MULTIPLIER IDEALS IN TWO-DIMENSIONAL LOCAL RINGS WITH RATIONAL SINGULARITIES

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Abstract. The aim of this paper is to study jumping numbers and multiplier ideals of any ideal in a two-dimensional local ring with a rational singularity. In particular we reveal which information encoded in a multiplier ideal determines the next jumping number. This leads to an algorithm to compute sequentially the jumping numbers and the whole chain of multiplier ideals in any desired range. As a consequence of our method we develop the notion of *jumping divisor* that allows to describe the jump between two consecutive multiplier ideals. In particular we find a unique minimal jumping divisor that is studied extensively.

1. Introduction

Let X be a complex algebraic variety with mild singularities and $\mathcal{O}_{X,O}$ the local ring of a point $O \in X$. To any ideal $\mathfrak{a} \subseteq \mathcal{O}_{X,O}$ one may associate a family of *multiplier ideals* $\mathcal{J}(\mathfrak{a}^\lambda)$ parametrized by positive rational numbers $\lambda \in \mathbb{Q}_{>0}$. Indeed, they form a nested sequence of ideals

$$\mathcal{O}_{X,O} \supsetneq \mathcal{J}(\mathfrak{a}^{\lambda_1}) \supsetneq \mathcal{J}(\mathfrak{a}^{\lambda_2}) \supsetneq \dots \supsetneq \mathcal{J}(\mathfrak{a}^{\lambda_i}) \supsetneq \dots$$

and the rational numbers $0 < \lambda_1 < \lambda_2 < \cdots$ where the multiplier ideals change are called *jumping numbers*. The first jumping number λ_1 is also known as the *log-canonical threshold*. Multiplier ideals and their associated jumping numbers have proven to be a powerful tool to understand the geometry of singularities. They are defined using a log-resolution of the pair $(X, \mathfrak{a})$. In fact, smaller or more dense jumping numbers can be thought to correspond to "worse" singularities.

The aim of this paper is to present a new approach to the understanding of multiplier ideals and jumping numbers of any ideal $\mathfrak{a}$ in the local ring $\mathcal{O}_{X,O}$ of a complex surface X having at worst a rational singularity at O. This is a case, especially when X is smooth, that has received a lot of attention in recent years because of the interesting properties these invariants satisfy (see the works of Favre-Jonsson [8], [9], Lipman-Watanabe[20] or Tucker [24]). This is also one of the few cases where explicit computations have been done.

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For simple complete ideals or irreducible plane curves in a smooth surface, Järviletho [15] and Naie [21] provide a closed formula for the set of jumping numbers in terms of some invariants of the singularity, the *Zariski exponents*. To give a closed formula for any general ideal is beyond the scope of this work. A formula for the log-canonical threshold already becomes quite complicated as one may see in the papers of Kuwata [16] and Galindo-Hernando-Monserrat [11].

For the case of any ideal in a surface with a rational singularity we must refer to the work of Tucker [25] where he gives a simple algorithm to compute the set of jumping numbers. To such purpose, he developed the notion of divisors that *(critically) contribute*, building upon previous work of Smith-Thompson [23]. We may interpret jumping numbers as being parametrized by contributing divisors and critical divisors are more economic to detect since the complete ideals they define are very close to their corresponding multiplier ideal. The algorithm he proposes uses a characterization of critical divisors that allows them to be found and consequently allows the corresponding jumping numbers to be computed.

A similar strategy is used by Hyry-Järvilehto in [14] where they proved that jumping numbers are parameterized by more general complete ideals[1]. Moreover, they provide a combinatorial criterion to detect a suitable ideal and its corresponding jumping number.

The aim of this paper is to understand the whole change between a multiplier ideal to the next one, and to reveal what information encoded in a multiplier ideal determines the next jumping number. This is done in our main result Theorem 3.5 and it gives rise to an algorithm to compute the ordered sequence of multiplier ideals in any desired range of the real line. The algorithm avoids considering candidates and computes sequentially at each step a jumping number and its associated multiplier ideal. This new algorithm improves in efficiency the computation of jumping numbers when compared with Tucker's algorithm.

Perhaps the most important contribution of our method lies in finding a divisor, that we name the *minimal jumping divisor*, tightly related to the aforementioned algorithm, which enables one to obtain a multiplier ideal from the previous one, and vice versa. This jumping divisor is studied, in particular its geometric structure on the dual graph, and it is compared with the previously known critically contributing divisors.

The structure of this paper is as follows: In Section §2 we introduce the basics of the theory of multiplier ideals and some of the tools in the theory of singularities that we will need in the rest of the paper. We pay special attention to the equivalence between complete ideals and antinef divisors developed by Lipman in [19] since this is the way we will present multiplier ideals. In particular we provide a new method to compute the antinef closure of any given divisor, generalizing previous versions of Casas-Alvero [5] and Reguera [22].

[1] Contributing divisors describe complete ideals nested in between consecutive multiplier ideals. The ideals considered in [14] are not necessarily nested.

In Section §3 we present the main result of this paper in Theorem 3.5. It gives a generalization of a well-known formula for the log-canonical threshold and allows us to compute a jumping number from the data given by the preceding a multiplier ideal. This leads to the desired algorithm that computes sequentially the chain of multiplier ideals.

In Section §4 we develop the theory of *jumping divisors* that allows us to describe the whole jump between two consecutive multiplier ideals. Quite surprisingly, the algorithm we develop in Section §3 allows us to construct the unique *minimal jumping divisor* associated to every jumping number. It is minimal in the sense that no proper subdivisor gives the jump between consecutive multiplier ideals. Moreover, we prove in Theorem 4.11 that minimal jumping divisors are *generically* invariant with respect to log-resolutions of the ideal and they satisfy some nice geometric properties when viewed in the dual graph.

Finally, in Section §5 we present the theory of jumping divisors in a more general framework that we develop using the results of Hyry-Järvilehto [14] and their relation with the theory of contributing divisors of Tucker [25]. The main result of this section is the fact that, among all the contributing divisors associated to a jumping number that give the same ideal, there is a minimal one. For example, critical divisors are of this type. It turns out that these minimal contributing divisors are all contained in the minimal jumping divisor and inherit the same invariance property with respect to log-resolutions of the ideal.

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2. Preliminaries

Let X be a normal surface and O a point where X has at worst a rational singularity. That is, there exists a desingularization $\pi : X' \to X$ such that the stalk at O of the higher direct image $R^1\pi_*\mathcal{O}_{X'}$ is zero. This property is then satisfied for any desingularization. The theory of rational singularities was introduced by Artin in [4] and further developed by Lipman in [19]. Another reference that we will follow closely is [22].

Let $\mathfrak{a} \subseteq \mathcal{O}_X$ be an ideal sheaf. Denote $\mathfrak{m} = \mathfrak{m}_{X,O} \subseteq \mathcal{O}_{X,O}$ the maximal ideal of the local ring $\mathcal{O}_{X,O}$ at O. Throughout this work we will often consider the case where $\mathfrak{a} \subseteq \mathfrak{m}$ is an $\mathfrak{m}$-primary ideal, which can be identified with an ideal sheaf that equals $\mathcal{O}_X$ outside the point O (we will use both languages interchangeably, depending on the context). Recall that a *log-resolution* of the pair $(X, \mathfrak{a})$ (or of $\mathfrak{a}$, for short) is a proper birational morphism $\pi : X' \to X$ such that

i) X' is smooth,
ii) the preimage of $\mathfrak{a}$ is locally principal, that is, $\mathfrak{a} \cdot \mathcal{O}_{X'} = \mathcal{O}_{X'}(-F)$ for some effective Cartier divisor F, and

iii) $F+E$ is a divisor with simple normal crossings support where $E = Exc\,(\pi)$ is the exceptional locus.

From now on, consider a given log-resolution of $\mathfrak{a}$. Since the point O has (at worst) a rational singularity, the exceptional locus E is a tree of smooth rational curves $E_1, \dots, E_r$. Furthermore, the matrix of intersections $(E_i \cdot E_j)_{1 \leqslant i,j \leqslant r}$ is negative-definite.

Let $\mathrm{Div}(X')$ be the group of integral divisors in X', i.e. divisors of the form $D = \sum_i d_i E_i$ where the E_i are pairwise different (non necessarily exceptional) prime divisors and $d_i \in \mathbb{Z}$. Among them, we will consider divisors in the lattice $\Lambda := \mathbb{Z}E_1 \oplus \cdots \oplus \mathbb{Z}E_r$ of exceptional divisors and we will simply refer them as divisors with *exceptional support*. Any divisor $D \in \mathrm{Div}(X')$ has a decomposition $D = D_{\mathrm{exc}} + D_{\mathrm{aff}}$ into its *exceptional* and *affine* part[2] according to its support. Our main example is the divisor F such that $\mathfrak{a} \cdot \mathcal{O}_{X'} = \mathcal{O}_{X'}(-F)$. In this case we will denote its exceptional and affine part as

$$F_{\mathrm{exc}} = \sum_{i=1}^{r} e_i E_i \quad \text{and} \quad F_{\mathrm{aff}} = \sum_{i=r+1}^{s} e_i E_i$$

where, by definition, the e_i are non-negative integers. Whenever $\mathfrak{a}$ is an $\mathfrak{m}$-primary ideal, the divisor F is just supported on the exceptional locus. i.e. $F = F_{\mathrm{exc}}$.

Remark 2.1. Let $C : f = 0$ be a curve defined by an element $f \in \mathcal{O}_{X,O}$. The *total transform* of C is the pull-back $\overline{C} := \pi^* C$ and its *strict transform* C' is the closure of $\pi^{-1}(C - \{O\})$. The total transform has a presentation $\overline{C} = C' + \overline{C}_{\mathrm{exc}} = C' + \sum d_i E_i$ where the weights $v_i(f) := d_i$ are the *values* of the curve C at E_i. Recall that $f \in \mathfrak{a}$ whenever $C' + \overline{C}_{\mathrm{exc}} \geqslant F$ and f is *generic* in $\mathfrak{a}$ if $\overline{C}_{\mathrm{exc}} = F_{\mathrm{exc}}$ and $C' - F_{\mathrm{aff}}$ has no singular points.

More generally, we will also consider $\mathbb{Q}$-divisors in $\mathrm{Div}_{\mathbb{Q}}(X') = \mathrm{Div}(X') \otimes_{\mathbb{Z}} \mathbb{Q}$ or divisors in the $\mathbb{Q}$-vector space $\Lambda_{\mathbb{Q}} := \mathbb{Q}E_1 \oplus \cdots \oplus \mathbb{Q}E_r$. The main example will be the *relative canonical divisor* K_π. Indeed, the definition of K_π is quite subtle if O is singular, because at first sight one can only define a canonical divisor K_X of X as a Weil divisor. Since rational singularities are in particular $\mathbb{Q}$-factorial, there exists a positive integer m such that mK_X is Cartier, which can be pulled back to X' and allows to define $K_\pi = K_{X'} - \frac{1}{m}\pi^*(mK_X)$. Alternatively,

$$K_\pi = \sum_{i=1}^{r} k_i E_i$$

is supported on the exceptional locus E, and must satisfy

$$(K_\pi + E_i) \cdot E_i = \left(\sum_{j=1}^{r} k_j E_j \cdot E_i \right) + E_i^2 = -2 \tag{2.1}$$

[2]We follow the terminology of Lipman-Watanabe [20]

for every exceptional component E_i because of the adjunction formula. This property indeed characterizes K_π because the intersection form on E is negative-definite, and therefore the system defined by equations (2.1) has a unique solution $(k_1, \dots, k_r)$. However, the k_i are not necessarily integral, and can even be negative. In the case that $k_i > -1$ (resp. $k_i \geqslant -1$) for all E_i, one says that X has a *log-terminal singularity* (resp. *log-canonical singularity*) at O.

For any $\mathbb{Q}$-divisor $D = \sum_i d_i E_i \in \mathrm{Div}_{\mathbb{Q}}(X')$, we denote its *round-down* and *round-up* as

$$\lfloor D \rfloor = \sum_i \lfloor d_i \rfloor E_i \quad \text{and} \quad \lceil D \rceil = \sum_i \lceil d_i \rceil E_i.$$

The *fractional part* of D is then $\{D\} = D - \lfloor D \rfloor = \sum_i \{d_i\} E_i$. In the sequel we will denote the value of each component E_i of D as $v_{E_i}(D) := d_i$. If no confusion arises, we will simply denote the value of the components as $v_i(D) := v_{E_i}(D)$.

2.1. **Dual graph.** The combinatorics of the log-resolution of $\mathfrak{a}$ can be encoded using the so-called *dual graph*. This is a rooted tree where the vertices represent the irreducible components $E_i \leqslant F$ and two vertices are joined by an edge if the corresponding divisors intersect.

Given any component E_i, we will denote by $\mathrm{Adj}\,(E_i)$ the set of components E_j, $j \neq i$, sharing an edge with E_i, i.e. $E_i \cdot E_j = 1$, and by

$$a\,(E_i) = \#\mathrm{Adj}\,(E_i) = E_i \cdot \left(F^{\mathrm{red}} - E_i\right)$$

the number of such components which is the *valence* of the vertex representing E_i, where F^{red} denotes de reduced divisor with the same support as F. An *end* of the dual graph is nothing but a vertex with valence 1, i.e. a vertex E_i such that $a\,(E_i) = 1$. More generally, for any effective subdivisor $D = E_{i_1} + \cdots + E_{i_m} \leqslant F$ we define

$$\mathrm{Adj}_D\,(E_i) = \{E_j \leqslant D \mid E_i \cdot E_j = 1\}$$

and $a_D\,(E_i) = \#\mathrm{Adj}_D\,(E_i)$. We denote by $v_D = m$ (resp. a_D) the number of components of D (resp. the number of intersections between two components of D). Since the dual graph is a tree it is clear that

$$\sum_{E_i \leqslant D} a_D\,(E_i) = 2a_D$$

and that $v_D - a_D$ equals the number of connected components of D. An end of the subgraph associated to D is a vertex with valence 1 or 0. The later meaning that E_i is an isolated component of D.

For any exceptional component E_i, we define the *excess* (of $\mathfrak{a}$) at E_i as $\rho_i = -F \cdot E_i$. It can be interpreted as the number of branches of the strict transform of a curve defined by a generic element $f \in \mathfrak{a}$ that intersect the component E_i. Indeed, if its total transform is $\overline{C} = C' + F$ then $0 = \overline{C} \cdot E_i = C' \cdot E_i + F \cdot E_i = C' \cdot E_i - \rho_i$, which proves the claim.

There are two kinds of exceptional divisors that will play a special role:

- A component E_i of E is a *rupture* component if $a(E_i) \geqslant 3$, that is, it intersects at least three more components of E (different from E_i).
- We say that E_i is *dicritical* if $\rho_i > 0$. By [19], dicritical components correspond to *Rees valuations*.

We also mention that non-exceptional components also correspond to *Rees valuations*.

2.2. **Complete ideals and antinef divisors.** Given an effective $\mathbb{Q}$-divisor $D = \sum d_i E_i \in \mathrm{Div}_{\mathbb{Q}}(X')$ we may consider its associated (sheaf) ideal $\pi_* \mathcal{O}_{X'}(-D) := \pi_* \mathcal{O}_{X'}(-\lceil D \rceil)$. Its stalk at O is

$$I_D := \{ f \in \mathcal{O}_{X,O} \mid v_i(f) \geqslant \lceil d_i \rceil \text{ for all } E_i \leqslant D \}.$$

This is a complete ideal of $\mathcal{O}_{X,O}$ that is $\mathfrak{m}$-primary whenever D has exceptional support, i.e. $D \in \Lambda_{\mathbb{Q}}$. Any two divisors $D, D' \in \mathrm{Div}_{\mathbb{Q}}(X')$ defining the same complete ideal $\pi_* \mathcal{O}_{X'}(-D) = \pi_* \mathcal{O}_{X'}(-D')$ are called *equivalent divisors*.

In the equivalence class of a given divisor one may find a unique maximal representative. First, recall that an effective divisor with integral coefficients $D \in \mathrm{Div}(X')$ is called *antinef* if $-D \cdot E_i \geqslant 0$, for every exceptional prime divisor E_i. It is worth to point out that the affine part of $D = D_{\mathrm{exc}} + D_{\mathrm{aff}}$ satisfies $D_{\mathrm{aff}} \cdot E_i \geqslant 0$. Therefore D is antinef whenever $-D_{\mathrm{exc}} \cdot E_i \geqslant D_{\mathrm{aff}} \cdot E_i$.

In the work of Lipman (see [19, §18]) one may find the following correspondence that we will heavily use throughout this work.

Theorem 2.2. *There is a one to one correspondence between antinef divisors in* $\mathrm{Div}(X')$ *and complete ideals in* $\mathcal{O}_{X,O}$ *whose log-resolution is dominated by* π. *In particular, antinef divisors in* Λ *correspond to* $\mathfrak{m}$*-primary complete ideals whose log-resolution is dominated by* π.

In order to find the representative in the equivalence class of a given divisor $D \in \mathrm{Div}_{\mathbb{Q}}(X')$ we will consider its so-called *antinef closure* $\widetilde{D}$. The existence of such divisor is a consequence of the following results that can be found in [19, §18], but we also refer to [24] and [20] for more insight.

Lemma 2.3. *For any effective* $\mathbb{Q}$*-divisor* $D \in \mathrm{Div}_{\mathbb{Q}}(X')$ *there exists a unique minimal integral antinef divisor* $\widetilde{D} \in \mathrm{Div}(X')$ *satisfying* $\widetilde{D} \geqslant D$ *that is called the* antinef closure *of* D. *In particular, any antinef divisor* D' *such that* $D' \geqslant D$ *must satisfy* $D' \geqslant \widetilde{D} \geqslant D$

Proposition 2.4. *An effective* $\mathbb{Q}$*-divisor* $D \in \mathrm{Div}_{\mathbb{Q}}(X')$ *and its antinef closure* $\widetilde{D} \in \mathrm{Div}(X')$ *are equivalent, i.e.*

$$\pi_* \mathcal{O}_{X'}(-D) = \pi_* \mathcal{O}_{X'}(-\widetilde{D}).$$

One of the advantages of working with antinef divisors is that they provide the following characterization for the inclusion (or strict inclusion) of two given complete ideals.

Proposition 2.5. *Let* D_1, D_2 *be two antinef divisors in* $\mathrm{Div}(X')$. *Then:*

i) $\pi_* \mathcal{O}_{X'}(-D_1) \supseteq \pi_* \mathcal{O}_{X'}(-D_2)$ *if and only if* $D_1 \leqslant D_2$.

ii) $\pi_*\mathcal{O}_{X'}(-D_1) \supsetneq \pi_*\mathcal{O}_{X'}(-D_2)$ *if and only if* $D_1 < D_2$.

For non-antinef divisors we can only claim the following implication:

Proposition 2.6. *Let* D_1, D_2 *be two divisors in* $\mathrm{Div}_{\mathbb{Q}}(X')$ *such that* $D_1 \leqslant D_2$. *Then:*

i) $\pi_*\mathcal{O}_{X'}(-D_1) \supseteq \pi_*\mathcal{O}_{X'}(-D_2)$.

ii) $\widetilde{D_1} \leqslant \widetilde{D_2}$.

The converses to these results are no longer true.

In general, the divisors that will be considered in this work are not antinef. In order to compute their antinef closure we will use an inductive procedure called *unloading* that was already described in the work of Enriques [7, IV.II.17] (for the case of smooth varieties) and Laufer's procedure to compute the fundamental cycle [17] (for varieties with rational singularities). Here we will present a new version that is a generalization of both the unloading procedures described by Casas-Alvero [5, §4.6] (for smooth varieties) and Reguera [22] (for varieties with rational singularities).

Unloading procedure: Let $D \in \mathrm{Div}_{\mathbb{Q}}(X')$ be any $\mathbb{Q}$-divisor. Its *excess* at the exceptional prime divisor E_i is the integer $\rho_i = -\lceil D \rceil \cdot E_i$. Denote by Θ the set of exceptional components $E_i \leqslant E$ with negative excesses, i.e.

$$\Theta := \{E_i \leqslant E \mid \rho_i = -\lceil D \rceil \cdot E_i < 0\}.$$

To *unload values* on this set is to consider the new divisor

$$D' = \lceil D \rceil + \sum_{E_i \in \Theta} n_i E_i,$$

where $n_i = \left\lceil \frac{\rho_i}{E_i^2} \right\rceil$. Notice that n_i is the least integer number such that

$$(\lceil D \rceil + n_i E_i) \cdot E_i = -\rho_i + n_i E_i^2 \leqslant 0.$$

Remark 2.7. Casas-Alvero considered at each step just one component with negative excess. Reguera also considered one component with negative excess but in her case she also imposed $n_i = 1$ at each step. In this sense, our approach is more economic from a computational point of view. Furthermore, our procedure allows unloading on divisors with affine part[3], which will enable us to treat in a unified way multiplier ideals of both curves and not necessarily $\mathfrak{m}$-primary complete ideals.

The correctness of the unloading procedure is a consequence of the following results.

Proposition 2.8. *Let* D' *be the divisor obtained from a divisor* $D \in \mathrm{Div}_{\mathbb{Q}}(X')$ *after one single unloading step. Then* $I_{D'} = I_D$.

Proof. It is clear from its construction that $I_{D'} \subseteq I_D$. Pick $f \in I_D$ and let $\overline{C} = C' + \overline{C}_{\mathrm{exc}}$ be the total transform of the curve C defined by $f = 0$. We have $v_i(f) \geqslant v_i(\lceil D \rceil) \geqslant v_i(D)$

[3]Our method also differs from the one considered by Lipman-Watanabe in [20].

for all E_i. Consider any exceptional divisor E_i where D has negative excess, from the inequality $(\overline{C}_{\text{exc}} - v_i(f)E_i) \cdot E_i \geqslant (\lceil D \rceil - v_i(\lceil D \rceil)E_i) \cdot E_i$ we deduce

$$-v_i(f)E_i \cdot E_i \geqslant (\lceil D \rceil - v_i(\lceil D \rceil)E_i) \cdot E_i$$

just because $\overline{C}_{\text{exc}} \cdot E_i \leqslant 0$. Equivalently $(\lceil D \rceil + (v_i(f) - v_i(\lceil D \rceil))E_i) \cdot E_i \leqslant 0$ so it follows that $n_i \leqslant v_i(f) - v_i(\lceil D \rceil)$. In particular $n_i + v_i(\lceil D \rceil) \leqslant v_i(f)$ and $f \in I_{D'}$.

□

Proposition 2.9. *The antinef closure $\widetilde{D}$ of a divisor $D \in \mathrm{Div}_{\mathbb{Q}}(X')$ is achieved after finitely many unloading steps.*

Proof. We want to show that the divisors in the sequence

$$D \leqslant D_1 = \lceil D \rceil < \cdots < D_t < D_{t+1} < \cdots$$

obtained during the unloading procedure are all contained in the antinef closure $\widetilde{D}$, then the result will follow since both D_1 and $\widetilde{D}$ have integral coefficients and the inequalities in the unloading sequence are strict. Clearly $D_1 \leqslant \widetilde{D}$ and suppose that $D_t \leqslant \widetilde{D}$. Notice that for any component $E_i \leqslant D_t$ with negative excess we have $(\widetilde{D} - D_t) \cdot E_i \leqslant -D_t \cdot E_i$. Then, if we denote $\widetilde{D} - D_t = \sum_i m_i E_i$, the previous inequality becomes

$$\begin{array}{rcl}(\widetilde{D} - D_t) \cdot E_i & = & (m_i E_i + \sum_{j \neq i} m_j E_j) \cdot E_i \\ & = & m_i E_i^2 + \sum_{j \neq i} m_j E_j \cdot E_i \quad \leqslant \quad -D_t \cdot E_i.\end{array}$$

Then, using that $\sum_{j \neq i} m_j E_j \cdot E_i \geqslant 0$, we get

$$m_i \geqslant \left\lceil \frac{-D_t \cdot E_i}{E_i^2} \right\rceil ,$$

where we used the fact that D_t and $\widetilde{D}$ have integer coefficients. It follows that D_{t+1} is also contained in $\widetilde{D}$. □

2.3. **Multiplier ideals.** Let $\pi : X' \to X$ be a log-resolution of an ideal $\mathfrak{a} \subseteq \mathcal{O}_X$ and let F be the divisor such that $\mathfrak{a} \cdot \mathcal{O}_{X'} = \mathcal{O}_{X'}(-F)$. The *multiplier ideal (sheaf)* associated to $\mathfrak{a}$ and some rational number $\lambda \in \mathbb{Q}_{>0}$ is defined as[4]

$$\mathcal{J}\left(\mathfrak{a}^{\lambda}\right) = \pi_* \mathcal{O}_{X'}\left(\lceil K_\pi - \lambda F \rceil\right).$$

For a detailed overview of the theory of multiplier ideals and the properties they satisfy, we must refer to the book of Lazarsfeld [18]. For more details in the case that X has rational singularities we also recommend to take a look at [24] and [25].

The definition of multiplier ideals is independent of the choice of log resolution. For simplicity, we will always fix a given resolution. Multiplier ideals are complete and they are invariants up to integral closure, i.e. $\mathcal{J}(\mathfrak{a}^{\lambda}) = \mathcal{J}(\overline{\mathfrak{a}}^{\lambda})$, therefore, without loss of generality, we may always assume that the ideal $\mathfrak{a}$ is complete. Moreover, if $\mathfrak{a}$ is $\mathfrak{m}$-primary it follows that its associated multiplier ideals $\mathcal{J}\left(\mathfrak{a}^{\lambda}\right)$ are $\mathfrak{m}$-primary as well.

[4]By an abuse of notation, we will also denote $\mathcal{J}\left(\mathfrak{a}^{\lambda}\right)$ its stalk at O so we will omit the word "sheaf" if no confusion arises.

Some other important properties of multiplier ideals that we will use in this work are:

- *Local vanishing theorem:* $R^i\pi_*\mathcal{O}_{X'}(\lceil K_\pi - \lambda F\rceil) = 0$ for all $i > 0$ and all $\lambda \in \mathbb{Q}_{>0}$.
- *Skoda's theorem:* $\mathcal{J}\left(\mathfrak{a}^\lambda\right) = \mathfrak{a}\cdot\mathcal{J}\left(\mathfrak{a}^{\lambda-1}\right)$ for all $\lambda > \dim \mathcal{O}_{X,O} = 2$.

For the case of principal ideals there is another version of Skoda's theorem that states that $\mathcal{J}\left(\mathfrak{a}^\lambda\right) = \mathfrak{a}\cdot\mathcal{J}\left(\mathfrak{a}^{\lambda-1}\right)$ for all $\lambda \geqslant 1$. In particular, we have peridiocity of jumping numbers.

Multiplier ideals come with an attached set of invariants that were studied systematically by Ein-Lazarsfeld-Smith-Varolin in [6]. Clearly

$$\lceil K_\pi - \lambda F\rceil \geqslant \lceil K_\pi - (\lambda+\varepsilon)F\rceil$$

for any $\varepsilon > 0$, with equality if ε is small enough. Therefore the multiplier ideals form a discrete nested sequence of ideals

$$\mathcal{O}_{X,O} \supseteq \mathcal{J}(\mathfrak{a}^{\lambda_0}) \supsetneq \mathcal{J}(\mathfrak{a}^{\lambda_1}) \supsetneq \mathcal{J}(\mathfrak{a}^{\lambda_2}) \supsetneq \dots \supsetneq \mathcal{J}(\mathfrak{a}^{\lambda_i}) \supsetneq \dots$$

indexed by an increasing sequence of rational numbers $0 = \lambda_0 < \lambda_1 < \lambda_2 < \dots$ such that for any $c \in [\lambda_i, \lambda_{i+1})$ it holds

$$\mathcal{J}(\mathfrak{a}^{\lambda_i}) = \mathcal{J}(\mathfrak{a}^c) \supsetneq \mathcal{J}(\mathfrak{a}^{\lambda_{i+1}}).$$

The λ_i are the so-called *jumping numbers* of the ideal $\mathfrak{a}$ and the first jumping number $\lambda_1 = \operatorname{lct}(\mathfrak{a})$ is the *log-canonical threshold* of $\mathfrak{a}$.

2.4. **Contributing divisors.** The jumps between multiplier ideals necessarily must occur at rational numbers $\lambda \in \mathbb{Q}$ which cause the strict inclusion of divisors

$$\lceil K_\pi - \lambda F\rceil < \lceil K_\pi - (\lambda-\varepsilon)F\rceil$$

for any ε. If we take a close look at $F = F_{\text{exc}} + F_{\text{aff}}$ these rational numbers must belong to the set of *candidate jumping numbers*

$$\left\{\frac{k_i+m}{e_i} \;\middle|\; m \in \mathbb{Z}_{>0}\right\}$$

Notice that for non-exceptional components $E_i \leqslant F_{\text{aff}}$ we have $k_i = 0$ and their corresponding candidates $\left\{\frac{m}{e_i} \;\middle|\; m \in \mathbb{Z}_{>0}\right\}$ are indeed jumping numbers.

It is easy to check that not every candidate jumping number (coming from the exceptional part) is necessarily a jumping number. To separate the wheat from the chaff, Tucker [25] developed the notion of *divisor that contributes* to a jumping number, building upon previous work by Smith-Thompson [23].

Definition 2.10. A positive rational number λ is a *candidate jumping number* for a reduced divisor $G \leqslant F$ if it satisfies $\lambda e_i - k_i \in \mathbb{Z}_{>0}$ for any component $E_i \leqslant G$.

Definition 2.11. [25, Def. 3.1] A reduced divisor $G \leqslant F$ for which λ is a candidate jumping number is said to *contribute* to λ if

$$\pi_*\mathcal{O}_{X'}(\lceil K_\pi - \lambda F\rceil + G) \supsetneq \mathcal{J}(\mathfrak{a}^\lambda)$$

Moreover, this contribution is *critical* if for any divisor $0 \leqslant G' < G$ we have

$$\pi_* \mathcal{O}_{X'}(\lceil K_\pi - \lambda F \rceil + G') = \mathcal{J}(\mathfrak{a}^\lambda).$$

Most often we will simply say that G is just a *contributing* or a *critical divisor* associated to λ. Critical divisors define complete ideals very close to a multiplier ideal in a precise sense that will be explained in the forthcoming Corollary 5.5 in Section §5. One may identify critical divisors with exceptional support through the following numerical characterization.

Proposition 2.12. [25, Thm. 4.3] *Let λ be a candidate jumping number for a reduced divisor $G \in \Lambda$ with connected support.*

- *If $G = E_i$ is prime, then E_i is a critical divisor for λ if and only if*
$$(\lceil K_\pi - \lambda F \rceil + E_i) \cdot E_i \geqslant 0.$$
- *If G is reducible, then G is a critical divisor for λ if and only if*
$$(\lceil K_\pi - \lambda F \rceil + G) \cdot E_i = 0$$
for all divisors E_i in the support of G.

Moreover, critical divisors with exceptional support satisfy a nice geometric property when viewed in the dual graph.

Proposition 2.13. [25, Cor. 4.2 & Thm 5.1] *Let G be a critical divisor for a jumping number λ. Then G is a connected chain in the dual graph of the log-resolution of $\mathfrak{a}$ whose ends must be either rupture or dicritical divisors.*

Using all these properties, Tucker provides a simple algorithm to compute the set of all jumping numbers (see [25, §6]). It boils down to the following steps:

Algorithm 2.14. *(Jumping Numbers)*

`Input:` *A log-resolution of an ideal $\mathfrak{a} \subseteq \mathcal{O}_{X,O}$.*
`Output:` *List of Jumping Numbers of $\mathfrak{a}$.*

- **Jumping number***:*
  - *Compute the candidate jumping numbers for F_{exc}.*
  - *Find all possible critical divisors using Prop. 2.13.*
  - *Find which candidate jumping numbers can be realized as jumping number associated to these critical divisors using Prop. 2.12.*
  - *Plug in those jumping numbers coming from F_{aff}.*

3. An algorithm to compute jumping numbers and multiplier ideals

The aim of this section is to compute the jumping numbers and their corresponding multiplier ideals of any given ideal $\mathfrak{a} \subseteq \mathcal{O}_{X,O}$. To such purpose, we fix a log-resolution $\pi : X' \longrightarrow X$ of our ideal $\mathfrak{a}$. The main ingredients we will have to deal with are the relative canonical divisor $K_\pi = \sum_{i=1}^{r} k_i E_i \in \Lambda_{\mathbb{Q}}$, and the divisor $F \in \mathrm{Div}(X')$ such that $\mathfrak{a}\mathcal{O}_{X'} = \mathcal{O}_{X'}(-F)$. Recall that we have a decomposition

$$F = F_{\mathrm{exc}} + F_{\mathrm{aff}} = \sum_{i=1}^{r} e_i E_i + \sum_{i=r+1}^{s} e_i E_i$$

in terms of its exceptional and affine support.

We will provide a very simple algorithm that allows one to construct sequentially the chain of multiplier ideals[5]

$$\mathcal{O}_{X,O} \supseteq \mathcal{J}(\mathfrak{a}^{\lambda_0}) \supsetneq \mathcal{J}(\mathfrak{a}^{\lambda_1}) \supsetneq \mathcal{J}(\mathfrak{a}^{\lambda_2}) \supsetneq \ldots \supsetneq \mathcal{J}(\mathfrak{a}^{\lambda_i}) \supsetneq \ldots$$

When X is a smooth surface, or even when X has a log-terminal singularity at O, the multiplier ideal associated to $\lambda_0 = 0$ is the whole ring, i.e. $\mathcal{O}_{X,O} = \mathcal{J}(\mathfrak{a}^{\lambda_0})$. In general, when X has a rational singularity we may have an strict inclusion $\mathcal{O}_{X,O} \supsetneq \mathcal{J}(\mathfrak{a}^{\lambda_0})$. The starting point of our method will be describing this ideal by means of the antinef closure $D_{\lambda_0} = \sum e_i^{\lambda_0} E_i$ of $\lfloor -K_\pi \rfloor$ that we compute using the unloading procedure described in Section §2.2.

As a consequence of our main result (see Theorem 3.5), the log-canonical threshold satisfies the following formula[6]

$$\lambda_1 = \mathrm{lct}(\mathfrak{a}) = \min_i \left\{ \frac{k_i + 1 + e_i^{\lambda_0}}{e_i} \right\}. \tag{3.1}$$

Then we describe its associated multiplier ideal $\mathcal{J}(\mathfrak{a}^{\lambda_1})$ just computing the antinef closure D_{λ_1} of $\lfloor \lambda_1 F - K_\pi \rfloor$ using the unloading procedure. Once we have the divisor D_{λ_1}, we use an extension of Formula 3.1 given by Theorem 3.5, that computes the next jumping number λ_2. Then we only have to follow the same strategy: the antinef closure D_{λ_2} of $\lfloor \lambda_2 F - K_\pi \rfloor$, i.e. the multiplier ideal $\mathcal{J}(\mathfrak{a}^{\lambda_2})$, will allow us to compute λ_3 and so on.

The main idea behind our method is a simple comparison between complete ideals. Whenever we have two antinef divisors it is easy to check whether their corresponding complete ideals satisfy a strict inclusion (see Proposition 2.5). To compare the ideals associated to an antinef and a non-antinef divisor is more subtle and this is the situation that we will have to deal with in this section.

To address this problem we will need some preliminary technical results.

Lemma 3.1. *Let D_1, D_2 be two divisors in $\mathrm{Div}(X')$ such that $D_1 \leqslant D_2$. Then, they have the same antinef closure $\widetilde{D_1} = \widetilde{D_2}$ if and only if $\widetilde{D_1} \geqslant D_2$.*

[5] In fact, we can compute the chain inside any desired fixed range $[c, c'] \subseteq \mathbb{R}$:

$$\mathcal{J}(\mathfrak{a}^{c}) = \mathcal{J}(\mathfrak{a}^{\lambda_0}) \supsetneq \mathcal{J}(\mathfrak{a}^{\lambda_1}) \supsetneq \ldots \supsetneq \mathcal{J}(\mathfrak{a}^{\lambda_r}) = \mathcal{J}(\mathfrak{a}^{c'}).$$

[6] When X is smooth, or even when it has log-terminal singularities, we have $D_{\lambda_0} = 0$ so one recovers the well-known formula for the log-canonical threshold.

Proof. Recall that, by Proposition 2.6, we already have $\widetilde{D_1} \leqslant \widetilde{D_2}$ just because $D_1 \leqslant D_2$.

Assume $\widetilde{D_1} \geqslant D_2$ then, by the definition of antinef closure (see Lemma 2.3), we also have $\widetilde{D_1} \geqslant \widetilde{D_2} \geqslant D_2$ and thus $\widetilde{D_1} = \widetilde{D_2}$. On the other hand, assume that $\widetilde{D_1} = \widetilde{D_2}$. Then, since the antinef closure of a divisor always contains it, we have $\widetilde{D_1} = \widetilde{D_2} \geqslant D_2$ as desired. ☐

Corollary 3.2. *Let D_1, D_2 be two divisors in $\mathrm{Div}(X')$ such that $D_1 \leqslant D_2$. Then, $\widetilde{D_1} < \widetilde{D_2}$ if and only if $v_i(\widetilde{D_1}) < v_i(D_2)$ for some E_i.*

Proof. As $D_1 \leqslant D_2$, the inclusion $\widetilde{D_1} \leqslant \widetilde{D_2}$ also holds. The result then follows from Lemma 3.1. ☐

Translated into the language of complete ideals, these results give a characterization of the jump between two nested ideals, which will be a key ingredient in the proof of our results.

Proposition 3.3. *Let D_1, D_2 be two divisors in $\mathrm{Div}(X')$ such that $D_1 \leqslant D_2$. Then:*

i) $\pi_*\mathcal{O}_{X'}(-D_1) = \pi_*\mathcal{O}_{X'}(-D_2)$ *if and only if* $\widetilde{D_1} \geqslant D_2$.
ii) $\pi_*\mathcal{O}_{X'}(-D_1) \supsetneq \pi_*\mathcal{O}_{X'}(-D_2)$ *if and only if* $v_i(\widetilde{D_1}) < v_i(D_2)$ *for some* E_i.

For convenience we also present this result in the form we will most commonly use it.

Corollary 3.4. *Let $\lambda' < \lambda$ be rational numbers. Let $D_{\lambda'} = \sum e_i^{\lambda'} E_i$ be the antinef closure of $\lfloor \lambda' F - K_\pi \rfloor$. Then:*

i) $\mathcal{J}(\mathfrak{a}^{\lambda'}) = \mathcal{J}(\mathfrak{a}^{\lambda})$ *if and only if* $\lfloor \lambda e_i - k_i \rfloor \leqslant e_i^{\lambda'}$ *for all* E_i.
ii) $\mathcal{J}(\mathfrak{a}^{\lambda'}) \supsetneq \mathcal{J}(\mathfrak{a}^{\lambda})$ *if and only if* $\lfloor \lambda e_i - k_i \rfloor > e_i^{\lambda'}$ *for some* E_i.

With the technical tools stated above we are ready for the main result of this section.

Theorem 3.5. *Let $\mathfrak{a} \subseteq \mathcal{O}_{X,O}$ be an ideal and let $D_{\lambda'} = \sum e_i^{\lambda'} E_i$ be the antinef closure of $\lfloor \lambda' F - K_\pi \rfloor$ for a given $\lambda' \in \mathbb{Q}_{>0}$. Then,*

$$\lambda = \min_i \left\{ \frac{k_i + 1 + e_i^{\lambda'}}{e_i} \right\}$$

is the jumping number consecutive to λ'.

Proof. Let us check first that $\lambda' < \lambda$. Indeed, by the definition of antinef closure, the integers $e_i^{\lambda'}$ satisfy $\lfloor \lambda' e_i - k_i \rfloor \leqslant e_i^{\lambda'}$ for any E_i, and hence:

$$\lambda' < \frac{k_i + 1 + e_i^{\lambda'}}{e_i}.$$

Thus, we have an inclusion of ideals $\mathcal{J}(\mathfrak{a}^{\lambda'}) \supseteq \mathcal{J}(\mathfrak{a}^{\lambda})$. Notice that for those divisors E_i where the minimum is achieved we have

$$\lfloor \lambda e_i - k_i \rfloor = 1 + e_i^{\lambda'} > e_i^{\lambda'}$$

so the above inclusion of ideals is strict by Corollary 3.4. To conclude that λ is the jumping number immediately after λ', we have to show that for any $c \in \mathbb{R}$ with $\lambda' \leqslant c < \lambda$ we have $\mathcal{J}(\mathfrak{a}^{\lambda'}) = \mathcal{J}(\mathfrak{a}^{c})$. Suppose the contrary, i.e., $\mathcal{J}(\mathfrak{a}^{\lambda'}) \supsetneq \mathcal{J}(\mathfrak{a}^{c})$. By Corollary 3.4, this c should satisfy $\lfloor \lambda e_i - k_i \rfloor > e_i^{\lambda'}$ or equivalently $c \geqslant \frac{k_i+1+e_i^{\lambda'}}{e_i}$ for some E_i, and this contradicts the fact that λ is the minimum of these rational numbers. ☐

The above result for the case $\lambda' = 0$ gives a mild generalization of the well-known formula for the log-canonical threshold in the smooth case. We point out that the antinef closure of $\lfloor -K_\pi \rfloor$ is 0 whenever X is smooth or, more generally, when it has log-terminal singularities.

Corollary 3.6. *Let $\mathfrak{a} \subseteq \mathcal{O}_{X,O}$ be an ideal. Let $D_{\lambda_0} = \sum e_i^{\lambda_0} E_i$ be the antinef closure of $\lfloor -K_\pi \rfloor$. Then,*

$$\operatorname{lct}(\mathfrak{a}) = \min_i \left\{ \frac{k_i + 1 + e_i^{\lambda_0}}{e_i} \right\}.$$

Another easy application of the results above is the following result that should be well-known to experts.

Corollary 3.7. *Let λ_1 be the log-canonical threshold of an ideal $\mathfrak{a} \subseteq \mathcal{O}_{X,O}$ and assume that X has at most a log-terminal singularity at O. Then $\mathcal{J}(\mathfrak{a}^{\lambda_1}) = \mathfrak{m}$.*

Proof. Since X has at most a log-terminal singularity, the log-canonical threshold is

$$\operatorname{lct}(\mathfrak{a}) = \lambda_1 = \min_i \left\{ \frac{k_i + 1}{e_i} \right\}$$

so it satisfies $\lambda_1 \leqslant \frac{k_i+1}{e_i}$ for any divisor E_i and equality is achieved at least for a given divisor. In particular, for all E_i we have

$$\lfloor \lambda_1 e_i - k_i \rfloor \leqslant 1.$$

It follows from Proposition 2.6 that $\mathfrak{m} \subseteq \mathcal{J}(\mathfrak{a}^{\lambda_1}) \subsetneq \mathcal{O}_{X,O}$ and we get the desired result. ☐

For non log-terminal singularities we may find examples where the codimension as $\mathbb{C}$-vector spaces of $\mathcal{J}(\mathfrak{a}^{\lambda_0}) \supsetneq \mathcal{J}(\mathfrak{a}^{\lambda_1})$ might be bigger than 1 (see Example 3.10).

Combining Theorem 3.5 and the unloading procedure described in Section §2.2 we can describe a very simple algorithm that allows us to compute the chain of multiplier ideals:

Algorithm 3.8. *(Jumping Numbers and Multiplier Ideals)*

`Input:` *A log-resolution of an ideal $\mathfrak{a} \subseteq \mathcal{O}_{X,O}$.*
`Output:` *List of Jumping Numbers of $\mathfrak{a}$ and its corresponding Multiplier Ideals.*

Set $\lambda_0 = 0$ and compute the antinef closure $D_{\lambda_0} = \sum e_i^{\lambda_0} E_i$ of $\lfloor -K_\pi \rfloor$ using the unloading procedure. From $j = 1$, incrementing by 1

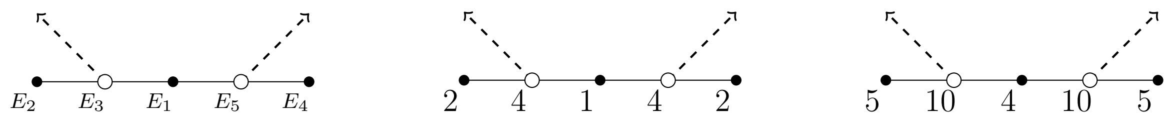
(Step j) · **Jumping number**: *Compute*

$$\lambda_j = \min_i \left\{ \frac{k_i + 1 + e_i^{\lambda_{j-1}}}{e_i} \right\}.$$

· **Multiplier ideal**: *Compute the antinef closure* $D_{\lambda_j} = \sum e_i^{\lambda_j} E_i$ *of* $\lfloor \lambda_j F - K_\pi \rfloor$ *using the unloading procedure.*

Notice that we may also find all the multiplier ideals in any given interval $[c', c]$ of the real line. In this case, our starting point would be computing the antinef closure $D_{c'}$ of $\lfloor c'F - K_\pi \rfloor$. To illustrate this method we consider an easy example in a smooth variety.

Example 3.9. Consider the ideal $\mathfrak{a} = (x^2y^2, x^5, y^5, xy^4, x^4y) \subseteq \mathbb{C}\{x, y\}$. We represent the relative canonical divisor K_π and the divisor F in the dual graph as follows:



Vertex ordering $\qquad$ K_π $\qquad$ F

The blank dots correspond to dicritical divisors and their excesses are represented by broken arrows[7]. For simplicity we will collect the values of any divisor in a vector. To start with we have $K_\pi = (1, 2, 4, 2, 4)$ and $F = (4, 5, 10, 5, 10)$. In the algorithm we will have to perform some unloading steps so we will have to consider the intersection matrix $M = (E_i \cdot E_j)_{1 \leqslant i,j \leqslant 5}$

$$M = \begin{pmatrix} -5 & 0 & 1 & 0 & 1 \\ 0 & -2 & 1 & 0 & 0 \\ 1 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & -2 & 1 \\ 1 & 0 & 0 & 1 & -1 \end{pmatrix}.$$

The algorithm is performed as follows:

- We start computing the log-canonical threshold:

$$\lambda_1 = \mathrm{lct}(\mathfrak{a}) = \min_i \left\{ \frac{k_i + 1}{e_i} \right\} = \min_i \left\{ \frac{2}{4}, \frac{3}{5}, \frac{5}{10}, \frac{3}{5}, \frac{5}{10} \right\} = \frac{1}{2}.$$

The divisor $\lfloor \frac{1}{2} F - K_\pi \rfloor = (1, 0, 1, 0, 1)$ is not antinef since it has excess -1 at E_2 and E_4. The first unloading step is to consider the divisor $\lfloor \frac{1}{2} F - K_\pi \rfloor + E_2 + E_4 = (1, 1, 1, 1, 1)$. This divisor has excess -1 at E_3 and E_5 so we need to perform a second unloading step to obtain the antinef closure $D_{\lambda_1} = (1, 1, 2, 1, 2)$.

[7]The broken arrows also represent the branches of the strict transform of a curve defined by a generic $f \in \mathfrak{a}$.

- The second Jumping Number is:

$$\lambda_2 = \min_i \left\{ \frac{k_i + 1 + e_i^{\lambda_1}}{e_i} \right\} = \min_i \left\{ \frac{2+1}{4}, \frac{3+1}{5}, \frac{5+2}{10}, \frac{3+1}{5}, \frac{5+2}{10} \right\} = \frac{7}{10}.$$

Then we get $\lfloor \frac{7}{10} F - K_\pi \rfloor = (1,1,3,1,3)$. It has excess -1 at E_1, E_2 and E_4 and we obtain the divisor $(2,2,3,2,3)$ after the first unloading step. This divisor has excess -1 at E_3 and E_5 and, after a second unloading step, we obtain the antinef closure $D_{\lambda_2} = (2,2,4,2,4)$.

- The third Jumping Number is:

$$\lambda_3 = \min_i \left\{ \frac{k_i + 1 + e_i^{\lambda_2}}{e_i} \right\} = \min_i \left\{ \frac{2+2}{4}, \frac{3+2}{5}, \frac{5+4}{10}, \frac{3+2}{5}, \frac{5+4}{10} \right\} = \frac{9}{10}.$$

Then we get $\lfloor \frac{9}{10} F - K_\pi \rfloor = (2,2,5,2,5)$ that has excess -1 at E_3 and E_5. After a single unloading step we get the antinef closure $D_{\lambda_3} = (2,3,5,3,5)$.

- The fourth Jumping Number is:

$$\lambda_4 = \min_i \left\{ \frac{k_i + 1 + e_i^{\lambda_3}}{e_i} \right\} = \min_i \left\{ \frac{2+2}{4}, \frac{3+3}{5}, \frac{5+5}{10}, \frac{3+3}{5}, \frac{5+5}{10} \right\} = 1.$$

Then we get $\lfloor F - K_\pi \rfloor = D_{\lambda_4} = (3,3,6,3,6)$ since this divisor is antinef.

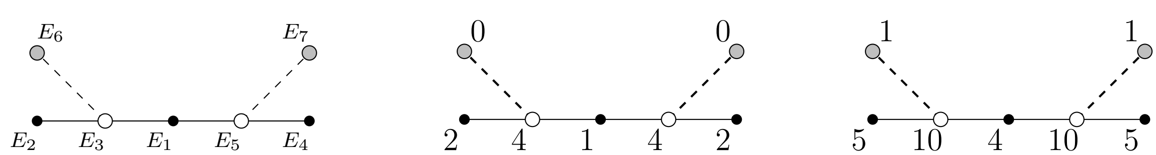
- The fifth Jumping Number is:

$$\lambda_5 = \min_i \left\{ \frac{k_i + 1 + e_i^{\lambda_4}}{e_i} \right\} = \min_i \left\{ \frac{2+3}{4}, \frac{3+3}{5}, \frac{5+6}{10}, \frac{3+3}{5}, \frac{5+6}{10} \right\} = \frac{11}{10}.$$

Then we get $\lfloor \frac{11}{10} F - K_\pi \rfloor = (3,3,7,3,7)$ and, after a single unloading step, we obtain the antinef closure $D_{\lambda_5} = (3,4,7,4,7)$.

Now we will compute the chain of multiplier ideals of the plane curve defined by $f = (x^2 - y^3)(y^2 - x^3) \in \mathbb{C}\{x,y\}$. The product of two cusps sharing the origin O is a generic element of the ideal $\mathfrak{a} = (x^2y^2, x^5, y^5, xy^4, x^4y)$ considered above, so $\mathcal{J}(f^\lambda) = \mathcal{J}(\mathfrak{a}^\lambda)$ for $\lambda < 1$. This example will illustrate how the non-exceptional components affect the unloading procedure and, consequently, the list of jumping numbers for $\lambda > 1$.

Denote the total transform of the curve defined by f simply as F. We represent the relative canonical divisor K_π and the divisor F in the dual graph as follows:



Vertex ordering K_π F

The gray dots will represent here the affine components belonging to the strict transform of the curve. The intersection matrix is now

$$M = \begin{pmatrix} -5 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & -2 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -2 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & -1 & 0 & 1 \end{pmatrix}.$$

The algorithm is performed as follows:

- The log-canonical threshold is:

$$\lambda_1 = \mathrm{lct}(\mathfrak{a}) = \min_i \left\{ \frac{k_i+1}{e_i} \right\} = \min_i \left\{ \frac{2}{4}, \frac{3}{5}, \frac{5}{10}, \frac{3}{5}, \frac{5}{10}, \frac{1}{1}, \frac{1}{1} \right\} = \frac{1}{2}.$$

We get $\lfloor \frac{1}{2} F - K_\pi \rfloor = (1,0,1,0,1,0,0)$ and, as in the previous example, its antinef closure is $D_{\lambda_1} = (1,1,2,1,2,0,0)$.

- The second Jumping Number is:

$$\lambda_2 = \min_i \left\{ \frac{k_i+1+e_i^{\lambda_1}}{e_i} \right\} = \min_i \left\{ \frac{2+1}{4}, \frac{3+1}{5}, \frac{5+2}{10}, \frac{3+1}{5}, \frac{5+2}{10}, \frac{1}{1}, \frac{1}{1} \right\} = \frac{7}{10}.$$

Then we get $\lfloor \frac{7}{10} F - K_\pi \rfloor = (1,1,3,1,3,0,0)$ and its antinef closure $D_{\lambda_2} = (2,2,4,2,4,0,0)$.

- The third Jumping Number is:

$$\lambda_3 = \min_i \left\{ \frac{k_i+1+e_i^{\lambda_2}}{e_i} \right\} = \min_i \left\{ \frac{2+2}{4}, \frac{3+2}{5}, \frac{5+4}{10}, \frac{3+2}{5}, \frac{5+4}{10}, \frac{1}{1}, \frac{1}{1} \right\} = \frac{9}{10}.$$

Then we get $\lfloor \frac{9}{10} F - K_\pi \rfloor = (2,2,5,2,5,0,0)$ and its antinef closure $D_{\lambda_3} = (2,3,5,3,5,0,0)$.

- The fourth Jumping Number is:

$$\lambda_4 = \min_i \left\{ \frac{k_i+1+e_i^{\lambda_3}}{e_i} \right\} = \min_i \left\{ \frac{2+2}{4}, \frac{3+3}{5}, \frac{5+5}{10}, \frac{3+3}{5}, \frac{5+5}{10}, \frac{1}{1}, \frac{1}{1} \right\} = 1.$$

Then we get $\lfloor F - K_\pi \rfloor = (3,3,6,3,6,1,1)$ but this divisor is not antinef because of the non-exceptional components. Namely, we have excess -1 at E_3 and E_5. To obtain the antinef closure $D_{\lambda_4} = (4,5,10,5,10,1,1)$ we need to perform seven unloading steps with the intermediate divisors:

· $(3,3,7,3,7,1,1)$ with excess -1 at E_2 and E_4.
· $(3,4,7,4,7,1,1)$ with excess -1 at E_3 and E_5.
· $(3,4,8,4,8,1,1)$ with excess -1 at E_1.
· $(4,4,8,4,8,1,1)$ with excess -1 at E_3 and E_5.
· $(4,4,9,4,9,1,1)$ with excess -1 at E_2 and E_4.
· $(4,5,9,5,9,1,1)$ with excess -1 at E_3 and E_5.

If we compare with the $\mathfrak{m}$-primary ideal $\mathfrak{a}$ we should notice that the affine components of $\lfloor F - K_\pi \rfloor$ force us to add more exceptional components when computing its antinef closure and consequently, this will give a different jumping number in the next step.

- The fifth Jumping Number is:

$$\lambda_5 = \min_i \left\{ \frac{k_i + 1 + e_i^{\lambda_4}}{e_i} \right\} = \min_i \left\{ \frac{2+4}{4}, \frac{3+5}{5}, \frac{5+10}{10}, \frac{3+5}{5}, \frac{5+10}{10}, \frac{2}{1}, \frac{2}{1} \right\} = \frac{3}{2}.$$

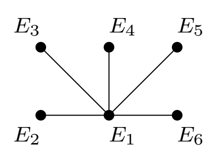
Then we get $\lfloor \frac{3}{2}F - K_\pi \rfloor = (5,5,11,5,11,1,1)$ and its antinef closure $D_{\lambda_5} = (5,6,12,6,12,1,1)$.

Consider a normal surface X with a singularity at O. Given a minimal resolution $\pi : X' \longrightarrow X$ of X, Artin [4] introduced the *fundamental cycle* as the unique smallest non-zero effective divisor with exceptional support that is antinef. Moreover he proved that the singularity is rational if and only if the arithmetical genus of the fundamental cycle is zero.

We have that π is also a minimal log-resolution of the maximal ideal $\mathfrak{m} \subseteq \mathcal{O}_{X,O}$ and the fundamental cycle is the divisor F such that $\mathfrak{m} \cdot \mathcal{O}_{X'} = \mathcal{O}_{X'}(-F)$. To compute its arithmetical genus we can use the formula $p_a(F) = 1 + \frac{(K_\pi + F)\cdot F}{2}$ (see [3]).

This characterization gives us a good source of examples of surfaces with rational singularities.

Example 3.10. Consider a surface X with a rational singularity at O whose minimal resolution $\pi : X' \longrightarrow X$ has six exceptional components $E_1, \ldots, E_6$ with the following dual graph and intersection matrix:



$$\begin{pmatrix} -4 & 1 & 1 & 1 & 1 & 1 \\ 1 & -5 & 0 & 0 & 0 & 0 \\ 1 & 0 & -5 & 0 & 0 & 0 \\ 1 & 0 & 0 & -5 & 0 & 0 \\ 1 & 0 & 0 & 0 & -5 & 0 \\ 1 & 0 & 0 & 0 & 0 & -5 \end{pmatrix}$$

The fundamental cycle is the divisor $F = (2,1,1,1,1,1)$ and the relative canonical divisor is $K_\pi = (-\frac{5}{3}, -\frac{14}{15}, -\frac{14}{15}, -\frac{14}{15}, -\frac{14}{15}, -\frac{14}{15})$ so the singularity is not even log-canonical.

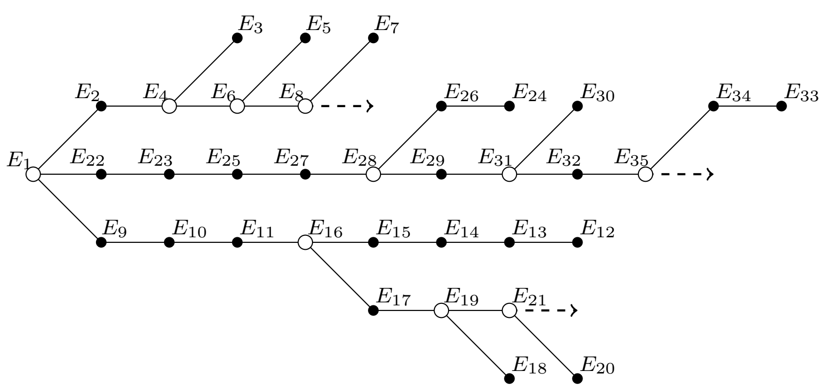
The multiplier ideals corresponding to $\lambda_0 = 0$ and $\lambda_1 = \mathrm{lct}(\mathfrak{m}) = \frac{4}{9}$ are given by the antinef divisors $D_{\lambda_0} = (2,1,1,1,1,1)$ and $D_{\lambda_1} = (3,1,1,1,1,1)$. Notice that $\mathcal{J}(\mathfrak{m}^{\lambda_0}) = \mathfrak{m}$ and, using the techniques of [1], we get that the codimension between these multiplier ideals is 4.

3.1. **Implementation.** We have implemented Algorithm 3.8 in the Computer Algebra system `Macaulay 2` [12]. The scripts of the source code as well as the output in full detail of some examples are available at the web page

`www.pagines.ma1.upc.edu/~jalvz/multiplier.html`

We implemented Tucker's Algorithm 2.14 as well in order to compare both approaches. Of course, once we have the list of jumping numbers we may use the unloading procedure of Section §2.2 to describe the corresponding multiplier ideals. We have also implemented this extended version of Tucker's algorithm and it turns out that our method is much faster.

For example, we have tested the case of an $\mathfrak{m}$-primary ideal $\mathfrak{a}$ whose corresponding dual graph has 35 vertices distributed in three branches only sharing the origin and each branch has three rupture divisors.



This example has 56986 jumping numbers in the interval $(0, 2]$. Using the extended version of Tucker's algorithm it takes 897.298 seconds to compute the whole list of jumping numbers and their corresponding multiplier ideals. Using our method it only takes 372.165 seconds, i.e. it is roughly 9 minutes faster.

The main difference between the two algorithms stems in the fact that Tucker needs to find first all the possible critical divisors. We will see in the next section that our algorithm can be understood as a method to find a unique and very precise contributing divisor.

The input that we use in both algorithms, i.e. the log-resolution $\pi : X' \to X$ of an ideal $\mathfrak{a} \subseteq \mathcal{O}_X$, is encoded using the intersection matrix and the vector of values for the divisor F such that $\mathfrak{a} \cdot \mathcal{O}_{X'} = \mathcal{O}_{X'}(-F)$. An algorithm to compute this data from a set of generators of the ideal $\mathfrak{a}$ has been described in [2]. An implementation in `Macaulay 2` will be available soon. For principal ideals this can be done using the `Singular` [13] package `alexpoly.lib`.

4. Jumping Divisors

The theory of critical divisors developed by Tucker [25] focuses on complete ideals very close to a given multiplier ideal. The aim of this section is to understand the whole jump between two consecutive multiplier ideals. To such purpose we introduce the following natural definition:

Definition 4.1. Let λ be a jumping numbers of an ideal $\mathfrak{a} \subseteq \mathcal{O}_{X,O}$. A reduced divisor $G \leqslant F$ for which λ is a candidate jumping number is called a *jumping divisor* for λ if

$$\mathcal{J}(\mathfrak{a}^{\lambda-\varepsilon}) = \pi_* \mathcal{O}_{X'}(\lceil K_\pi - \lambda F \rceil + G),$$

for ε small enough. We say that a jumping divisor is minimal if no proper subdivisor is a jumping divisor for λ, i.e.

$$\mathcal{J}(\mathfrak{a}^{\lambda-\varepsilon}) \supsetneq \pi_* \mathcal{O}_{X'}(\lceil K_\pi - \lambda F \rceil + G')$$

for any $0 \leqslant G' < G$.

Remark 4.2. Any reduced divisor $G \leqslant F$ for which λ is a candidate jumping number defines an ideal nested between two consecutive multiplier ideals

$$\mathcal{J}(\mathfrak{a}^{\lambda-\varepsilon}) \supseteq \pi_* \mathcal{O}_{X'}(\lceil K_\pi - \lambda F \rceil + G) \supseteq \mathcal{J}(\mathfrak{a}^{\lambda}).$$

Hence, a jumping divisor for λ is a contributing divisor to λ. In particular, a minimal jumping divisor can be understood as the minimal contribution which defines the preceding multiplier ideal.

It is a striking fact that the methods used in the previous section, in particular our main result Theorem 3.5, will allow us to construct the unique minimal jumping divisor associated to a jumping number. In fact, we will see in Corollary 4.7 that the only jumping divisors are those reduced divisors $D \leqslant F$ satisfying $G_\lambda \leqslant D \leqslant H_\lambda$, where G_λ and H_λ are defined as follows:

Definition 4.3. Let λ be a jumping number of an ideal $\mathfrak{a} \subseteq \mathcal{O}_{X,O}$. Let $D_{\lambda-\varepsilon} = \sum e_i^{\lambda-\varepsilon} E_i$ be the antinef closure of $\lfloor (\lambda - \varepsilon)F - K_\pi \rfloor$ for ε small enough. Then we define:

- *Maximal jumping divisor:* Is the reduced divisor $H_\lambda \leqslant F$ supported on those components E_i for which $\lambda e_i - k_i \in \mathbb{Z}_{>0}$. Equivalently
$$H_\lambda = \lceil K_\pi - (\lambda - \varepsilon)F \rceil - \lceil K_\pi - \lambda F \rceil.$$
- *Minimal jumping divisor:* Is the reduced divisor $G_\lambda \leqslant F$ supported on those components E_i for which
$$\lambda = \frac{k_i + 1 + e_i^{\lambda-\varepsilon}}{e_i},$$
i.e. supported on those divisors where the minimum considered in Theorem 3.5 is achieved.

It is clear that H_λ is a jumping divisor and $G_\lambda \leqslant H_\lambda$. In fact, any reduced divisor $G \leqslant F$ that contributes to λ satisfies $G \leqslant H_\lambda$. We will prove next that G_λ deserves the given name.

Proposition 4.4. *Let λ be a jumping number of an ideal $\mathfrak{a} \subseteq \mathcal{O}_{X,O}$. The reduced divisor G_λ is a jumping divisor.*

Proof. Since $G_\lambda \leqslant H_\lambda$, we have $\lfloor (\lambda-\varepsilon)F - K_\pi \rfloor \leqslant \lfloor \lambda F - K_\pi \rfloor - G_\lambda$ and therefore

$$\mathcal{J}(\mathfrak{a}^{\lambda-\varepsilon}) \supseteq \pi_* \mathcal{O}_{X'}(\lceil K_\pi - \lambda F \rceil + G_\lambda).$$

For the reverse inclusion, let $D_{\lambda-\varepsilon} = \sum e_i^{\lambda-\varepsilon} E_i$ be the antinef closure of $\lfloor (\lambda-\varepsilon)F - K_\pi \rfloor$. We want to check that $\lfloor \lambda F - K_\pi \rfloor - G_\lambda \leqslant D_{\lambda-\varepsilon}$. To this purpose we only need to consider the following cases:

- If $E_i \leqslant G_\lambda$ then we have $\lambda = \frac{k_i+1+e_i^{\lambda-\varepsilon}}{e_i}$. In particular $\lfloor \lambda e_i - k_i \rfloor - 1 = e_i^{\lambda-\varepsilon}$.
- If $E_i \not\leqslant G_\lambda$ then we have $\lambda < \frac{k_i+1+e_i^{\lambda-\varepsilon}}{e_i}$. Thus $\lfloor \lambda e_i - k_i \rfloor < 1 + e_i^{\lambda-\varepsilon}$ and the result follows.

□

The unicity of the jumping divisor G_λ is a consequence of the following more general statement

Theorem 4.5. *Let λ be a jumping number of an ideal $\mathfrak{a} \subseteq \mathcal{O}_{X,O}$. Any contributing divisor $G \leqslant F$ associated to λ satisfies either:*

- $\mathcal{J}(\mathfrak{a}^{\lambda-\varepsilon}) = \pi_*\mathcal{O}_{X'}(\lceil K_\pi - \lambda F \rceil + G) \supsetneq \mathcal{J}(\mathfrak{a}^{\lambda})$ *if and only if* $G_\lambda \leqslant G$, *or*
- $\mathcal{J}(\mathfrak{a}^{\lambda-\varepsilon}) \supsetneq \pi_*\mathcal{O}_{X'}(\lceil K_\pi - \lambda F \rceil + G) \supsetneq \mathcal{J}(\mathfrak{a}^{\lambda})$ *otherwise.*

Proof. Since $G \leqslant H_\lambda$, we have $\lfloor (\lambda-\varepsilon)F - K_\pi \rfloor \leqslant \lfloor \lambda F - K_\pi \rfloor - G$ and therefore

$$\mathcal{J}(\mathfrak{a}^{\lambda-\varepsilon}) \supseteq \pi_* \mathcal{O}_{X'}(\lceil K_\pi - \lambda F \rceil + G).$$

Now assume $G_\lambda \leqslant G$. Then $\lfloor \lambda F - K_\pi \rfloor - G \leqslant \lfloor \lambda F - K_\pi \rfloor - G_\lambda$, and using the fact that G_λ is a jumping divisor we obtain the equality $\mathcal{J}(\mathfrak{a}^{\lambda-\varepsilon}) = \pi_*\mathcal{O}_{X'}(\lceil K_\pi - \lambda F \rceil + G)$.

If $G_\lambda \not\leqslant G$ we may consider a component $E_i \leqslant G_\lambda$ such that $E_i \not\leqslant G$. Notice that we have

$$v_i(D_{\lambda-\varepsilon}) = e_i^{\lambda-\varepsilon} = \lambda e_i - k_i - 1 < \lambda e_i - k_i = v_i(\lfloor \lambda F - K_\pi \rfloor - G)$$

where $D_{\lambda-\varepsilon} = \sum e_i^{\lambda-\varepsilon} E_i$ is the antinef closure of $\lfloor (\lambda-\varepsilon)F - K_\pi \rfloor$. Therefore, by Proposition 3.3, we get the strict inclusion

$$\mathcal{J}(\mathfrak{a}^{\lambda-\varepsilon}) \supsetneq \pi_* \mathcal{O}_{X'}(\lceil K_\pi - \lambda F \rceil + G).$$

□

Corollary 4.6. *Let λ be a jumping number of an ideal $\mathfrak{a} \subseteq \mathcal{O}_{X,O}$. Then G_λ is the unique minimal jumping divisor associated to λ.*

Notice that Theorem 4.5 also describes all the jumping divisors associated to a given jumping number. Namely, we have

Corollary 4.7. *Let λ be a jumping number of an ideal $\mathfrak{a} \subseteq \mathcal{O}_{X,O}$. Then, any reduced divisor in the interval $G_\lambda \leqslant D \leqslant H_\lambda$ is a jumping divisor.*

It is clear from its definition that maximal jumping divisors are periodic, i.e. $H_\lambda = H_{\lambda+1}$ for any jumping number λ. On the other hand, critical divisors do not satisfy any periodicity condition. One may find examples where a divisor G is a critical divisor for the jumping number λ but not for $\lambda+1$ and vice versa. For minimal jumping divisors we have:

Proposition 4.8. *Let λ be a jumping number of an ideal $\mathfrak{a} \subseteq \mathcal{O}_{X,O}$ and G_λ its associated minimal jumping divisor. Then we have:*

i) *If $\lambda \leqslant 1$ then $G_\lambda \leqslant G_{\lambda+1}$.*
ii) *If $\lambda > 1$ then $G_\lambda = G_{\lambda+1}$.*

Proof. Assume that there exists a prime divisor $E_i \leqslant G_\lambda$ such that $E_i \nleqslant G_{\lambda+1}$. Then, for a sufficiently small $\varepsilon > 0$ we have

$$\lambda = \frac{k_i + 1 + e_i^{\lambda-\varepsilon}}{e_i} \quad \text{and} \quad \lambda + 1 < \frac{k_i + 1 + e_i^{(\lambda-\varepsilon)+1}}{e_i}$$

where $D_{\lambda-\varepsilon} = \sum e_i^{\lambda-\varepsilon} E_i$ denotes the antinef closure of $\lfloor(\lambda-\varepsilon)F - K_\pi\rfloor$ and equivalently, $D_{(\lambda-\varepsilon)+1} = \sum e_i^{(\lambda-\varepsilon)+1} E_i$ is the antinef closure of $\lfloor((\lambda-\varepsilon)+1)F - K_\pi\rfloor$.

Therefore

$$\frac{k_i + 1 + e_i^{\lambda-\varepsilon}}{e_i} + 1 < \frac{k_i + 1 + e_i^{(\lambda-\varepsilon)+1}}{e_i}$$

or equivalently $e_i^{\lambda-\varepsilon} + e_i < e_i^{(\lambda-\varepsilon)+1}$. Then we have $\mathfrak{a} \cdot \mathcal{J}(\mathfrak{a}^{\lambda-\varepsilon}) \nsubseteq \mathcal{J}(\mathfrak{a}^{(\lambda-\varepsilon)+1})$ so we get a contradiction.

For $\lambda > 1$ we have an equality $e_i^{\lambda-\varepsilon} + e_i = e_i^{(\lambda-\varepsilon)+1}$ because of Skoda's theorem so the result follows. □

Let $\lambda' < \lambda$ be two consecutive jumping numbers of an ideal $\mathfrak{a} \subseteq \mathcal{O}_{X,O}$. It is quite surprising that the minimal jumping divisor G_λ gives such nice approach to the understanding of the jump from $\mathcal{J}(\mathfrak{a}^\lambda)$ to its preceding multiplier ideal $\mathcal{J}(\mathfrak{a}^{\lambda'})$. Taking into account that its construction is based on Theorem 3.5, where λ is obtained from the antinef divisor associated to $\mathcal{J}(\mathfrak{a}^{\lambda'})$, it would seem more natural to consider the jump in the other direction. It turns out that the jump from $\mathcal{J}(\mathfrak{a}^{\lambda'})$ to $\mathcal{J}(\mathfrak{a}^\lambda)$ does not behave that nicely.

Proposition 4.9. *Let $\lambda' < \lambda$ be two consecutive jumping numbers of an ideal $\mathfrak{a} \subseteq \mathcal{O}_{X,O}$ and $D_{\lambda'}$ be the antinef closure of $\lfloor\lambda' F - K_\pi\rfloor$. Then we have:*

i) $\mathcal{J}(\mathfrak{a}^{\lambda'}) \supsetneq \pi_*\mathcal{O}_{X'}(-D_{\lambda'} - G_\lambda) = \mathcal{J}(\mathfrak{a}^\lambda)$.
ii) $\mathcal{J}(\mathfrak{a}^{\lambda'}) \supsetneq \pi_*\mathcal{O}_{X'}(\lceil K_\pi - (\lambda-\varepsilon)F\rceil - G_\lambda) = \mathcal{J}(\mathfrak{a}^\lambda)$

Proof. Let $D_{\lambda'} = \sum e_i^{\lambda'} E_i$, $D_\lambda = \sum e_i^\lambda E_i$ be the antinef closures of $\lfloor\lambda' F - K_\pi\rfloor$ and $\lfloor\lambda F - K_\pi\rfloor$ respectively.

i) Since G_λ is a jumping divisor we have $\lfloor\lambda F - K_\pi\rfloor - G_\lambda \leqslant D_{\lambda'}$, and hence $\lfloor\lambda F - K_\pi\rfloor \leqslant D_{\lambda'} + G_\lambda$. This gives the inclusion $\pi_*\mathcal{O}_{X'}(-D_{\lambda'} - G_\lambda) \subseteq \mathcal{J}(\mathfrak{a}^\lambda)$.

In order to check the reverse inclusion $\pi_*\mathcal{O}_{X'}(-D_{\lambda'} - G_\lambda) \supseteq \mathcal{J}(\mathfrak{a}^\lambda)$, it is enough, using Proposition 3.3, to prove $v_i(D_{\lambda'} + G_\lambda) \leqslant v_i(D_\lambda) = e_i^\lambda$ for any component E_i. We have $e_i^{\lambda'} \leqslant e_i^\lambda$ just because $\mathcal{J}(\mathfrak{a}^{\lambda'}) \supsetneq \mathcal{J}(\mathfrak{a}^\lambda)$ and the inequality is strict when $E_i \leqslant G_\lambda$, so the result follows.

ii) Let D' be the antinef closure of $\lfloor(\lambda-\varepsilon)F - K_\pi\rfloor + G_\lambda$. Since $G_\lambda \leqslant H_\lambda$ we have

$$\lfloor(\lambda-\varepsilon)F - K_\pi\rfloor + G_\lambda \leqslant \lfloor\lambda F - K_\pi\rfloor \leqslant D_\lambda$$

so the inclusion $\pi_*\mathcal{O}_{X'}(\lceil K_\pi - (\lambda-\varepsilon)F\rceil - G_\lambda) \supseteq \mathcal{J}(\mathfrak{a}^\lambda)$ holds. In order to prove the reverse inclusion we will introduce an auxiliary divisor $D = \sum d_i E_i \in \Lambda$ defined as follows:

- $d_i = \lfloor(\lambda-\varepsilon)e_i - k_i\rfloor + 1$ if $E_i \leqslant G_\lambda$,
- $d_i = e_i^{\lambda'}$ if $E_i \leqslant H_\lambda$ but $E_i \nleqslant G_\lambda$,
- $d_i = \lfloor(\lambda-\varepsilon)e_i - k_i\rfloor$ otherwise.

Clearly we have $\lfloor(\lambda-\varepsilon)F - K_\pi\rfloor + G_\lambda \leqslant D$, but we also have $\lfloor\lambda F - K_\pi\rfloor \leqslant D$. Indeed,

- For $E_i \leqslant G_\lambda$ we have $\lfloor\lambda e_i - k_i\rfloor = \lambda e_i - k_i = \lfloor(\lambda-\varepsilon)e_i - k_i\rfloor + 1 = d_i$.
- If λ is a candidate for E_i but $E_i \nleqslant G_\lambda$, $\lfloor\lambda e_i - k_i\rfloor = \lambda e_i - k_i < 1 + e_i^{\lambda'}$, hence $\lfloor\lambda e_i - k_i\rfloor \leqslant e_i^{\lambda'} = d_i$.
- Otherwise $\lfloor\lambda e_i - k_i\rfloor = \lfloor(\lambda-\varepsilon)e_i - k_i\rfloor = d_i$.

Therefore, taking antinef closures, we have $D' \leqslant D_\lambda \leqslant \widetilde{D}$. On the other hand $D \leqslant D'$. Namely, $v_i(D') \geqslant e_i^{\lambda'}$ at any E_i because $\lfloor\lambda' F - K_\pi\rfloor \leqslant \lfloor(\lambda-\varepsilon)F - K_\pi\rfloor + G_\lambda$. Moreover, $v_i(D') \geqslant \lfloor(\lambda-\varepsilon)e_i - k_i\rfloor + \delta_i^{G_\lambda}$ by definition of antinef closure. Here, $\delta_i^{G_\lambda} = 1$ if $E_i \leqslant G_\lambda$ and zero otherwise. Thus $v_i(D') \geqslant v_i(D)$ as desired. As a consquence $\widetilde{D} \leqslant D'$, which together with the previous $D' \leqslant D_\lambda \leqslant \widetilde{D}$, gives $\widetilde{D} = D' = D_\lambda$ and the result follows. □

Remark 4.10. Contrary to the case of Theorem 4.5, G_λ may not be minimal. In fact, we will see in Example 5.8 a divisor $G < G_\lambda$ satisfying:

$$\mathcal{J}(\mathfrak{a}^{\lambda'}) = \pi_*\mathcal{O}_{X'}(-D_{\lambda'}) \supsetneq \pi_*\mathcal{O}_{X'}(-D_{\lambda'} - G) = \mathcal{J}(\mathfrak{a}^\lambda)\,.$$

Despite the fact that the antinef closure of both $\lfloor(\lambda-\varepsilon)F - K_\pi\rfloor$ and $\lfloor\lambda' F - K_\pi\rfloor$ is $D_{\lambda'}$, it is quite remarkable that the above jumping property does not hold taking $\lfloor\lambda' F - K_\pi\rfloor$, i.e. the equality $\pi_*\mathcal{O}_{X'}(\lfloor\lambda' F - K_\pi\rfloor - G_\lambda) = \mathcal{J}(\mathfrak{a}^\lambda)$ is not always true.

4.1. **Invariance of the minimal jumping divisor with respect to the log-resolution.** Multiplier ideals and jumping numbers are known to be independent of the chosen log-resolution of the initial ideal $\mathfrak{a} \subseteq \mathcal{O}_{X,O}$. The aim of this section is to prove that the minimal jumping divisor is *generically* independent of the log-resolution in a sense that we will make precise below. As a consequence of Proposition 5.6 and Corollary 5.5 in Section §5, critical divisors will also be generically independent of the log-resolution. This is a remarkable fact since, as it was pointed out by Tucker in [25, Remark 3.4], there is no reason to believe that critical divisors (and by extension minimal jumping divisors) are independent of the resolution since they depend on all the divisorial valuations appearing in F.

We start fixing some notation that we will use in this section. Let $\pi' : X' \longrightarrow X$ be the minimal log-resolution of an ideal $\mathfrak{a} \subseteq \mathcal{O}_{X,O}$. Any other log-resolution $\pi : Y \longrightarrow X$ factors through π', i.e. there is a birational morphism $g : Y \longrightarrow X'$ such that $\pi = \pi' \circ g$ (see [19, Theorem 4.1]).

For a given jumping number λ of $\mathfrak{a}$ we will denote G'_λ the minimal jumping divisor of π' and $E'_1, \ldots, E'_r$ the exceptional components of $E' = Exc(\pi')$. If G_λ and $E_1, \ldots, E_s$ are the minimal jumping divisor and the exceptional components of $E = Exc(\pi)$ for any other log-resolution π, we will enumerate them setting E_i equal to the strict transform by g of E'_i for $1 \leqslant i \leqslant r$. If no confusion arise, we will use the same symbol to denote a divisor $D = \sum_{i=1}^r d_i E'_i$ on X' or its strict transform $D = \sum_{i=1}^r d_i E_i$ on Y.

Theorem 4.11. *With the previous notations, G_λ is independent of the log-resolution π if and only if π does not include any blowing-up at points in the intersection of two components of the minimal jumping divisor G'_λ of the minimal log-resolution.*

Actually, from the proof of this result, we can express the minimal jumping divisor of any resolution. To such purpose we need to fix some notation:

A reduced divisor with exceptional support $D = E_{i_1} + \cdots + E_{i_m} \leqslant E$ is a *chain with ends* E_{i_1} and E_{i_m} if $a_D(E_{i_1}) = a_D(E_{i_m}) = 1$ and $a_D(E_{i_k}) = 2$ for any other $1 < k < m$. Given $E_{j_1}, E_{j_2} \leqslant E$, we say that the chain above *connects* E_{j_1} and E_{j_2} if $E_{j_1} \in \mathrm{Adj}\,(E_{i_1})$ and $E_{j_2} \in \mathrm{Adj}\,(E_{i_m})$. Observe that if E_{j_1} and E_{j_2} are adjacent in E, a chain connecting them will be $D = 0$.

Corollary 4.12. *Keeping the above notations we have*

$$G_\lambda = G'_\lambda + \sum_{\substack{E'_i + E'_j \leqslant G'_\lambda \\ E'_i \cdot E'_j = 1}} D_{ij} \tag{4.1}$$

where D_{ij} is a chain connecting E_i and E_j.

Consider *generic* log-resolutions as those obtained from a minimal one by further blowing-ups at simple (and hence generic) points on the exceptional components. Then, Theorem 4.11 states that generic log-resolutions have the same minimal jumping divisor. This *generictiy* may be formulated, when X is smooth, in terms of valuations in the valuative tree $\mathcal{V}$ of Favre-Jonsson [8]. Consider the dual graphs Γ and Γ' of E and E' respectively, embedded in the valuative tree $\mathcal{V}$ as in [8, Chapter 6] and let ν_i denote the divisorial valuation centered at E_i.

Corollary 4.13. *The minimal jumping divisor G_λ of π equals the minimal jumping divisor G'_λ if and only if Γ has no vertex inside any segment $]\nu_i, \nu_j[$ for which E'_i and E'_j are adjacent in E' and belong to G'_λ.*

Proof of Theorem 4.11. Let $\lambda' < \lambda$ be two consecutive jumping numbers of $\mathfrak{a}$. We will argue by induction on the number of blowing-ups needed to reach Y from a minimal resolution. In order to simplify the notation, we will assume throughout this proof that X' also dominates a minimal log-resolution and that Y is obtained from X' by one blowing-up $g : Y \longrightarrow X'$ at a closed point $p \in X'$ giving the exceptional component E_s. Assume that (4.1) holds on X' and let us prove it on Y. Notice that, keeping the notation used in this section, we are in the case $r + 1 = s$.

Let $F' = \sum_{i=1}^{r} e_i E'_i$ and $F = \sum_{i=1}^{s} e_i E_i$ be the divisors in X' and Y respectively such that $\mathfrak{a}\mathcal{O}_{X'} = \mathcal{O}_{X'}(-F')$ and $\mathfrak{a}\mathcal{O}_Y = \mathcal{O}_Y(-F)$. We also consider the antinef divisors $D'_{\lambda'} = \sum_{i=1}^{r} e_i^{\lambda'} E'_i$ and $D_{\lambda'} = \sum_{i=1}^{s} e_i^{\lambda'} E_i$ for which $\mathcal{J}(\mathfrak{a}^{\lambda'}) = \pi'_* \mathcal{O}_{X'}(-D'_{\lambda'}) = \pi_* \mathcal{O}_Y(-D_{\lambda'})$ sharing the first r coefficients since multiplier ideals are independent of the log-resolution. Moreover, by Theorem 3.5

$$\lambda = \min_{1 \leqslant i \leqslant r} \left\{ \frac{k_i + 1 + e_i^{\lambda'}}{e_i} \right\} = \min_{1 \leqslant i \leqslant s} \left\{ \frac{k_i + 1 + e_i^{\lambda'}}{e_i} \right\},$$

clearly demonstrating that the strict transform of G'_λ is contained in G_λ. In particular, $\lambda e_i - k_i = 1 + e_i^{\lambda'}$ if and only if $E_i \leqslant G_\lambda$ and $\lambda e_i - k_i < 1 + e_i^{\lambda'}$ otherwise.

We distinguish two cases:

i) The closed point p lies only on one exceptional divisor E'_j. Then we have $e_s = e_j$, $k_s = k_j + 1$ and $e_s^{\lambda'} = e_j^{\lambda'}$ and thus

$$v_s(\lfloor \lambda F - K_\pi \rfloor) = \lfloor \lambda e_s - k_i \rfloor = \lfloor \lambda e_j - k_j \rfloor - 1 \leqslant e_j^{\lambda'} = e_s^{\lambda'}.$$

Hence E_s can not belong to G_λ.

ii) The closed point p lies on the intersection of two exceptional divisors E'_{j_1} and E'_{j_2}. Then we have $e_s = e_{j_1} + e_{j_2}$, $k_s = k_{j_1} + k_{j_2} + 1$ and $e_s^{\lambda'} = e_{j_1}^{\lambda'} + e_{j_2}^{\lambda'}$ so

$$v_s(\lfloor \lambda F - K_\pi \rfloor) = \lfloor \lambda e_s - k_s \rfloor = \lfloor \lambda e_{j_1} - k_{j_1} + \lambda e_{j_2} - k_{j_2} \rfloor - 1 \leqslant e_{j_1}^{\lambda'} + e_{j_2}^{\lambda'} + 1 = e_s^{\lambda'} + 1,$$

and equality holds if and only if $E'_{j_1} + E'_{j_2} \leqslant G_\lambda$. In particular, E_s does not belong to G_λ whenever none or just one of the components E'_{j_1}, E'_{j_2} belong to G'_λ. □

4.2. **Geometric properties of minimal jumping divisors in the dual graph.** Assume that a critical divisor G associated to a jumping number λ has exceptional support. One of the key ingredients in Tucker's algorithm for the computation of jumping numbers is that G satisfies some nice geometric conditions when viewed in the dual graph: G is a connected chain and its ends must be either rupture or dicritical divisors (see Proposition 2.13). Then, it is natural to ask whether jumping divisors satisfy analogous properties.

Throughout this section we will also assume that the minimal jumping divisor G_λ has exceptional support. Then, it may have several connected components in the dual graph and these components are not necessarily chains. However, we can still control the ends of each component. To prove the main result of this section (see Theorem 4.17) we need some preliminary results first. Keep the notations of Section §2.

Lemma 4.14. *Let λ be a jumping number of an ideal $\mathfrak{a} \subseteq \mathcal{O}_{X,O}$. For any component E_i of the minimal jumping divisor G_λ we have*

$$(\lceil K_\pi - \lambda F \rceil + G_\lambda) \cdot E_i = -2 + \lambda \rho_i + \sum_{E_j \in \operatorname{Adj}(E_i)} \{\lambda e_j - k_j\} + a_{G_\lambda}(E_i).$$

Proof. For any $E_i \leqslant G_\lambda$ we have

$$\begin{aligned}(\lceil K_\pi - \lambda F \rceil + G_\lambda) \cdot E_i &= ((K_\pi - \lambda F) + \{-K_\pi + \lambda F\} + G_\lambda - E_i + E_i) \cdot E_i = \\ &= (K_\pi + E_i) \cdot E_i - \lambda F \cdot E_i + \{\lambda F - K_\pi\} \cdot E_i + (G_\lambda - E_i) \cdot E_i.\end{aligned}$$

Let us now compute each summand separately. Firstly, the adjunction formula gives $(K_\pi + E_i) \cdot E_i = -2$ because $E_i \cong \mathbb{P}^1$. As for the second and fourth terms, the equality $-\lambda F \cdot E_i = \lambda \rho_i$ follows from the definition of the excesses, and clearly $a_{G_\lambda}(E_i) = (G_\lambda - E_i) \cdot E_i$ because $E_i \leqslant G_\lambda$.

Therefore it only remains to prove that

$$\{\lambda F - K_\pi\} \cdot E_i = \sum_{E_j \in \operatorname{Adj}(E_i)} \{\lambda e_j - k_j\}, \tag{4.2}$$

which is also quite immediate. Indeed, writing

$$\{\lambda F - K_\pi\} = \sum_{j=1}^{r} \{\lambda e_j - k_j\} E_j,$$

equality (4.2) follows by observing that (for $j \neq i$), $E_j \cdot E_i = 1$ if and only if $E_j \in \operatorname{Adj}(E_i)$, and the term corresponding to $j = i$ vanishes because we have $\lambda e_i - k_i \in \mathbb{Z}$. ☐

Remark 4.15. It is important to notice that $(\lceil K_\pi - \lambda F \rceil + G_\lambda) \cdot E_i \in \mathbb{Z}$, that is $-2 + \sum_{E_j \in \operatorname{Adj}(E_i)} \{\lambda e_j - k_j\} + \lambda \rho_i + a_{G_\lambda}(E_i) \in \mathbb{Z}$.

The following result is an analogue of the numerical conditions that critical divisors satisfy (see Proposition 4.19). Unfortunately it does not provide a characterization of minimal jumping divisors.

Proposition 4.16. *Let λ be a jumping number of an ideal $\mathfrak{a} \subseteq \mathcal{O}_{X,O}$. For any component $E_i \leqslant G_\lambda$ of the minimal jumping divisor G_λ we have*

$$(\lceil K_\pi - \lambda F \rceil + G_\lambda) \cdot E_i \geqslant 0.$$

Proof. Let G_λ be the minimal jumping divisor. Given a prime divisor $E_i \leqslant G_\lambda$ we consider the short exact sequence

$$\begin{aligned}0 \longrightarrow \mathcal{O}_{X'}(\lceil K_\pi - \lambda F \rceil + G_\lambda - E_i) \longrightarrow \mathcal{O}_{X'}(\lceil K_\pi - \lambda F \rceil + G_\lambda) \longrightarrow \\ \longrightarrow \mathcal{O}_{E_i}(\lceil K_\pi - \lambda F \rceil + G_\lambda) \longrightarrow 0\end{aligned}$$

Pushing it forward to X we get

$$\begin{aligned}0 \longrightarrow \pi_* \mathcal{O}_{X'}(\lceil K_\pi - \lambda F \rceil + G_\lambda - E_i) \longrightarrow \pi_* \mathcal{O}_{X'}(\lceil K_\pi - \lambda F \rceil + G_\lambda) \longrightarrow \\ \longrightarrow H^0(E_i, \mathcal{O}_{E_i}(\lceil K_\pi - \lambda F \rceil + G_\lambda)) \otimes \mathbb{C}_O,\end{aligned}$$

where $\mathbb{C}_O$ denotes the skyscraper sheaf supported at O with fibre $\mathbb{C}$. The minimality of G_λ (see Theorem 4.5) implies that

$$\pi_* \mathcal{O}_{X'} \left(\lceil K_\pi - \lambda F \rceil + G_\lambda - E_i \right) \neq \pi_* \mathcal{O}_{X'} \left(\lceil K_\pi - \lambda F \rceil + G_\lambda \right).$$

Thus $H^0 \left(E_i, \mathcal{O}_{E_i} \left(\lceil K_\pi - \lambda F \rceil + G_\lambda \right) \right) \neq 0$, or equivalently $\left(\lceil K_\pi - \lambda F \rceil + G_\lambda \right) \cdot E_i \geqslant 0$. □

With the above ingredients we can provide the following geometric property of minimal jumping divisors when viewed in the dual graph.

Theorem 4.17. *Let G_λ be the minimal jumping divisor associated to a jumping number λ of an ideal $\mathfrak{a} \subseteq \mathcal{O}_{X,O}$. Then the ends of a connected component of G_λ must be either rupture or dicritical divisors.*

Proof. Assume that an end E_i of a connected component of G_λ is neither a rupture nor a dicritical divisor. It means that E_i has no excess, i.e. $\rho_i = 0$, and that it has one or two adjacent divisors, say E_j and E_l, in the dual graph but at most one of them belongs to G_λ.

For the case that E_i has two adjacent divisors E_j and E_l the formula given in lemma 4.14 reduces to $\left(\lceil K_\pi - \lambda F \rceil + G_\lambda \right) \cdot E_i = -2 + \{\lambda e_j - k_j\} + \{\lambda e_l - k_l\} + \lambda \rho_i + a_{G_\lambda}(E_i)$. Then:

- If E_i has valence one in G_λ, e.g. $E_l \nleqslant G_\lambda$ then

$$\left(\lceil K_\pi - \lambda F \rceil + G_\lambda \right) \cdot E_i = -2 + \{\lambda e_l - k_l\} + 1 < 0.$$

- If E_i is an isolated component of G_λ, i.e., $E_j, E_l \nleqslant G_\lambda$ then

$$\left(\lceil K_\pi - \lambda F \rceil + G_\lambda \right) \cdot E_i = -2 + \{\lambda e_j - k_j\} + \{\lambda e_l - k_l\} < 0.$$

If E_i has just one adjacent divisor E_j, i.e. E_i is an end of the dual graph, the formula reduces to $\left(\lceil K_\pi - \lambda F \rceil + G_\lambda \right) \cdot E_i = -2 + \{\lambda e_j - k_j\} + \lambda \rho_i + a_{G_\lambda}(E_i)$. Then:

- If E_i has valence one in G_λ then $\left(\lceil K_\pi - \lambda F \rceil + G_\lambda \right) \cdot E_i = -2 + 1 < 0$
- If E_i is an isolated component of G_λ then

$$\left(\lceil K_\pi - \lambda F \rceil + G_\lambda \right) \cdot E_i = -2 + \{\lambda e_j - k_j\} < 0.$$

In any case we get a contradiction with Proposition 4.16. □

Remark 4.18. It follows from [26, Theorem 3.3] that the minimal jumping divisor associated to the log-canonical threshold is connected in the case that X is smooth.

As a consequence we may also give the following refinement of Proposition 4.16.

Proposition 4.19. *Let λ be a jumping number of an $\mathfrak{m}$-primary ideal $\mathfrak{a} \subseteq \mathcal{O}_{X,O}$. If $E_i \leqslant G_\lambda$ is neither a rupture nor a dicritical component of the minimal jumping divisor G_λ we have*

$$\left(\lceil K_\pi - \lambda F \rceil + G_\lambda \right) \cdot E_i = 0.$$

Proof. Assume that $E_i \leqslant G_\lambda$ is neither a rupture or a dicritical component. In particular, it is not the end of a connected component of G_λ. Thus, E_i has exactly two adjacent components E_j and E_l in G_λ, and its excess is $\rho_i = 0$. The formula given in Lemma 4.14 reduces to

$$(\lceil K_\pi - \lambda F \rceil + G_\lambda) \cdot E_i = -2 + \lambda \rho_i + \{\lambda e_j - k_j\} + \{\lambda e_l - k_l\} + a_{G_\lambda}(E_i).$$

Notice that $a_{G_\lambda}(E_i) = 2$, and also $\{\lambda e_j - k_j\} = \{\lambda e_l - k_l\} = 0$ because E_j and E_l are components of G_λ, so finally $(\lceil K_\pi - \lambda F \rceil + G_\lambda) \cdot E_i = 0$. ☐

5. Minimal contributing divisors

The theory of minimal jumping divisors introduced in Section §4 can be included in a more general framework that we will describe in this section. To such purpose we will give our own perspective of the work of Hyry-Järviletho [14] and its relation with the theory of contributing divisors of Tucker [25].

Let λ be a jumping number of an ideal $\mathfrak{a} \subseteq \mathcal{O}_{X,O}$. Recall that a reduced divisor $G \leqslant F$ that contributes to λ defines an ideal nested between two consecutive multiplier ideals

$$\mathcal{J}(\mathfrak{a}^{\lambda-\varepsilon}) \supseteq \pi_* \mathcal{O}_{X'}(\lceil K_\pi - \lambda F \rceil + G) \supsetneq \mathcal{J}(\mathfrak{a}^{\lambda}).$$

We may interpret that λ is *parametrized* by the set of nested ideals defined by contributions but this is far from being a one-to-one correspondence. An easy way to detect such a nested ideal is finding a suitable critical divisor using Tucker's algorithm. The approach given in the previous sections is more economical in the sense that each jumping number is parametrized by its unique minimal jumping divisor G_λ or equivalently, its preceding multiplier ideal.

Hyry-Järviletho [14] give a similar approach where jumping numbers are parametrized by general antinef divisors[8], or equivalently complete ideals not necessarily nested in the chain of multiplier ideals. We should point out that their results also hold for the case that X has rational singularities since their arguments are based on divisorial considerations. Given any antinef divisor $D = \sum d_i E_i \in \mathrm{Div}(X')$, they considered the following notions:

· *Jumping number corresponding to D:*

$$\lambda_D := \min_i \left\{ \frac{k_i + 1 + d_i}{e_i} \right\}.$$

· *Support of a jumping number corresponding to D:*

$$S_D := \left\{ i \;\middle|\; \lambda_D = \frac{k_i + 1 + d_i}{e_i} \right\}.$$

· *Contributing divisor associated to D:*

$$G_D := \sum_{i \in S_D} E_i.$$

[8]Hyry-Järviletho only consider the case of $\mathfrak{m}$-primary ideals on smooth surfaces and consequently antinef divisors with exceptional support but their ideas also hold in general

Hyry-Järviletho proved in [14, Proposition 1] that all jumping numbers of $\mathfrak{a}$ can be obtained in this way: as λ_D for a suitable antinef divisor $D \in \mathrm{Div}(X')$ (or equivalently a complete ideal I_D). Moreover, they give in [14, Theorem 1] a combinatorial criterion that detects the existence of such antinef divisors. The simplest parametrizations they used to describe the set of jumping numbers are given by antinef divisors corresponding to critical divisors (see [14, Theorem 2]).

In general, the complete ideal I_D associated to an antinef divisor $D \in \mathrm{Div}(X')$ satisfies $\mathcal{J}(\mathfrak{a}^{\lambda_D - \varepsilon}) \supseteq I_D$ but does not necessarily contain $\mathcal{J}(\mathfrak{a}^{\lambda_D})$. However, if I_D is nested in between two consecutive multiplier ideals

$$\mathcal{J}(\mathfrak{a}^{\lambda-\varepsilon}) \supseteq I_D \supsetneq \mathcal{J}(\mathfrak{a}^{\lambda})$$

then it must satisfy $\lambda = \lambda_D$.

Remark 5.1. One can also interpret this framework through the generalized version of log-canonical thresholds already introduced by Järviletho in [15]. Namely, the *log-canonical threshold* with respect to any other ideal $\mathfrak{b} \subseteq \mathcal{O}_{X,O}$ is defined as follows:

$$\mathrm{lct}_{\mathfrak{b}}(\mathfrak{a}) := \inf\{c \in \mathbb{Q}_{>0} \mid \mathcal{J}(\mathfrak{a}^c) \not\supset \mathfrak{b}\}$$

Notice that whenever I_D is the complete ideal associated to an antinef divisor $D \in \mathrm{Div}(X')$, then $\lambda_D = \mathrm{lct}_{I_D}(\mathfrak{a})$.

Hyry-Järviletho [14, Lemma 11] proved that if $D \in \mathrm{Div}(X')$ is an antinef divisor then G_D is a contributing divisor for λ_D. In fact, the contributing divisors obtained in this way satisfy some nice properties as we will see next.

Proposition 5.2. *Let G be a contributing divisor associated to a jumping number λ. Let D be the antinef closure of $\lfloor \lambda F - K_\pi \rfloor - G$. Then $G_D \leqslant G$.*

Proof. Let $D = \sum d_i E_i$ be the antinef closure of $\lfloor \lambda F - K_\pi \rfloor - G$. Since I_D is a nested ideal in the chain of multiplier ideals, then we have

$$\lambda = \lambda_D = \min_i \left\{ \frac{k_i + 1 + d_i}{e_i} \right\}.$$

Hence $\lambda e_i - k_i \leqslant 1 + d_i$ and equality holds if and only if $i \in S_D$. In order to prove $G_D \leqslant G$ we will show that $E_i \nleqslant G$ implies $E_i \nleqslant G_D$. Indeed, if $E_i \nleqslant G$ and $E_i \leqslant G_D$ then $\lfloor \lambda e_i - k_i \rfloor \leqslant d_i$ (just because $\lfloor \lambda F - K_\pi \rfloor - G \leqslant D$ by Lemma 2.3) and $\lambda e_i - k_i - 1 = d_i$ so we get a contradiction. □

Proposition 5.3. *Let $\lambda = \lambda_{D'}$ be a jumping number associated to an antinef divisor $D' \in \mathrm{Div}(X')$. Let D be the antinef closure of $\lfloor \lambda F - K_\pi \rfloor - G_{D'}$. Then we have $D \leqslant D'$, $\lambda_D = \lambda_{D'}$, $S_D = S_{D'}$ and $G_D = G_{D'}$.*

Proof. Using the definition of antinef closure (see Lemma 2.3), in order to get $D \leqslant D'$ we only need to prove that $\lfloor \lambda F - K_\pi \rfloor - G_{D'} \leqslant D'$. Set $D' = \sum d'_i E_i$. By hypothesis

$$\lambda = \lambda_{D'} = \min_i \left\{ \frac{k_i + 1 + d'_i}{e_i} \right\}$$

therefore we have $\lfloor \lambda e_i - k_i \rfloor \leqslant d'_i$ if $i \notin S_{D'}$, whereas $\lfloor \lambda e_i - k_i \rfloor - 1 = d'_i$ if $i \in S_{D'}$ as desired.

Notice then that we have $\mathcal{J}(\mathfrak{a}^{\lambda-\varepsilon}) \supseteq I_D \supseteq I_{D'}$ so, given the fact that $I_{D'} \nsubseteq \mathcal{J}(\mathfrak{a}^{\lambda})$, we get $\lambda_D = \lambda$. Now, the inclusion of divisors $D \leqslant D'$ having the same minimum $\lambda_D = \lambda_{D'}$, gives the inclusion of supports $S_D \supseteq S_{D'}$ and equivalently $G_D \geqslant G_{D'}$. On the other hand, taking $G = G_{D'}$ in Proposition 5.2, we get the reverse inequality of divisors $G_D \leqslant G_{D'}$ so we are done. □

The main result of this section is that we can find a minimal contributing divisor among all contributing divisors defining the same nested ideal.

Theorem 5.4. *Let G be a contributing divisor associated to a jumping number λ. Let D be the antinef closure of $\lfloor \lambda F - K_\pi \rfloor - G$, which gives a nested ideal*

$$\mathcal{J}(\mathfrak{a}^{\lambda-\varepsilon}) \supseteq I_D = \pi_* \mathcal{O}_{X'}\left(\lceil K_\pi - \lambda F \rceil + G\right) \supsetneq \mathcal{J}(\mathfrak{a}^{\lambda}).$$

Then we also have $I_D = \pi_* \mathcal{O}_{X'}\left(\lceil K_\pi - \lambda F \rceil + G_D\right)$. Furthermore, G_D is the minimal contributing divisor associated to λ that defines the same ideal I_D, that is:

- *Any contribution G' to λ defining $I_D = \pi_* \mathcal{O}_{X'}\left(\lceil K_\pi - \lambda F \rceil + G'\right)$ must satisfy $G_D \leqslant G'$.*
- *Any proper subdivisor $G' < G_D$ defines an strictly included ideal*

$$I_D \supsetneq \pi_* \mathcal{O}_{X'}\left(\lceil K_\pi - \lambda F \rceil + G'\right).$$

Proof. Let D' be the antinef closure of $\lfloor \lambda F - K_\pi \rfloor - G_D$. We will see first that $D = D'$ thus giving the desired equality of ideals

$$I_D = \pi_* \mathcal{O}_{X'}\left(\lceil K_\pi - \lambda F \rceil + G\right) = \pi_* \mathcal{O}_{X'}\left(\lceil K_\pi - \lambda F \rceil + G_D\right) = I_{D'}.$$

In virtue of Proposition 5.2, we have $G_D \leqslant G$ so $\lfloor \lambda F - K_\pi \rfloor - G \leqslant \lfloor \lambda F - K_\pi \rfloor - G_D$ and $D \leqslant D'$. The reverse inequality $D \geqslant D'$ is a consequence of Proposition 5.3.

To show that G_D is the minimal contributor to the jumping number λ that defines the same ideal I_D we will prove the following equivalent result:

Claim: Any contributor G' to λ for which $I_D \supseteq \pi_* \mathcal{O}_{X'}\left(\lceil K_\pi - \lambda F \rceil + G'\right)$ also satisfies the reverse inclusion $I_D \subseteq \pi_* \mathcal{O}_{X'}\left(\lceil K_\pi - \lambda F \rceil + G'\right)$ if and only if $G_D \leqslant G'$.

Proof of Claim: Suppose first that $G_D \leqslant G'$. Then $\lfloor \lambda F - K_\pi \rfloor - G' \leqslant \lfloor \lambda F - K_\pi \rfloor - G_D$ and hence $D'' \leqslant D' = D$, where D'' is the antinef closure of $\lfloor \lambda F - K_\pi \rfloor - G'$. Therefore $I_D \subseteq I_{D''}$ as wanted.

Assume now that $G_D \nleqslant G'$ and pick a component $E_i \leqslant G_D$ such that $E_i \nleqslant G'$. By hypothesis $I_D \supseteq I_{D''}$ and equivalently $D \leqslant D''$ but in fact $D < D''$ since

$$v_i(D) = \lambda e_i - k_i - 1 < \lambda e_i - k_i = v_i(\lfloor \lambda F - K_\pi \rfloor - G') \leqslant v_i(D'').$$

The result follows then from Proposition 3.3. □

It turns out that critical divisors are also minimal in the above sense as we can see in the following generalization of [14, Proposition 3].

Corollary 5.5. *Let G be a contributing divisor associated to a jumping number λ. Let D be the antinef closure of $\lfloor \lambda F - K_\pi \rfloor - G$. Then G is a critical divisor if and only if $G_D = G$ and I_D and $\mathcal{J}(\mathfrak{a}^\lambda)$ do not admit strictly nested ideals between them defined by contributors to λ.*

Proof. Assume first that $G_D = G$. Then, by Theorem 5.4, any proper subdivisor $0 \leqslant G' < G$ defines an ideal strictly included in $I_D \supsetneq \pi_* \mathcal{O}_{X'}(\lceil K_\pi - \lambda F \rceil + G') \supseteq \mathcal{J}(\mathfrak{a}^\lambda)$. Since I_D and $\mathcal{J}(\mathfrak{a}^\lambda)$ do not admit strictly nested ideals between them coming from contributors, we get $\pi_* \mathcal{O}_{X'}(\lceil K_\pi - \lambda F \rceil + G') = \mathcal{J}(\mathfrak{a}^\lambda)$ so G is a critical divisor.

Assume now that G is a critical divisor. By Proposition 5.2 we have $G_D \leqslant G$. Both divisors define the same ideal by Theorem 5.4 so they must be equal otherwise we would have a contradiction with the fact that G is a critical divisor.

Finally we will see that there is no contributing divisor G' associated to λ defining a strictly nested ideal

$$I_D \supsetneq \pi_* \mathcal{O}_{X'}(\lceil K_\pi - \lambda F \rceil + G') \supsetneq \mathcal{J}(\mathfrak{a}^\lambda).$$

Assume that such G' exists and let D' be the antinef closure of $\lfloor \lambda F - K_\pi \rfloor - G'$. Then the inclusion of divisors $D < D'$ having the same minimum $\lambda_D = \lambda_{D'} = \lambda$ implies $S_{D'} \subseteq S_D$ and $G_{D'} \leqslant G_D$. Since $G = G_D$ is minimal, applying Theorem 5.4, we must have $G = G_D = G_{D'} \leqslant G'$ contradicting the starting hypothesis of inclusion of ideals. ☐

The minimal jumping divisor introduced in Section §4 fits nicely in this theory. Given a jumping number λ of an $\mathfrak{m}$-primary ideal $\mathfrak{a} \subseteq \mathcal{O}_{X,O}$, let $D_{\lambda-\varepsilon}$ be the antinef closure of $\lfloor (\lambda - \varepsilon) F - K_\pi \rfloor$ for $\varepsilon > 0$ small enough. Then we have $\lambda = \lambda_{D_{\lambda-\varepsilon}}$ and the unique minimal jumping divisor is $G_\lambda = G_{D_{\lambda-\varepsilon}}$.

In general, a divisor $G \in \Lambda$ that contributes to the jumping number λ might not be contained in G_λ. For minimal contributing divisors we have the following:

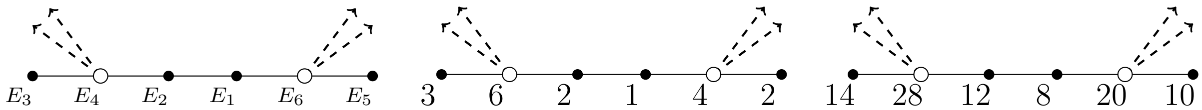
Proposition 5.6. *Let λ be a jumping number of an ideal $\mathfrak{a} \subseteq \mathcal{O}_{X,O}$ and G_λ be its associated minimal jumping divisor. Then $G_D \leqslant G_\lambda$ for any antinef divisor $D \in \mathrm{Div}(X')$ such that $\lambda = \lambda_D$.*

Proof. Let D' be the antinef closure of $\lfloor \lambda F - K_\pi \rfloor - G_D$. By Proposition 5.3 we have $G_D = G_{D'}$ and $\lambda = \lambda_D = \lambda_{D'}$. Since the ideals $\mathcal{J}(\mathfrak{a}^{\lambda-\varepsilon}) \supseteq I_{D'}$ are nested, their corresponding antinef divisors satisfy $D_{\lambda-\varepsilon} \leqslant D'$ and they reach the same minimum $\lambda_{D_{\lambda-\varepsilon}} = \lambda_{D'} = \lambda$. Hence, $S_{D'} \subseteq S_{D_{\lambda-\varepsilon}}$ which implies $G_D = G_{D'} \leqslant G_\lambda$ as we wanted. ☐

Corollary 5.7. *Let λ be a jumping number of an ideal $\mathfrak{a} \subseteq \mathcal{O}_{X,O}$. Then we have $G \leqslant G_\lambda$ for any critical divisor G associated to λ.*

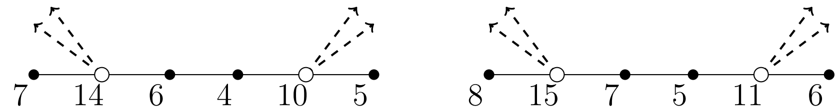
The reduced sum of all critical divisors equals the jumping divisor G_λ for simple complete ideals (see [10, Thm. 2.3] for the smooth case). However this is no longer true in general.

Example 5.8. Let X be a smooth surface and consider the $\mathfrak{m}$-primary ideal $\mathfrak{a} \subseteq \mathcal{O}_{X,O}$ whose dual graph is



Vertex ordering $\qquad$ K_π $\qquad$ F

The multiplier ideals corresponding to the consecutive jumping numbers $\frac{5}{7} < \frac{3}{4}$ are:



$\mathcal{J}(\mathfrak{a}^{\frac{5}{7}}) \quad \supseteq \quad \mathcal{J}(\mathfrak{a}^{\frac{3}{4}})$

The minimal jumping divisor corresponding to $\lambda = \frac{3}{4}$ is $G_{\frac{3}{4}} = E_1 + E_2 + E_4 + E_6$ but the only critical divisors are E_4 and E_6. In particular

$$\mathcal{J}(\mathfrak{a}^{\frac{5}{7}}) \supsetneq \pi_* \mathcal{O}_{X'}(\lceil K_\pi - \frac{3}{4}F \rceil + E_4 + E_6).$$

It is worth pointing out that

$$\pi_* \mathcal{O}_{X'}(-D_{\frac{5}{7}} - E_4 - E_6) = \pi_* \mathcal{O}_{X'}(-D_{\frac{5}{7}} - G_{\frac{3}{4}}) = \mathcal{J}(\mathfrak{a}^{\frac{3}{4}})$$

where $D_{\frac{5}{7}}$ is the antinef closure of $\lfloor \frac{5}{7}F - K_\pi \rfloor$. So minimality is not always achieved for the divisor G_λ in Proposition 4.9.

In general, not every nested ideal between two consecutive multiplier ideals is given by a contributing divisor. The following result identifies them precisely.

Proposition 5.9. *Any nested ideal $\mathcal{J}(\mathfrak{a}^{\lambda-\varepsilon}) \supseteq I_{D'} \supsetneq \mathcal{J}(\mathfrak{a}^{\lambda})$ comes from a contributing divisor G associated to λ, i.e. $I_{D'} = \pi_* \mathcal{O}_{X'}(\lceil K_\pi - \lambda F \rceil + G)$, if and only if $D' = D$ where D is the antinef closure of $\lfloor \lambda F - K_\pi \rfloor - G$ and in this case $G = G_{D'}$.*

Proof. Let D' be the antinef closure of $\lfloor \lambda F - K_\pi \rfloor - G$. By Proposition 5.3 we have $D \leqslant D'$. On the other hand, Proposition 5.2 implies $G_{D'} \leqslant G$ which gives

$$\lfloor \lambda F - K_\pi \rfloor - G \leqslant \lfloor \lambda F - K_\pi \rfloor - G_{D'}$$

and hence $D' \leqslant D$ so we get the desired result. The reverse implication is straightforward. ☐

Proposition 5.10. *Let I_D be the ideal associated to an antinef divisor $D \in \Lambda$. Then, I_D is a multiplier ideal for the ideal $\mathfrak{a} \subseteq \mathcal{O}_{X,O}$ if and only if D is contained in the antinef closure of $\lfloor (\lambda_D - \varepsilon)F - K_\pi \rfloor$. If this is the case, D is also the antinef closure of $\lfloor \lambda_D F - K_\pi \rfloor - G_D$.*

Proof. By definition, we have $\lfloor (\lambda_D - \varepsilon)F - K_\pi \rfloor \leqslant D$ because $\mathcal{J}(\mathfrak{a}^{\lambda_D - \varepsilon}) \supseteq I_D$. We also have $I_D \nsubseteq \mathcal{J}(\mathfrak{a}^{\lambda_D})$ so the only possibility for I_D of being a multiplier ideal is when $\mathcal{J}(\mathfrak{a}^{\lambda_D - \varepsilon}) = I_D$ so, applying Lemma 3.1, D must be contained in the antinef closure of$\lfloor (\lambda_D - \varepsilon)F - K_\pi \rfloor$. The rest of the statement follows from Theorem 5.4. ☐

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