semigroups[☆]

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Abstract

We introduce the concept of fundamental gap of a numerical semigroup. We give lower and upper bounds for the number of fundamental gaps of a numerical semigroup in terms of its Frobenius number. Finally we derive several applications from the properties of fundamental gaps.

MSC: 20M14; 20M30

1. Introduction

A numerical semigroup is a subset S of $\mathbb{N}$ that is closed under addition, $0 \in S$ and generates $\mathbb{Z}$ as a group; this last condition is equivalent to $\gcd(S) = 1$ (here $\mathbb{N}$ and $\mathbb{Z}$ denote the set of nonnegative integers and integers, respectively; gcd stands for greatest common divisor). For a given subset A of $\mathbb{N}$, we write $\langle A \rangle$ to refer to the set

$$\left\{ \sum_{i=1}^{k} a_i n_i \mid k \in \mathbb{N}, a_1, \ldots, a_k \in \mathbb{N}, n_1, \ldots, n_k \in A \right\},$$

which clearly is closed under addition and $0 \in \langle A \rangle$. Moreover, $\gcd(A) = 1$ if and only if $\langle A \rangle$ is a numerical semigroup. Usually, A is known as a system of generators of $\langle A \rangle$.

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For a semigroup S, A is a minimal system of generators if $\langle A\rangle = S$ and no proper subset of A generates S.

From this definition it is easy to obtain the following properties (see [1] or [9]) of any numerical semigroup S.

(1) The set $\mathbb{N}\setminus S$ has finitely many elements; its maximum is known as the *Frobenius number* of S and it is denoted by $\mathrm{g}(S)$. If $S=\mathbb{N}$, then $\mathrm{g}(S)=-1$.
(2) S has a unique minimal system of generators $\{n_1<\cdots<n_p\}$. The element n_1 is the least positive integer belonging to S and it is denoted by $\mathrm{m}(S)$, the *multiplicity* of S.

The elements of $\mathbb{N}\setminus S$, denoted by $\mathscr{H}(S)$, are called *gaps* of S. Notice that $\mathscr{H}(S)$ clearly determines S. Moreover, if $x\in\mathscr{H}(S)$ and y is a nonnegative integer such that $y|x$ (to be read as "y divides x"), then y also belongs to $\mathscr{H}(S)$. Thus $\mathscr{H}(S)$, and consequently S, are determined by the elements $x\in\mathscr{H}(S)$ such that $kx\notin\mathscr{H}(S)$ for all $k\geqslant 2$. For this reason these elements will be called fundamental gaps of S.

In the literature the most common way to present a numerical semigroup is by one of its sets of generators (usually by its minimal system of generators). The problem of such a representation is that of membership, that is, given $n\in\mathbb{N}$, it can be hard to decide whether or not n belongs to S. Associated to this problem arises the Frobenius problem, that is, determine the largest nonnegative integer that cannot be expressed as a linear combination (with coefficients in $\mathbb{N}$) of a given finite set of nonnegative integers, or in other words, find the Frobenius number of a numerical semigroup given by one of its systems of generators. There are many papers dealing with this problem and no general formula is known (see for instance [2–5,11]).

In this paper we propose an alternative way of representing a numerical semigroup through its set of fundamental gaps. If S is a numerical semigroup and $x_1,\ldots,x_r$ are its fundamental gaps, then $S=\mathbb{N}\setminus\mathscr{D}(x_1,\ldots,x_r)$, where

$$\mathscr{D}(x_1,\ldots,x_r)=\{x\in\mathbb{N} : x\,|\,x_i \text{ for some } i\in\{1,\ldots,r\}\}.$$

In this way, the membership problem becomes trivial. Nevertheless the problem of deciding whether or not $\mathbb{N}\setminus\mathscr{D}(x_1,\ldots,x_r)$ is a numerical semigroup with $\{x_1,\ldots,x_r\}$ an arbitrary subset of $\mathbb{N}\setminus\{0\}$ is in principle not easy to solve.

The contents of this work are organized as follows. In Section 2 we describe those numerical semigroups having only a fundamental gap. In Section 3 we prove that $\mathbb{N}\setminus\mathscr{D}(x_1,\ldots,x_r)$ is a numerical semigroup if and only if the set of all numerical semigroups having no intersection with $\{x_1,\ldots,x_r\}$ has a maximum with respect to set inclusion. We study also some properties of the fundamental gaps of a numerical semigroup. The aim of Sections 4 and 5 is to give lower and upper bounds for the number of fundamental gaps of a numerical semigroup in terms of its Frobenius number. Finally in Section 6, taking advantage of the properties of fundamental gaps, we describe a practical method to construct the set of numerical semigroups containing a given numerical semigroup. An easy modification of this method will allow us to compute all numerical semigroups containing a given numerical semigroup with fixed Frobenius number. In particular this yields a procedure for calculating the set of all numerical semigroups with fixed Frobenius number and that of all symmetric and pseudo-symmetric numerical semigroups with fixed Frobenius number. We give a table with the total amount of numerical semigroups with Frobenius number g, with $g \in \{1,\dots,39\}$. A similar table is presented for irreducible numerical semigroups (symmetric and pseudo-symmetric numerical semigroups). In [10], a method for computing the set of numerical semigroups with fixed Frobenius number is presented. However the method presented there relies on finding the set of nonnegative integer solutions of Diophantine linear equations, and nowadays these procedures are much time consuming than the ones presented in this paper.

2. Numerical semigroups determined by its Frobenius number

Given an integer x denote by $\mathscr{S}(x)$ the set of all numerical semigroups not containing x. We say that a numerical semigroup S is *determined by its Frobenius number* if $S = \max(\mathscr{S}(\mathrm{g}(S)))$ (this maximum is taken with respect to set inclusion). In this section we give a characterization of those numerical semigroups determined by their Frobenius numbers. This characterization can be also deduced from the results presented later in the paper, but we include it here since it serves as a starting point and motivation for the rest of the sections of this work.

Lemma 1. *If g is a positive integer, then the following conditions are equivalent*:

(1) *$\mathscr{S}(g)$ has a maximum*,
(2) *$\mathbb{N}\backslash\mathscr{D}(g)$ is a numerical semigroup*.

Proof. (1) *implies* (2). Let $a, b \in \mathbb{N}\backslash\mathscr{D}(g)$ and let $S = \max(\mathscr{S}(g))$. As $a \nmid g$ and $b \nmid g$, we have that both $S_1 = \langle a, g+1, \rightarrow\rangle$ and $S_2 = \langle b, g+1, \rightarrow\rangle$ are elements of $\mathscr{S}(g)$ (the symbol $\rightarrow$ is used here to express that the elements greater than $g+1$ are also in the set). Hence $S_1 \cup S_2 \subseteq S$ and $a+b \in S$. Since $g \notin S$, we deduce that $a+b \nmid g$, which implies that $a+b \in \mathbb{N}\backslash\mathscr{D}(g)$.

(2) *implies* (1). We prove that $S = \mathbb{N}\backslash\mathscr{D}(g) = \max(\mathscr{S}(g))$. Let $\bar{S} \in \mathscr{S}(g)$ and take $x \in \bar{S}$. Then $g \notin \bar{S}$ and thus $x \nmid g$. It follows that $x \in S$. Hence $\bar{S} \subseteq S$. $\square$

Lemma 2. *Let g be an integer greater than six. Then there exist $a, b \notin \mathscr{D}(g)$ such that $g = a + b$.*

Proof. For g odd, it suffices to take $a = 2$, $b = g-2$. Now assume that g is even and let m be the least positive integer not dividing g. If $m > g/2$, then both $g/2-1$ and $g/2$ divide g. Thus $g/2(g/2-1) | g$, since $g/2(g/2-1)$ is the least common multiple of $g/2-1$ and $g/2$. But this leads to $g/2(g/2-1) \leqslant g$, or equivalently, to $g \leqslant 6$, in contradiction to $g > 6$. Therefore $m < g/2$ and $g-m > g/2$, which in particular implies that $g-m \nmid g$. Taking $a = m$ and $b = g-m$, we conclude the proof. $\square$

With this result it is easy to prove the following consequence.

Corollary 3. *Let g be a positive integer. Then $\mathbb{N}\backslash\mathscr{D}(g)$ is a semigroup if and only if $g\in\{1,2,3,4,6\}$.*

Using now Lemma 1 and that $\mathrm{g}(\mathbb{N})=-1$, we get a characterization for those numerical semigroups determined by its Frobenius number.

Corollary 4. *A numerical semigroup is determined by its Frobenius number if and only if $\mathrm{g}(S)\in\{-1,1,2,3,4,6\}$.*

3. Fundamental gaps

Let S be a numerical semigroup. A subset X of $\mathbb{N}$ *H-determines* S if S is the maximum (with respect to set inclusion) numerical semigroup such that $X\subseteq\mathscr{H}(S)$. In view of this definition, if X H-determines S, then X has finitely many elements and $0\notin X$. Note that a numerical semigroup S is determined by its Frobenius number if $X=\{\mathrm{g}(S)\}$ H-determines it. The following result is the natural generalization of Lemma 1.

Proposition 5. *Let X be a finite subset of $\mathbb{N}\backslash\{0\}$. The following conditions are equivalent*:

(1) *the set X H-determines a numerical semigroup*,
(2) *$\mathbb{N}\backslash\mathscr{D}(X)$ is a numerical semigroup.*

If these conditions hold, then X H-determines $\mathbb{N}\backslash\mathscr{D}(X)$.

Proof. (1) *implies* (2). Let S be the numerical semigroup H-determined by X. Since $X\subseteq\mathscr{H}(S)$, we have that $\mathscr{D}(X)\subseteq\mathscr{H}(S)$ and thus $S\subseteq\mathbb{N}\backslash\mathscr{D}(X)$. Take $a\in\mathbb{N}\backslash\mathscr{D}(X)$. Then $S'=\langle a,\max(X)+1,\rightarrow\rangle$ is a numerical semigroup such that $X\subseteq\mathscr{H}(S')$ and from the definition of S, we have that $S'\subseteq S$. Hence $a\in S$ and this proves that $\mathbb{N}\backslash\mathscr{D}(X)=S$. In particular we obtain that $\mathbb{N}\backslash\mathscr{D}(X)$ is a numerical semigroup.

(2) *implies* (1). Obviously, $\mathbb{N}\backslash\mathscr{D}(X)$ is the numerical semigroup H-determined by X. $\square$

Proposition 6. *Let S be a numerical semigroup and let X be a subset of $\mathscr{H}(S)$. The following conditions are equivalent*:

(1) *X H-determines S*,
(2) *for every $a\in\mathbb{N}$, if $a\in\mathscr{H}(S)$ and $\{2a,3a\}\subset S$, then $a\in X$.*

Proof. If X H-determines S, then by applying Proposition 5, we have that $S=\mathbb{N}\backslash\mathscr{D}(X)$ and consequently $\mathscr{H}(S)=\mathscr{D}(X)$. If a is an element of $\mathscr{H}(S)$, then there exists $x\in X$ such that $a\,|\,x$ and thus $ka=x$ for some $k\in\mathbb{N}$. If in addition we assume that $\{2a,3a\}\subset S$, we have that $la\in S$ for every positive integer l greater than one. Therefore $k=1$ and $a=x\in X$.

For the other implication, in view of Proposition 5, it suffices to prove that $S = \mathbb{N}\backslash\mathscr{D}(X)$. By hypothesis $X \subseteq \mathscr{H}(S)$ and thus $\mathscr{D}(X) \subseteq \mathscr{H}(S)$. Hence $S \subseteq \mathbb{N}\backslash\mathscr{D}(X)$. If a is a nonnegative integer not belonging to S, then $a \in \mathscr{H}(S)$. Let $k = \max\{n \in \mathbb{N} : na \in \mathscr{H}(S)\}$ ($\mathscr{H}(S)$ is finite, $0 \notin \mathscr{H}(S)$ and thus $k \in \mathbb{N}$). It follows that $ka \in \mathscr{H}(S)$ and $\{2ka, 3ka\} \subset S$, whence $ka \in X$ and consequently $a \in \mathscr{D}(X)$. This proves $S = \mathbb{N}\backslash\mathscr{D}(X)$. □

This last result motivates the following definition. A gap x of a numerical semigroup S is *fundamental* if $\{2x, 3x\} \subset S$ (or equivalently, $kx \in S$ for all $k > 1$). We denote by $\mathscr{F}\mathscr{H}(S)$ the set of fundamental gaps of S. With this new notation we can reformulate Proposition 6.

Corollary 7. *Let S be a numerical semigroup and let X be a subset of $\mathscr{H}(S)$. Then X H-determines S if and only if $\mathscr{F}\mathscr{H}(S) \subseteq X$.*

Hence for a numerical semigroup S, $\mathscr{F}\mathscr{H}(S)$ is just the smallest (with respect to set inclusion) subset of $\mathscr{H}(S)$ H-determining S. Note that if $x, y \in \mathscr{F}\mathscr{H}(S)$ and $x \neq y$, then $x \nmid y$.

Proposition 8. *Let X be a finite subset of $\mathbb{N}\backslash\{0\}$. The following conditions are equivalent*:

(1) *there exists a numerical semigroup S such that $\mathscr{F}\mathscr{H}(S) = X$*,
(2) *$\mathbb{N}\backslash\mathscr{D}(X)$ is a numerical semigroup and $x \nmid y$ for all $x, y \in X$ such that $x \neq y$.*

Proof. (1) *implies* (2). This implication has been already proved.

(2) *implies* (1). If $S = \mathbb{N}\backslash\mathscr{D}(X)$ is a numerical semigroup, then X H-determines it. Moreover, by applying Corollary 7 we get that $\mathscr{F}\mathscr{H}(S) \subseteq X$. From the hypothesis $x \nmid y$ for all $x, y \in X$, $x \neq y$, it follows that for every $x \in X$, we have that $\{2x, 3x\} \cap \mathscr{D}(X)$ is empty. Hence $x \in \mathscr{H}(S)$ and $\{2x, 3x\} \subset S$. This means that $x \in \mathscr{F}\mathscr{H}(S)$. □

Theorem 9. *Let $\mathscr{S}$ be the set of all numerical semigroups and let $\mathscr{D}$ be the set of all finite subsets of $\mathbb{N}\backslash\{0\}$ fulfilling condition (2) of Proposition 8. Then the map*

$$\varphi : \mathscr{S} \to \mathscr{D}, \varphi(S) = \mathscr{F}\mathscr{H}(S)$$

is bijective. Moreover, its inverse is

$$\psi : \mathscr{D} \to \mathscr{S}, \psi(X) = \mathbb{N}\backslash\mathscr{D}(X).$$

Proof. The map φ is onto in view of Proposition 8 and into due to Proposition 5 and Corollary 7. The fact that $\psi = \varphi^{-1}$ follows from Proposition 5 and Corollary 7. □

4. A lower bound for the number of fundamental gaps of a numerical semigroup

Let S be a numerical semigroup. Our aim in this section is to give a lower bound for the cardinality of $\mathscr{FH}(S)$ in terms of $\mathrm{g}(S)$.

Lemma 10. *Let $a, b \in \mathbb{N}\backslash\{0\}$ be such that $a > 3b$ and take $i \in \{1, \ldots, b\}$. If $a - i < x \leqslant 2a$ and $a - i | x$, then $x = 2a - 2i$.*

Proof. From the hypothesis we deduce that

$$1 < \frac{x}{a-i} \leqslant \frac{2a}{a-i} \leqslant \frac{2a}{a-b} < 3.$$

Hence if $a - i | x$, then $x/(a - i) = 2$ and consequently $x = 2a - 2i$. □

Proposition 11. *Let $a, b \in \mathbb{N}\backslash\{0\}$ be such that $a > 3b$ and let $x_1, \ldots, x_t$ be positive integers such that $x_1 < \cdots < x_{t-1} < x_t = 2a$. If $S = \mathbb{N}\backslash\mathscr{D}(x_1, \ldots, x_t)$ is a semigroup, then $t > b$.*

Proof. For every $i \in \{1, \ldots, b\}$, we have that $2a = (a - i) + (a + i)$. If S is a semigroup, then either $a - i \notin S$ or $a + i \notin S$. Hence for every $i \in \{1, \ldots, b\}$, there exists $j_i \in \{1, \ldots, t\}$ such that $a + i \in \mathscr{D}(x_{j_i})$ or $a - i \in \mathscr{D}(x_{j_i})$. Using Lemma 10 and that $a + i > x_{j_i}/2$, we obtain that in this setting $x_{j_i} \in \{a + i, a - i, 2a - 2i\}$. Observe that

$$a - b < \cdots < a - 1 < a + 1 < \cdots < a + b < 2a - 2b < \cdots < 2a - 2.$$

Thus for $l \in \{1, \ldots, b\}$, $l \neq i$, we get that $\{a + l, a - l, 2a - 2l\} \cap \{a + i, a - i, 2a - 2i\}$ is empty. Therefore $x_{j_i} \neq x_{j_l}$. Hence the set $\{x_1, \ldots, x_{t-1}\}$ has at least b elements and consequently $t > b$. □

Proposition 12. *Let $a, b \in \mathbb{N}\backslash\{0\}$ be such that $a > 3b$ and let $x_1, \ldots, x_t$ be positive integers such that $x_1 < \cdots < x_{t-1} < x_t = 2a - 1$. If $S = \mathbb{N}\backslash\mathscr{D}(x_1, \ldots, x_t)$ is a semigroup, then $t > b$.*

Proof. The proof is analogous to that of Proposition 11, but using for $i \in \{1, \ldots, b\}$, the expressions $2a - 1 = (a - i) + (a + i - 1)$ and the sets $\{a + i - 1, a - i, 2a - 2i\}$ instead. □

For a given real number x, denote by $\lceil x \rceil$ the smallest integer greater than or equal to x.

Theorem 13. *Let $X = \{x_1 < \cdots < x_t\}$ be a subset of $\mathbb{N}\backslash\{0\}$. If $\mathbb{N}\backslash\mathscr{D}(X)$ is a semigroup, then $t \geqslant \lceil x_t/6 \rceil$.*

Proof. We distinguish two cases.

(1) If $x_t = 2a$ for some positive integer a, then by Proposition 11, we know that $t > b$, for any b such that $3b < a$, or equivalently, $b < x_t/6$. Thus $t \geqslant \lceil x_t/6 \rceil$.

(2) If $x_t = 2a - 1$ for some positive integer a, then by using now Proposition 12, we obtain that $t > b$, if $3b < a$, that is to say, $6b \leqslant x_t$, or in other words, $b \leqslant x_t/6$. Hence $t \geqslant \lceil x_t/6 \rceil$. □

Note that if $S = \mathbb{N}\backslash\mathscr{D}(X)$, with $X = \{x_1 < \cdots < x_t\}$, is a numerical semigroup, then $\mathrm{g}(S) = x_t$. As a consequence of the last theorem and this fact, we obtain the following result.

Corollary 14. *Let S be a numerical semigroup. Then*

$$\#\mathscr{F}\mathscr{H}(S) \geqslant \left\lceil \frac{\mathrm{g}(S)}{6} \right\rceil .$$

Remark 15. By Corollary 14, if S is a semigroup, then $g(S) \leqslant 6 \times \#\mathscr{F}\mathscr{H}(S)$. To obtain all the semigroups with $\#\mathscr{F}\mathscr{H}(S)$ equal to t we compute all the sequences $\{x_1 < \cdots < x_t\}$ verifying $1 < x_1,\ x_t \leqslant 6 \times t$ and such that $x_i \nmid x_j$ if $i < j$. Among these sequences we consider those verifying that $\mathbb{N}\backslash\mathscr{D}(x_1, \ldots, x_t)$ is a semigroup. The following results have been obtained using this method:

- If $\#\mathscr{F}\mathscr{H}(S) = 1$, then $g(S) \leqslant 6 \times 1$. The sequences are formed by one element and this element only has to satisfy that $0 < x_1 \leqslant 6$. We already know by Section 2 that all the semigroups fulfilling that $\#\mathscr{F}\mathscr{H}(S) = 1$ are

$$\mathbb{N}\backslash\mathscr{D}(1), \mathbb{N}\backslash\mathscr{D}(2), \mathbb{N}\backslash\mathscr{D}(3), \mathbb{N}\backslash\mathscr{D}(4), \mathbb{N}\backslash\mathscr{D}(6);$$

  five semigroups in total.
- If $\#\mathscr{F}\mathscr{H}(S) = 2$, then $g(S) \leqslant 6 \times 2$. Now we consider sequences of two elements $1 < x_1 < x_2 \leqslant 12$ fulfilling that $x_1 \nmid x_2$. We obtained that the semigroups with $\#\mathscr{F}\mathscr{H}(S) = 2$ are

$$\mathbb{N}\backslash\mathscr{D}(2,3), \mathbb{N}\backslash\mathscr{D}(3,4), \mathbb{N}\backslash\mathscr{D}(2,5), \mathbb{N}\backslash\mathscr{D}(3,5),$$

$$\mathbb{N}\backslash\mathscr{D}(4,5), \mathbb{N}\backslash\mathscr{D}(4,6), \mathbb{N}\backslash\mathscr{D}(5,6), \mathbb{N}\backslash\mathscr{D}(4,7), \mathbb{N}\backslash\mathscr{D}(6,7),$$

$$\mathbb{N}\backslash\mathscr{D}(3,8), \mathbb{N}\backslash\mathscr{D}(5,8), \mathbb{N}\backslash\mathscr{D}(6,8), \mathbb{N}\backslash\mathscr{D}(4,9), \mathbb{N}\backslash\mathscr{D}(8,9),$$

$$\mathbb{N}\backslash\mathscr{D}(6,10), \mathbb{N}\backslash\mathscr{D}(5,12), \mathbb{N}\backslash\mathscr{D}(7,12), \mathbb{N}\backslash\mathscr{D}(10,12).$$

  This makes eighteen semigroups with $\#\mathscr{F}\mathscr{H}(S) = 2$.

Finally, doing the same for $\#\mathscr{F}\mathscr{H}(S) = 3$ and $\#\mathscr{F}\mathscr{H}(S) = 4$ we obtained that the number of these semigroups are 63 and 224, respectively.

Remark 16. There are nonnegative integers g for which there is no numerical semigroup S with $\mathrm{g}(S) = g$ and $\#\mathscr{F}\mathscr{H}(S) = \lceil g/6 \rceil$. Actually, if S is a semigroup such that $\mathrm{g}(S) = g = 5 + 6k$ for some nonnegative integer k, then $\#\mathscr{F}\mathscr{H}(S) > \lceil g/6 \rceil = k + 1$. This can be shown by taking into account that

$$g = (2 + 3k) + (3 + 3k) = \cdots = (2 + 2k) + (3 + 4k)$$

and that

$$2+2k<\cdots<2+3k<3+3k<\cdots<3+4k<4+4k<\cdots<4+6k,$$

and then proceed as in Proposition 11.

5. An upper bound for the number of fundamental gaps of a numerical semigroup

In this section we give an upper bound for the number of fundamental gaps of a numerical semigroup. The trivial example of semigroups reaching this bound is described in the following result.

Lemma 17. *Let a be a positive integer and let S be the numerical semigroup $S=\langle a+1, a+2, \rightarrow\rangle$. Then $\mathrm{g}(S)=a$ and $\#\mathscr{F}\mathscr{H}(S)=\lceil a/2\rceil$.*

Proof. From the definition of fundamental gap, it follows that $x\in\mathscr{F}\mathscr{H}(S)$ if and only if $2x\geqslant a+1$, or equivalently, $x\geqslant\lceil(a+1)/2\rceil$. Hence

$$\mathscr{F}\mathscr{H}(S)=\left\{\left\lceil\frac{a+1}{2}\right\rceil,\left\lceil\frac{a+1}{2}\right\rceil+1,\ldots,a\right\}$$

and $\#\mathscr{F}\mathscr{H}(S)=\lceil a/2\rceil$. □

If we remove the multiplicity of a numerical semigroup, then the fundamental gaps of the resulting numerical semigroup are as shown in the next two lemmas.

Lemma 18. *Let S be a numerical semigroup and assume that $m=\mathrm{m}(S)$. Then $S\backslash\{m\}$ is a numerical semigroup and $\mathscr{F}\mathscr{H}(S\backslash\{m\})\subseteq\mathscr{F}\mathscr{H}(S)\cup\{m\}$. Furthermore, $m\in\mathscr{F}\mathscr{H}(S\backslash\{m\})$.*

Proof. Clearly $S\backslash\{m\}$ is a numerical semigroup. If $x\in\mathscr{F}\mathscr{H}(S\backslash\{m\})$, then $\{2x,3x\}\in S\backslash\{m\}$ and $x\notin S\backslash\{m\}$. Hence $x=m$ or $x\in\mathscr{F}\mathscr{H}(S)$. As $m\notin S\backslash\{m\}$ and $\{2m,3m\}\subset S\backslash\{m\}$, we conclude that $m\in\mathscr{F}\mathscr{H}(S\backslash\{m\})$. □

Lemma 19. *Let S be a numerical semigroup with $m=\mathrm{m}(S)$. Then $\mathscr{F}\mathscr{H}(S)\subseteq\mathscr{F}\mathscr{H}(S\backslash\{m\})\cup\{m/2\}$. Moreover, $m/2\in\mathscr{F}\mathscr{H}(S)$ if and only if $3m/2\in S$.*

Proof. If $x\in\mathscr{F}\mathscr{H}(S)$, then $x\notin S$ and $\{2x,3x\}\subset S$. Hence either $x\in\mathscr{F}\mathscr{H}(S\backslash\{m\})$ or $m\in\{2x,3x\}$. Note that m cannot be $3x$, since if this were the case, $2x$ would be an element in S smaller than m, the multiplicity of S. Therefore either $x\in\mathscr{F}\mathscr{H}(S\backslash\{m\})$ or $m=2x$, that is, $x\in\mathscr{F}\mathscr{H}(S\backslash\{m\})\cup\{m/2\}$.

Now assume that $m/2\in\mathscr{F}\mathscr{H}(S)$. This occurs if and only if $\{2m/2,3m/2\}\subset S$ and this is equivalent to $3m/2\in S$. □

Gathering Lemmas 18 and 19 together, we obtain the following result.

Corollary 20. *Let S be a numerical semigroup and let m be its multiplicity. Then*

$$\#\mathscr{F}\mathscr{H}(S)=\begin{cases}\#\mathscr{F}\mathscr{H}(S\backslash\{m\}) & if\ \frac{3}{2}m\in S,\\ \#\mathscr{F}\mathscr{H}(S\backslash\{m\})-1 & if\ \frac{3}{2}m\notin S.\end{cases}$$

Given a numerical semigroup S, define recursively S_i as follows:

- $S_0 = S$,
- $S_{n+1} = S_n\backslash\{\mathrm{m}(S_n)\}$.

Clearly, there exists $k\in\mathbb{N}$ such that $S_k = \langle \mathrm{g}(S)+1,\rightarrow\rangle$. Hence

$$\langle \mathrm{g}(S)+1,\rightarrow\rangle = S_k \subseteq \cdots \subseteq S_1 \subseteq S_0 = S$$

and by Corollary 20 and Lemma 17 we have that

$$\#\mathscr{F}\mathscr{H}(S)=\#\mathscr{F}\mathscr{H}(S_0)\leqslant\cdots\leqslant\#\mathscr{F}\mathscr{H}(S_k)=\left\lceil\frac{\mathrm{g}(S)}{2}\right\rceil.$$

With this, it is easy to obtain the following consequence.

Theorem 21. *Let S be a numerical semigroup. Then*

$$\#\mathscr{F}\mathscr{H}(S)\leqslant\left\lceil\frac{\mathrm{g}(S)}{2}\right\rceil.$$

Furthermore, the following conditions are equivalent:

1. $\#\mathscr{F}\mathscr{H}(S)=\lceil \mathrm{g}(S)/2\rceil$,
2. *for every* $x\in S$ *such that* $x<\mathrm{g}(S)$, *we have that* $3x/2\in S$,
3. *if* a *is a minimal generator of* S *and* $a<\mathrm{g}(S)$, *then* $3a/2\in S$.

6. The over-semigroups of a numerical semigroup

In this section we generalize the process of removing the multiplicity of a numerical semigroup in order to obtain a new numerical semigroup. We also take advantage of the inverse process: add a new element to a numerical semigroup so that the resulting set is again a numerical semigroup. With all this, we give a method for building recursively the set of all numerical semigroups that contain a given semigroup. Note that for a given numerical semigroup S, its set of gaps $\mathscr{H}(S)$ is finite and thus the set of numerical semigroups properly containing S is finite.

For a given numerical semigroup S and an element s of S, the set $S\backslash\{s\}$ is again a semigroup provided that $s\neq a+b$ with $a,b\in S\backslash\{0\}$, or in other words, $S\backslash\{s\}$ is a semigroup if and only if s is a minimal generator of S. If we remove s from S, then s becomes a fundamental gap of $S\backslash\{s\}$, and eventually, some fundamental gaps of S

are no longer fundamental gaps of $S\backslash\{s\}$. Those gaps are precisely the elements x in $\mathscr{F}\mathscr{H}(S)$ such that $x \mid s$. In this way, we get the following result.

Proposition 22. *Let S be a numerical semigroup and let s be an element of S. Then $S\backslash\{s\}$ is a numerical semigroup if and only if s is a minimal generator of S. Moreover, if $S\backslash\{s\}$ is a numerical semigroup, then*

$$\mathscr{F}\mathscr{H}(S\backslash\{s\}) = (\mathscr{F}\mathscr{H}(S) \cup \{s\})\backslash\{x \in \mathscr{F}\mathscr{H}(S) : x \mid s\}.$$

Remark 23. The multiplicity m of a numerical semigroup S belongs to the set of minimal generators of S. Thus $S\backslash\{m\}$ is a numerical semigroup as we pointed out in Lemma 18. Moreover, if $x \in \mathscr{F}\mathscr{H}(S)$ is such that $x \mid m$, by the definition of fundamental gap and since $m = \min(S\backslash\{0\})$, we get that $2x = m$. In particular this implies that at most there is an element in $\mathscr{F}\mathscr{H}(S)$ dividing m and if there is one, then m is even (see Lemmas 18 and 19).

The inverse process is to add a new element to a given numerical semigroup S. Assume that the element we want to add is $y \in \mathscr{H}(S)$. Obviously $S \cup \{y\}$ is a numerical semigroup if and only if $ky \in S$ for all $k > 1$ and $y + s \in S$ for all $s \in S\backslash\{0\}$. Thus y must be a fundamental gap of S. The second condition imposed on y states that y is what is known in the literature as a pseudo-Frobenius number of S (see for instance [7]).

Given two nonnegative integers x_1, x_2, we write

$$x_1 \leqslant_S x_2 \quad \text{if } x_2 - x_1 \in S.$$

The reader can check that the binary relation $\leqslant_S$ is an order relation (it is reflexive, transitive and antisymmetric).

Proposition 24. *Let S be a numerical semigroup and let $y \in \mathscr{H}(S)$. Then $S \cup \{y\}$ is a numerical semigroup if and only if $y \in \mathrm{Maximals}_{\leqslant_S}(\mathscr{F}\mathscr{H}(S))$. Moreover, if $S \cup \{y\}$ is a numerical semigroup and $p_1, \ldots, p_n$ are the primes dividing y, then*

$$\mathscr{F}\mathscr{H}(S \cup \{y\}) = (\mathscr{F}\mathscr{H}(S)\backslash\{y\}) \cup \left\{ \frac{y}{p_i} : \frac{y}{p_i} \notin \mathscr{D}(\mathscr{F}\mathscr{H}(S)\backslash\{y\}) \right\}.$$

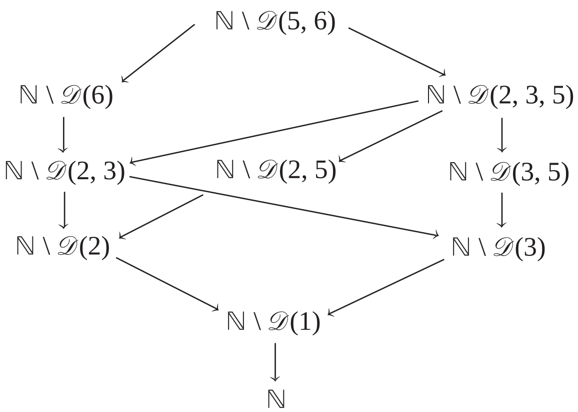
Proof. Assume that $S \cup \{y\}$ is a numerical semigroup. Then as we mentioned above, $y \in \mathscr{F}\mathscr{H}(S)$. Note that $y + s \notin \mathscr{F}\mathscr{H}(S)$ for every $s \in S\backslash\{0\}$ (otherwise $y + s \notin S$). Thus $y \in \mathrm{Maximals}_{\leqslant_S}(\mathscr{F}\mathscr{H}(S))$.

Now suppose that $y \in \mathrm{Maximals}_{\leqslant_S}(\mathscr{F}\mathscr{H}(S))$. Since $y \in \mathscr{F}\mathscr{H}(S)$, then $ky \in S$ for all $k > 1$. If $y + s \notin S$ for some $s \in S\backslash\{0\}$, then $y + s \mid x$ with $x \in \mathscr{F}\mathscr{H}(S)$. It follows that $k(y+s) = x$ for some $k \in \mathbb{N}$. Since $ky \in S$ for $k > 1$ and ks is obviously in S, we get that $k(y+s) \in S$ for $k > 1$. This forces k to be one, which means that $y + s = x$, in contradiction with $y \in \mathrm{Maximals}_{\leqslant_S}(\mathscr{FH}(S))$. This proves that $S \cup \{y\}$ is a semigroup. Since $\mathbb{N} \backslash (S \cup \{y\})$ is finite and $0 \in S \cup \{y\}$, the semigroup $S \cup \{y\}$ is a numerical semigroup.

Finally, if $S \cup \{y\}$ is a numerical semigroup, then we must remove y from $\mathscr{FH}(S \cup \{y\})$, but we do not have to remove its proper divisors, and this is achieved by adjoining $\{y/p_i : y/p_i \notin \mathscr{D}(\mathscr{FH}(S) \backslash \{y\})\}$. $\square$

By using the results presented so far in this section, for a given semigroup S, we can construct the tree of all numerical semigroups containing S in the following way. Take as root S, and once a node T is constructed, its sons are $T \cup \{y\}$ where y ranges in $\mathrm{Maximals}_{\leqslant_T}(\mathscr{FH}(T))$. Clearly, we encounter a leaf whenever $T = \mathbb{N}$, because otherwise $\mathrm{g}(T)$ is always in $\mathrm{Maximals}_{\leqslant_T}(\mathscr{FH}(T))$.

Example 25. Let us consider the semigroup $S = \mathbb{N} \backslash \mathscr{D}(5,6)$. We construct its associated graph. We have that 5 is prime and $6 = 2 \cdot 3$, and both are maximals of $\{5,6\}$ with respect to $\leqslant_S$. Thus our semigroup has two children $\mathbb{N} \backslash \mathscr{D}(6)$ (by removing 5) and $\mathbb{N} \backslash \mathscr{D}(2,3,5)$ (from the decomposition of 6). Proceeding in this way, the resulting graph is



A numerical semigroup is *irreducible* if it cannot be expressed as the intersection of two numerical semigroups properly containing it. In [8], it is shown that S is irreducible if and only if S is maximal (with respect to set inclusion) in the set of all numerical semigroups with Frobenius number $\mathrm{g}(S)$. From [4], it follows then that the set of irreducible numerical semigroups with odd Frobenius number is the set of symmetric numerical semigroups, and that of the numerical semigroups with even Frobenius number corresponds to the set of pseudo-symmetric numerical semigroups (see [1] for the definition of symmetric and pseudo-symmetric numerical semigroup). Using now Proposition 24 we can achieve a new characterization of symmetric and pseudo-symmetric numerical semigroups in terms of their sets of fundamental gaps.

Proposition 26. *Let S be a numerical semigroup, $S \neq \mathbb{N}$. The following conditions are equivalent*:

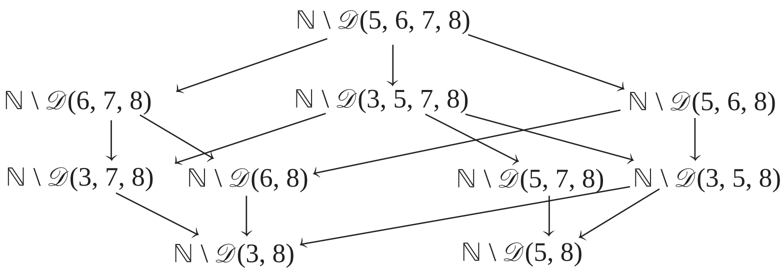
(1) *S is irreducible*,
(2) $\mathrm{Maximals}_{\leqslant_S}(\mathscr{F}\mathscr{H}(S)) = \{\mathrm{g}(S)\}$.

Proof. (1) *implies* (2). By Proposition 24, if there were $x, y \in \mathrm{Maximals}_{\leqslant_S}(\mathscr{F}\mathscr{H}(S))$, such that $x \neq y$, then $S \cup \{x\}$ and $S \cup \{y\}$ would be numerical semigroups. Clearly $S = (S \cup \{x\}) \cap (S \cup \{y\})$, contradicting that S is irreducible. Hence the cardinality of $\mathrm{Maximals}_{\leqslant_S}(\mathscr{F}\mathscr{H}(S))$ is one. Thus $\mathrm{Maximals}_{\leqslant_S}(\mathscr{F}\mathscr{H}(S)) = \{\mathrm{g}(S)\}$.

(2) *implies* (1). If S is not irreducible, then there exists $y \in \mathscr{H}(S)$, $y \neq \mathrm{g}(S)$, such that $S \cup \{y\}$ is a numerical semigroup (see [6, Lemma 3.2]). By Proposition 24, this implies that $y \in \mathrm{Maximals}_{\leqslant_S}(\mathscr{F}\mathscr{H}(S))$. □

If in Proposition 22 we impose s to be less than $\mathrm{g}(S)$, then $\mathrm{g}(S) = \mathrm{g}(S \setminus \{s\})$. Observe that such an s exists unless S is of the form $\langle a, a+1, \rightarrow \rangle$ for some positive integer a (it suffices to take $s = \mathrm{m}(S)$). As for Proposition 24, taking $y \neq \mathrm{g}(S)$, we have that $\mathrm{g}(S) = \mathrm{g}(S \cup \{y\})$. Therefore Propositions 22 and 24 can be used to compute all numerical semigroups with fixed Frobenius number containing a given numerical semigroup. From Proposition 26 it is deduced that a leaf in the corresponding tree is just an irreducible semigroup. If we start from $\langle g+1, g+2, \rightarrow \rangle$, then the process yields the set of all numerical semigroups with Frobenius number g.

Example 27. We study the set of numerical semigroups with Frobenius number eight. They all contain the semigroup $\{0, 9, 10, \rightarrow\} = \mathbb{N} \setminus \mathscr{D}(5, 6, 7, 8)$. Using the above reasoning we can obtain all of them from this semigroup. Given one of these semigroups S, what we do is to use Proposition 24 to obtain all the semigroups containing S and having the same Frobenius number. From the above semigroup we get the following graph



Hence there are 10 semigroups with Frobenius number eight and two of them are irreducible. Repeating this method for a given number g and using as starting point the semigroup $\{0, g+1, g+2, \rightarrow\} = \mathbb{N} \setminus \mathscr{D}(\lceil g+1/2 \rceil, \ldots, g)$ we can construct the associated graph to this semigroup which will contain all the semigroups with Frobenius number equal to g. Performing this computations we obtain the following table ($\mathrm{ns}(g)$ denotes the number of numerical semigroups with Frobenius number g, nis(g) the number of those that are irreducible).

g	ns(g)	nis(g)	g	ns(g)	nis(g)	g	ns(g)	nis(g)
1	1	1	14	103	6	27	16,132	45
2	1	1	15	200	7	28	16,267	37
3	2	1	16	205	7	29	34,903	83
4	2	1	17	465	15	30	31,822	36
5	5	2	18	405	7	31	70,854	109
6	4	1	19	961	20	32	68,681	70
7	11	3	20	900	11	33	137,391	101
8	10	2	21	1828	18	34	140,661	106
9	21	3	22	1913	20	35	292,081	174
10	22	3	23	4096	36	36	270,258	77
11	51	6	24	3578	14	37	591,443	246
12	40	2	25	8273	44	38	582,453	182
13	106	8	26	8175	35	39	1,156,012	227

The implementation of the algorithms are available in Haskell and in C. The source files can be obtained under e-mail request to any of the authors.

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