Splash and almost-splash stationary solutions to the Euler equations

Diego Córdoba[*], Alberto Enciso[†] and Nastasia Grubic[‡]

Instituto de Ciencias Matemáticas
Consejo Superior de Investigaciones Científicas
28049 Madrid, Spain

Abstract

In this paper we discuss the existence of stationary incompressible fluids with splash singularities. Specifically, we show that there are stationary solutions to the Euler equations with two fluids whose interfaces are arbitrarily close to a splash, and that there are stationary water waves with splash singularities.

1 Introduction

The aim of this paper is to seek for stationary splash singularities for a two-fluid interface. We consider the two-fluid incompressible irrotational Euler equations in $\mathbb{R}^2$ where the interface

$$\mathcal{S} = \{z(\alpha) = (z_1(\alpha), z_2(\alpha)) \,|\, \alpha \in \mathbb{R}\}$$

separates the plane in two regions Ω_j, with $j = 1, 2$. Each Ω_j denotes the region occupied by the two different fluids with velocities $v^j = (v_1^j, v_2^j)$, different constant densities ρ_j and pressures p^j, which must satisfy the following equations:

$$\rho_j(v^j \cdot \nabla)v^j = -\nabla p^j - g\rho_j\, e_2 \quad \text{in} \quad \Omega_j, \tag{1a}$$

$$\nabla \cdot v^j = 0 \quad \text{and} \quad \nabla^\perp v^j = 0 \quad \text{in} \quad \Omega_j, \tag{1b}$$

$$v^j \cdot (\partial_\alpha z)^\perp = 0 \quad \text{on} \quad \mathcal{S}, \tag{1c}$$

$$p^1 - p^2 = -\sigma K \quad \text{on} \quad \mathcal{S}. \tag{1d}$$

---

[*]dcg@icmat.es
[†]aenciso@icmat.es
[‡]nastasia.grubic@icmat.es

Here $j \in \{1, 2\}$, $\sigma > 0$ is the surface tension coefficient, e_2 is the second vector of a Cartesian basis and K is the curvature of the interface.

For the time dependent case of water waves (that is, $\rho_1 = 0$, the fluid is irrotational and no surface tension), Castro–Cordoba–Fefferman–Gancedo–Gomez-Serrano [6] showed the formation in finite time of splash and splat singularities. A splash singularity is when the interface remain smooth but self-intersects at a point and a splat singularity is when it self-intersects along an arc. The system of water waves in $\mathbb{R}^2$ can be written in terms of the boundary $\mathcal{S}(t) = \{z(\alpha, t) = (z_1(\alpha, t), z_2(\alpha, t)) \,|\, \alpha \in \mathbb{R}\}$ and the measure of the vorticity in the boundary $\nabla^\perp \cdot v = \omega(\alpha, t)\delta(x - z(\alpha, t))$. Notice that for these type of singularities to happen the amplitude of the vorticity $\omega(\alpha, t)$ has to blow-up at the point of the self-intersection. With a different approach Coutand–Shkoller proved in [11] that these two type of singularities develop when you consider non zero vorticity. The presence of surface tension, see [5], does not prevent the formation of a splash and a splat singularity.

In a recent work by Fefferman–Ionescu–Lie [17] they showed that the presence of a second fluid $(\rho_2, \rho_1 > 0)$ prevents both splash and splat singularities to form in finite time. The condition $\rho_2, \rho_1 > 0$ is used to show a critical L^∞ bound for the measure of the vorticity in the boundary. With a different approach, similar results have been obtained by Coutand-Shkoller [12] for the vortex sheet.

The main goal of this paper is not to study the dynamics in which a non-intersecting smooth curve self-intersects in finite-time but to show the existence of stationary solutions in which have a splash singularity. It is worth recalling that in the last few years the properties of stationary solutions to the Euler equation have attracted considerable attention, which has led to recent results concerning e.g. Liouville theorems [19, 7], connections with coadjoint orbits in the space of volume-preserving diffeomorphisms [8], the existence of weak solutions [9], mixing properties for the flow [18] and the existence of knotted vortex lines and thin vortex tubes [15, 16].

Specifically, in this paper we will prove two related results concerning the existence of splash singularities in stationary solutions of the Euler equations. In both cases we will assume that the fluid is *periodic*, i.e., that the pressure and velocity of the fluid and the geometry of the interface remain invariant under the translation

$$(x, y) \mapsto (x + 2\pi, y).$$

Suitable boundary conditions are also specified as $y \to \pm\infty$; full details are given in Section 2.

The first result asserts that there are almost-splash stationary solutions to the Euler equations with two fluids. To make precise what we understand by "almost splash", consider a curve $z(\alpha)$ parametrized by an arc-length parameter α. For positive but arbitrarily small η, we say that it is an η-*splash curve*

if

$$\inf_{\alpha < \beta} \frac{|z(\alpha) - z(\beta)|}{\min\{\alpha - \beta, 1\}} \leq \eta$$

and if moreover this condition is only satisfied when α and β respectively lie in intervals $[\alpha_1, \alpha_2]$ and $[\beta_1, \beta_2]$ whose lengths are bounded by some quantity that tends to zero as $\eta \searrow 0$ (modulo periodicity). The precise (but fixed) bound that we take is not important.

Intuitively speaking, this means that there are two points in the curve whose chord distance (that is, their distance as measured in the ambient space $\mathbb{R}^2$) is at most η while their distance as measured along the curve is not small. The condition that the curve be parametrized by arc-length ensures that this actually corresponds to the intuition that the curve is close to self-intersecting, while the assertion about the condition being satisfied only for α and β in two small intervals guarantees that the shape of the η-splash curves actually tend to a "splash" as $\eta \searrow 0$, and not to other sort of self-intersecting curve such as a splat curve. In particular, a splash curve can be defined as an η-splash curve with $\eta = 0$.

With this definition, our first result can then be stated as follows:

Theorem 1. *Let us fix the density of the second fluid $\rho_2 > 0$ and consider any $\eta > 0$. For any small enough $\rho_1 \geq 0$ and g there is some positive surface tension coefficient σ such that there is a periodic solution to the two-fluid problem (1) for which the interface $\mathcal{S}$ is an η-splash curve.*

It should be noticed that Theorem 1 does not contradict the result of Fefferman–Ionescu–Lie [17] on the absence of splash singularities when ρ_1 is positive. Of course, the result for the 2D Euler equations has also a bearing on the three-dimensional case with periodic boundary conditions, as it corresponds to 3D solutions that are invariant under translations along the third axis.

Our second theorem concerns the existence of stationary splash singularities for water waves, that is, for $\rho_1 = 0$. Essentially, what we prove is that in this case one can take $\eta = 0$. These solutions will necessarily have an unbounded amplitude of the vorticity.

Theorem 2. *Let us fix the density of the second fluid $\rho_2 > 0$ and assume that $\rho_1 = 0$. Then for any small enough g there is some positive surface tension coefficient σ such that there is a periodic solution to the water waves problem (1) for which the interface $\mathcal{S}$ has a splash singularity.*

The idea of the proof of this result is to perturb a family of exact stationary water waves introduced by Crapper [13]. The idea of perturbing Crapper waves to obtain more general stationary water waves was introduced by Akers, Ambrose and Wright in [1] and exploited further in [14]. Global bifurcation results for traveling waves à la Rabinowitz, also considering the presence of two fluids, were presented in [2].

In this paper we are interested in the existence of splash or almost-splash solutions to the Euler equations with two fluids (of course, this involves ensuring that these curves remain weak solutions to the equations (1)). To this end we will make use of the fact that the family of Crapper waves, which depends on a parameter A, includes a splash curve for certain value of the parameter, A_0, so we will try to control the effects of the perturbation in a neighborhood of this solution. In a perturbation argument, this involves controlling the perturbations not only of the solutions with parameter A_0, but also with parameters slightly smaller or larger. Notice the connection between the Crapper solutions with $A > A_0$ and the Euler equations is uncertain, so in the case of water waves we will eventually resort to the trick of "opening up" the domain using a suitable conformal transformation. The proofs of Theorems 1 and 2 are respectively given in Sections 3 and 4, while in Section 2 we reformulate the problem in a way that is most convenient for our purposes.

2 Setting up the problem

2.1 The vorticity formulation

Through the paper the interface

$$\mathcal{S} = \{z(\alpha) = (z_1(\alpha), z_2(\alpha)) \,|\, \alpha \in \mathbb{R}\}$$

will be parametrized by a function $z(\alpha)$ that satisfies the periodicity conditions

$$z_1(\alpha + 2\pi) = z_1(\alpha) + 2\pi, \quad z_2(\alpha + 2\pi) = z_2(\alpha)$$

and is symmetric with respect to the y-axis:

$$z_1(-\alpha) = -z_1(\alpha), \quad z_2(-\alpha) = z_2(\alpha).$$

The interface $\mathcal{S}$ separates $\mathbb{R}^2$, which we will often identify with the complex plane $\mathbb{C}$, into two disjoint unbounded domains Ω_j $(j = 1, 2)$, each of which is occupied by a fluid of constant (but distinct) density ρ_j. In particular, the interface is given by the set of discontinuity of the density function

$$\rho = \begin{cases} \rho_1, & z \in \Omega_1, \\ \rho_2, & z \in \overline{\Omega}_2 = \mathbb{R}^2 \setminus \Omega_1. \end{cases}$$

We assume that the densities satisfy $\rho_2 > 0$ and $\rho_1 \geq 0$. The fluid flow, governed by the stationary Euler equations

$$\rho(v \cdot \nabla)v = -\nabla p - g\rho\, e_2, \tag{2}$$

where p is the pressure and g the gravity constant, is assumed incompressible

$$\nabla \cdot v = 0. \tag{3}$$

Both equations are to be understood weakly, since the density ρ is discontinuous. However, on either side of the interface, the velocity is smooth, so restrictions $v^j = v|_{\Omega_j}$ satisfy the above equations strongly in their respective domains. Away from the interface, we assume that

$$v^2 \to -1 \quad \text{uniformly as} \quad y \to -\infty \tag{4}$$

and

$$v^1 \to 0 \quad \text{uniformly as} \quad y \to \infty. \tag{5}$$

Here and in what follows, we identify $\mathbb{R}^2$ with the complex plane whenever it is notationally convenient and use the notation $z = x + iy$. These conditions guarantee that the mean value of the vorticity amplitude is also well behaved in the limit $\rho_1 \to 0$. We also impose the kinematic boundary condition

$$v^j \cdot (\partial_\alpha z)^\perp = 0 \quad \text{on} \quad \mathcal{S}, \tag{6}$$

and we assume the velocity is irrotational in the interior of each domain, i.e.,

$$\nabla^\perp \cdot v^j = 0 \quad \text{in} \quad \Omega_j. \tag{7}$$

These are the equations (1) mentioned in the Introduction and the pertinent boundary conditions.

In particular, the vorticity

$$\omega = \nabla^\perp \cdot v$$

can be seen as a measure supported on $\mathcal{S}$. With some abuse of notation, we will write this measure as

$$\omega(z) = \omega(\alpha)\delta(z - z(\alpha)),$$

so $\omega(\alpha)$ will stand for the amplitude of the vorticity along the interface. By the incompressibility condition, the velocity can be written as a normal gradient of a stream function $v = \nabla^\perp \psi$, which therefore satisfies the equation

$$\triangle \psi = \omega.$$

Hence away from the interface we have

$$\psi(z) = \frac{1}{2\pi} \int \ln|z - z(\alpha)| \, \omega(\alpha) d\alpha.$$

This function is harmonic in the interior of each domain and continuous over the interface. Condition (6) implies it is actually constant on $\mathcal{S}$ so without loss of generality we can assume that it is zero: $\psi|_\mathcal{S} = 0$. To obtain the velocities v^j, we take the normal gradient of ψ in the respective domain. Approaching the interface from inside of each Ω_j in the normal direction, we have

$$v^2(z(\alpha)) = BR(z, \omega)(\alpha) + \frac{1}{2} \frac{\omega(\alpha)}{|\partial_\alpha z(\alpha)|^2} \partial_\alpha z(\alpha),$$

$$v^1(z(\alpha)) = BR(z, \omega)(\alpha) - \frac{1}{2} \frac{\omega(\alpha)}{|\partial_\alpha z(\alpha)|^2} \partial_\alpha z(\alpha),$$

5

where $BR(z, \omega)$ denotes the Birkhoff–Rott integral of ω along the curve $z(\alpha)$, that is,

$$BR(z, \omega)(\alpha) = \frac{1}{2\pi} PV \int \frac{(z(\alpha) - z(\beta))^{\perp}}{|z(\alpha) - z(\beta)|^2} \, \omega(\beta) d\beta.$$

In particular, the amplitude of the vorticity measures the jump in the tangential component of the velocity along the interface,

$$(v^2 - v^1) \cdot \partial_\alpha z = \omega.$$

It remains to rewrite the Euler equation (2) and kinematic boundary condition (6) in terms of the vorticity amplitude $\omega(\alpha)$ and the parametrization z. The latter now reads

$$BR(z, \omega) \cdot \partial_\alpha^{\perp} z = 0 \quad \text{on} \quad \mathcal{S}, \tag{8}$$

while the Euler equation can be integrated to obtain the Bernoulli equation

$$\rho_j \left(\frac{1}{2} |v^j|^2 + gy \right) + p_j = \rho_j k_j \quad \text{in} \quad \Omega_j,$$

for k_j constant. Let us take $k_1 = k_2 = \frac{1}{2} + \kappa$. This way, we can understand κ as a perturbation of the constant arising in the pure capillary wave problem (cf. [20]). We assume that on the interface, the pressures are related by

$$p_2 = p_1 + \sigma K,$$

where $\sigma > 0$ is the surface tension coefficient and $K(z)$ is the curvature of the free boundary, i.e.,

$$K(z) = \frac{\partial_\alpha z_1 \partial_\alpha^2 z_2 - \partial_\alpha z_2 \partial_\alpha^2 z_1}{\left[(\partial_\alpha z_1)^2 + (\partial_\alpha z_2)^2 \right]^{3/2}}.$$

Subtracting the two equations, we then obtain

$$\frac{1}{|\partial_\alpha z(\alpha)|^2} \left(\left(v^2 \cdot \partial_\alpha z \right)^2 + \frac{\rho_1}{\rho_2 - \rho_1} \left((v^2 - v^1) \cdot \partial_\alpha z \right) \left((v^2 + v^1) \cdot \partial_\alpha z \right) \right) = F(z)$$

where $F(z)$ is given by

$$F(z) = -\frac{2\rho_2}{\rho_2 - \rho_1} q K(z) - 2g z_2 + 1 + 2\kappa$$

and $q := \frac{\sigma}{\rho_2}$. Setting

$$\epsilon := \frac{2\rho_1}{\rho_2 - \rho_1},$$

and then using the expressions for the velocity on the interface, we have

$$\left(\omega + 2 BR(z, \omega) \cdot \partial_\alpha z \right)^2 + 4\epsilon \omega \, BR(z, \omega) \cdot \partial_\alpha z = 4 |\partial_\alpha z|^2 F(z).$$

6

Finally, collecting all the non-local terms on the right-hand side, we have

$$\left((1+\epsilon)\omega + 2BR(z,\omega)\cdot\partial_\alpha z\right)^2 = 4\,|\partial_\alpha z|^2 F(z) + \epsilon\,(2+\epsilon)\,\omega^2, \qquad (9)$$

where, in terms of ϵ the function $F(z)$ can be written as

$$F(z) := -(2+\epsilon)q\,K(z) - 2gz_2 + 1 + 2\kappa. \qquad (10)$$

2.2   Choice of parametrization for z

We still have one degree of freedom left. The equations are only determined up to a parametrization of the interface. To fix the parametrization, we use the hodograph transform with respect to the lower fluid. Since we will only consider the lower fluid, in this section we omit the subscript two in all the quantities. The idea is to use the analytic function $w = \phi + i\psi$ as an independent variable instead of $z = x + iy$. This will allow us to transform the free boundary problem into an equation on a fixed domain. In fact, as long as $\mathcal{S}$ is non-self-intersecting, it can be shown that

$$w = \phi + i\psi : \Omega \to \mathbb{C}_-$$

is a conformal bijection, which extends to a homeomorphism on $\overline{\Omega} \to \overline{\mathbb{C}}_-$ (cf. [22]). Here, $\mathbb{C}_-$ denotes the lower half-plane and we have used that

$$\psi(z(\alpha)) = 0.$$

Recall that ψ is constant on the interface by the kinematic boundary condition (6). Moreover, the periodicity assumption on the interface and condition (4) imply

$$\psi(x + 2\pi, y) = \psi(x, y),$$
$$\phi(x + 2\pi, y) = \phi(x, y) + 2\pi,$$

and we can actually restrict our attention to a period (cf. [22]). In particular, the inverse mapping

$$\phi \mapsto w^{-1}(\phi, 0)$$

gives a useful (although not arc-length) parametrization of the interface $\mathcal{S}$. We will henceforth assume that the curve $z(\alpha)$ is parametrized in this manner, which is equivalent to requiring

$$\phi(z(\alpha)) = \alpha. \qquad (11)$$

Hence we obtain a simple relation for the velocity and tangent vector on the interface:

$$v^2 \cdot \partial_\alpha z = \nabla\phi \cdot \partial_\alpha z = 1.$$

This translates into a particularly nice expression in terms of the amplitude of vorticity:

$$\omega + 2BR(z,\omega) \cdot \partial_\alpha z = 2.$$

Using the above relations and rearranging, equation (9) reads

$$2 - 2|\partial_\alpha z|^2 F(z) = \epsilon \omega(\omega - 2).$$

We now have a system of three equations in terms of the vorticity and the parametrization, and the problem can be reformulated as:

Problem A. *Find 2π-periodic functions $\omega(\alpha)$ and $z(\alpha) - \alpha$ satisfying*

$$2|\partial_\alpha z|^2 F(z) + \epsilon \omega(\omega - 2) = 2,$$
$$2BR(z,\omega) \cdot \partial_\alpha z + \omega = 2,$$
$$BR(z,\omega) \cdot \partial_\alpha^\perp z = 0,$$

where F is given by (10).

This problem can be simplified further, since the last two equations of (12) have been combined to construct the parametrization. In fact, we can write the velocity in terms of the polar coordinates associated with w as

$$(\nabla \phi) \circ w^{-1} = e^{-if}.$$

This defines a function

$$f = \theta + i\tau : \mathbb{C}_- \to \mathbb{C}$$

that is analytic in $\mathbb{C}_-$ and continuous in $\overline{\mathbb{C}}_-$. Note that the boundary conditions ensure

$$f \to 0 \quad \text{as} \quad \psi \to -\infty.$$

Equation (11) now implies

$$\partial_\alpha z = |\partial_\alpha z| e^{i\theta} = e^{if}.$$

This has two advantages. First, the expression for curvature takes the simple form

$$K = e^\tau \frac{d\theta}{d\alpha}.$$

Second, as θ and τ are a 2π-periodic conjugate functions, they must be related on $\mathcal{S}$ by the periodic Hilbert transform, that is, $\tau(\alpha)$ can be written as a function of $\theta(\alpha)$ via

$$\tau(\alpha) = H\theta(\alpha) = \frac{1}{2\pi} P.V. \int_{-\pi}^{\pi} \cot \frac{\alpha - \alpha'}{2} \theta(\alpha') d\alpha'.$$

We therefore take $\theta(\alpha)$ as our main unknown and consider z as a function of θ via the integral operator

$$z(\alpha) = I(\theta)(\alpha) := \int_{-\pi}^{\alpha} e^{-H\theta(\alpha') + i\theta(\alpha')} d\alpha'. \tag{13}$$

It remains to rewrite $F(z)$ in terms of θ. We have

$$F(z) = -2q\Big(1 + \frac{\epsilon}{2}\Big)e^{\tau}\frac{d\theta}{d\alpha} - 2gI(\theta) + 1 + 2\kappa,$$

so after a short calculation, the first equation in (12) reads

$$q\Big(1 + \frac{\epsilon}{2}\Big)\frac{d\theta}{d\alpha} + \sinh\tau = -ge^{-\tau}I(\theta) + \kappa e^{-\tau} + \frac{e^{\tau}}{4}\epsilon\,\omega(\omega - 2)$$

Using $\tau = H\theta$, the problem reduces to

Problem B. *Find 2π-periodic functions $\theta(\alpha)$ and $\omega(\alpha)$, such that θ is odd, ω is even and they satisfy*

$$q\Big(1 + \frac{\epsilon}{2}\Big)\frac{d\theta}{d\alpha} + \sinh H\theta + ge^{-H\theta}I(\theta) - \kappa e^{-H\theta} - \frac{\epsilon}{4}e^{H\theta}\omega(\omega - 2) = 0,$$
$$2BR(z,\omega)\cdot\partial_{\alpha}z + \omega = 2,$$

where $z := I(\theta)$ is defined by (13).

Note that the integration constant in (13) does not affect the Birkhoff–Rott integral. However, it serves to fix the origin and contributes a constant to the Bernoulli equation.

2.3 Crapper solutions

The equations that appear in Problem B depend on four parameters g, ϵ, κ and q, which respectively. Recall that q represents the surface tension, g is the gravity and ϵ indicates the presence of the upper fluid. Setting ϵ to zero, the equations decouple and we recover the capillary-gravity wave problem as studied in [1]. Once a solution θ of the Bernoulli equation is known, the vorticity can be recovered from the second equation as long as the underlying interface is non self-intersecting (cf. Proposition 3 below). If, in addition, we set $g = 0$, we recover the pure capillary waves problem as formulated by Levi-Civita (see e.g. [20]), namely,

Problem C. *Find a 2π-periodic, analytic function $f = \theta + i\tau$ on the lower half-plane that satisfies*

$$q\frac{d\theta}{d\alpha} = -\sinh H\theta \tag{15}$$

on the boundary and tends to zero at infinity.

This problem admits a family of exact solutions depending on the parameter q. In fact, Crapper has shown [13] that the family of analytic functions

$$f_A(w) := 2i\log\frac{1 + Ae^{-iw}}{1 - Ae^{-iw}} \tag{16}$$

9

has all the required properties. Here the parameter A, which is a real number smaller than 1 in absolute value, depends on q via

$$q = \frac{1 + A^2}{1 - A^2}.$$

To obtain the wave profiles, recall that by construction we have

$$\partial_\alpha z = e^{-\tau_A + i\theta_A} = \left(\frac{1 - Ae^{-i\alpha}}{1 + Ae^{-i\alpha}}\right)^2$$

on the boundary, so integrating we obtain the parametrization of the interface

$$z_A(\alpha) = \alpha + \frac{4i}{1 + Ae^{-i\alpha}} - 4i.$$

The constant has been chosen to have $z_A \equiv 0$ for $A = 0$. It actually suffices to consider $A \geq 0$, since the transformation $A \mapsto -A$ corresponds to a translation $\alpha \to \alpha + \pi$. For sufficiently large values of parameter A these solutions can no longer be represented as a graph of a function, and eventually self-intersect. It is not hard to see that the curve z_A can be represented as the graph of a function (that is, as $y = h(x)$) if and only if

$$A < \sqrt{2} - 1,$$

and that $z_A(\alpha)$ does not have self-intersections if and only if

$$A < A_0 \approx 0.45467.$$

For $A = A_0$, the curve $z_A(\alpha)$ exhibits a splash, while for A slightly larger than A_0 the curve intersects at exactly two points, and the intersection is transverse. (What is not clear, of course, is the connection between curves with $A > A_0$ and the Euler equations.)

3   Proof of Theorem 1

In this section, we construct stationary solutions to the Euler equations with two fluids whose interface is arbitrarily close to a splash. For this, we use the implicit function theorem to perturb with respect to the density of the upper fluid (and the gravity) a Crapper solution that is arbitrarily close to the splash. This strategy has been employed to perturb Crapper solutions with respect to gravity in the water wave setting [1] and then to deal with presence of a constant vorticity in a fluid of finite depth [14].

Let H_{even}^s and H_{odd}^s respectively denote the subspace of even and odd functions in the Sobolev space $H^s(\mathbb{T})$. We define

$$G = (G_1, G_2) : H_{\text{odd}}^2 \times H_{\text{even}}^1 \times \mathbb{R}^3 \to H_{\text{even}}^1 \times H_{\text{even}}^1$$

to be the differentiable mapping of components

$$G_1(\theta, \omega; \epsilon, g, \kappa) := q\left(1 + \frac{\epsilon}{2}\right)\frac{d\theta}{d\alpha} + \sinh H\theta + ge^{-H\theta}I(\theta) - \frac{\epsilon}{4}e^{H\theta}\omega(\omega - 2) - \kappa e^{-H\theta}$$

$$G_2(\theta, \omega; \epsilon, g, \kappa) := \omega + 2BR(z, \omega) \cdot \partial_\alpha z - 2,$$

where z is considered to be a function of θ via $z = I(\theta)$ (cf. (13)).

In order to use the implicit function theorem on Crapper solutions θ_A we first need to construct the corresponding vorticity ω_A. The fact that ω_A is well defined is a consequence of the following proposition, where 1 stands for the identity operator. For the proof of this result see e.g. [3] or [10].

Proposition 3. *Let $z \in H^3$ and assume z is a curve without self-intersections. Then*

$$\mathcal{A}(z)(\omega) = 2BR(z, \omega) \cdot \partial_\alpha z$$

defines a compact linear operator

$$\mathcal{A}(z) : H^1 \to H^1$$

whose eigenvalues are strictly smaller than 1 in absolute value. In particular, the operator $1 + \mathcal{A}(z)$ is invertible.

We will sometimes denote points in the domain of definition of G by

$$\xi = (\theta, \omega; \epsilon, g, \kappa).$$

In particular, for the Crapper solutions we set

$$\xi_A = (\theta_A, \omega_A; 0, 0, 0)$$

so by construction we have

$$G(\xi_A) = 0.$$

The Fréchet derivative of G with respect to (θ, ω) has the simple triangular form

$$D_{\theta, \omega}G(\xi_A) = \begin{pmatrix} \Gamma & 0 \\ D_\theta G_2 & D_\omega G_2 \end{pmatrix},$$

where Γ is just the Fréchet derivative of the pure capillary wave operator (15),

$$\Gamma u = q\frac{du}{d\sigma} + \cosh(H\theta_A)\,Hu,$$

and $(D_\omega G_2)T$ is exactly

$$(D_\omega G_2)T = (I + \mathcal{A}(z_A))T$$

since G_2 is linear in ω. By Proposition 3, we already know that $D_\omega G_2$ is invertible on H^1_{even} as long as z_A does not self-intersect. The structure of Γ is already known as well. In fact, we have the following result, which we borrow from [1] and whose proof we sketch here for completeness:

11

Lemma 4. *The operator* $\Gamma : H^1_{\text{odd}} \to L^2_{\text{even}}$ *is injective, but not surjective. Its cokernel is spanned by* $\cos\theta_A$.

Proof. First note that Γ is a compact perturbation of Fredholm operator of index -1, and as such it is another Fredholm operator of index -1 itself. In order to see this, notice that Γ can be written as a sum

$$\Gamma = \Gamma_1 + K_1, \qquad \Gamma_1 u := q\frac{du}{d\sigma}, \qquad K_1 u := \cosh(H\theta_A)\,Hu,$$

where Γ_1 is Fredholm of index -1 and K_1 is compact. Indeed, Γ_1 is injective on H^1_{odd}, since constants are the only trivial solutions. On the other hand, the dimension of the cokernel is one, since $\Gamma_1 T = f$ has a periodic odd solution if and only if f is even and of zero mean ($\int_{-\pi}^{\pi} f(\sigma)d\sigma = 0$). The operator K_1 is compact because the embedding $H^1_{\text{odd}} \hookrightarrow L^2$ is compact by the Rellich–Kondrachov theorem.

Γ is injective when restricted to H^1_{odd}, since zero is an eigenvalue of $\Gamma : H^1 \to L^2$ with geometric multiplicity one (cf. [20]). The only eigenfunction is given by $d\theta_A/d\sigma$, which is even.

Hence the cokernel is at most one-dimensional. To show that it is spanned by $\cos\theta_A$, note that for any function θ we have, with $\epsilon = g = 0$,

$$\int_{-\pi}^{\pi} G_1(\xi)\cos\theta\,d\alpha = \int_{-\pi}^{\pi}\left[q\frac{d\theta}{d\alpha} + \sinh H\theta\right]\cos\theta\,d\alpha = 0 \qquad (17)$$

since the first term is just the integral of the derivative of $q\sin\theta$, so it integrates to zero. To show that we also have $\int_{-\pi}^{\pi}\sinh H\theta\,\cos\theta\,d\alpha = 0$, it suffices to employ the identity

$$\int_{-\pi}^{\pi} e^{\pm H\theta(\alpha)+i\theta(\alpha)}d\alpha = 2\pi,$$

which follows from the Cauchy integral theorem.

Hence taking the derivative with respect to θ at θ_A in the identity (17) we obtain

$$0 = \int_{-\pi}^{\pi}\Gamma u\,\cos\theta_A\,d\alpha + \int_{-\pi}^{\pi}\left[q\frac{d\theta_A}{d\alpha} + \sinh H\theta_A\right]\sin\theta_A\,u\,d\alpha.$$

The expression in square brackets zero by construction, so we obtain

$$\langle \Gamma u, \cos\theta_A \rangle = 0$$

for all u, where the angle brackets denote the L^2 product. $\square$

In view of the above, we obtain the following result:

Proposition 5. *Let* $A < A_0$. *Then the Fréchet derivative of* G *with respect to* (θ, ω) *at* ξ_A *is injective, i.e.,*

$$\ker D_{\theta,\omega}G = \{0\}.$$

12

On the other hand, the equation

$$\begin{pmatrix} \Gamma & 0 \\ D_\theta G_2 & D_\omega G_2 \end{pmatrix} \begin{pmatrix} u \\ T \end{pmatrix} = \begin{pmatrix} f \\ g \end{pmatrix}$$

has a solution if and only if

$$\langle f, \cos\theta_A \rangle = 0.$$

Proof. $D_{\theta,\omega}G$ is injective since both Γ and $D_\omega G_2$ are. On the other hand, if $\langle f, \cos\theta_A \rangle = 0$, then there exists u such that

$$\Gamma u = f,$$

by Lemma 4. Since $D_\omega G_2$ is invertible by Lemma 3, there exists T such that

$$(D_\omega G_2)T = g - (D_\theta G_2)u,$$

so the claim follows. $\qquad\qquad\square$

Since $D_{\theta,\omega}G$ is not surjective, we cannot use the implicit function theorem directly to prove Theorem 1. Therefore, following [1], we use an adaptation of the Lyapunov–Schmidt reduction argument. So let Π denote the L^2 projector onto the linear span of $\cos\theta_A$, i.e.,

$$\Pi u := \langle \cos\theta_A, u \rangle \frac{\cos\theta_A}{\|\cos\theta_A\|_2^2}.$$

Then $Q := 1 - \Pi$ is the projector on $\Gamma(H^2_{\text{odd}})$ and $\tilde{G} = (QG_1, G_2)$ is a mapping

$$\tilde{G} : H^2_{\text{odd}} \times H^1_{\text{even}} \times \mathbb{R}^3 \to \Gamma(H^2_{\text{odd}}) \times L^2,$$

whose Fréchet derivative with respect to (θ, ω) at ξ_A is an isomorphism by construction. We can then apply the implicit function theorem on $\tilde{G}$ to obtain a smooth function $\Theta_A(\epsilon, g, \kappa)$ satisfying $\Theta_A(0, 0, 0) = (\theta_A, \omega_A)$ and

$$\tilde{G}(\Theta_A(\epsilon, g, \kappa); \epsilon, g, \kappa) = 0$$

for all (ϵ, g, l) in a small neighbourhood of $(0, 0, 0)$. This function does not necessarily satisfy the original equation $G = 0$, since the projection ΠG_1 on $\cos\theta_A$ is not necessarily zero. So consider

$$f(\epsilon, g; \kappa) = \langle \cos\theta_A, G_1(\Theta_A(\epsilon, g, \kappa); \epsilon, g, \kappa) \rangle$$

and note that $f(0, 0; 0) = 0$. For the derivative at $(0, 0, 0)$, we have

$$\partial_\kappa f(0, 0; 0) = \langle \cos\theta_A, \Gamma\partial_\kappa\Theta_A + \partial_\kappa G_1 \rangle = -2\pi,$$

where we have used that the first term is zero by construction and that the second gives

$$\langle \cos\theta_A, \, \partial_\kappa G_1 \rangle = \int_{-\pi}^{\pi} e^{-H\theta_A(\sigma)} \cos\theta_A(\sigma) d\sigma = -2\pi$$

by the Cauchy integral theorem. We can apply the implicit function theorem once again to obtain a smooth function $\kappa_*(\epsilon, g)$ satisfying $f(\epsilon, g; \kappa_*(\epsilon, g)) = 0$ and $\kappa_*(0,0) = 0$. So we have proved the following

Theorem 6. *Let $A < A_0$. Then there exist $\epsilon_A > 0$, $g_A > 0$, $\kappa_A > 0$, a unique smooth function*

$$\kappa_* : B_{\epsilon_A}(0) \times B_{g_A}(0) \to B_{\kappa_A}(0)$$

such that $\kappa_*(0,0) = 0$ and a unique smooth function

$$\Theta_A : B_{\epsilon_A}(0) \times B_{g_A}(0) \times B_{\kappa_A}(0) \to H_{\mathrm{odd}}^2 \times H_{\mathrm{even}}^1$$

such that $\Theta_A(0,0,0) = (\theta_A, \omega_A)$ and

$$G\big(\Theta_A(\epsilon, g, \kappa_*(\epsilon, g)); \, \epsilon, g, \kappa_*(\epsilon, g)\big) = 0.$$

Remark 7. The restriction $A < A_0$ is necessary if the upper fluid is present, since absolute values of eigenvalues of $\mathcal{A}(z)$ approach 1 as the arc-chord condition fails (cf. [10]). If the upper fluid is neglected, the equations (14) decouple and we recover the capillary-gravity waves problem as studied in [1]. In particular, if we were only concerned with the function G_1 (as opposed to $G = (G_1, G_2)$), the proof would go through for all $A \in (-1, 1)$.

Remark 8. Note that we could have proved an analogous result with the function G defined on $H_{\mathrm{odd}}^{k+1} \times H_{\mathrm{even}}^k$.

Theorem 1 is a straightforward consequence of Theorem 6. Indeed, the latter ensures that, given small enough ϵ and g and assuming that the corresponding Crapper interface z_A does not have any self-intersections, there is a solution of the equations with the aforementioned values of ϵ and g that depends smoothly on these parameters. Choosing the Crapper interface z_A to be an η-splash curve, by continuity we infer that for small ϵ and g the resulting interface z must be an η'-splash curve, with $\eta' - \eta$ tending to zero as $|\epsilon| + |g| \to 0$. The result then follows.

4   Proof of Theorem 2

In this section, we only consider capillary-gravity water waves, so we assume throughout that $\epsilon = 0$. The problem is then governed by the Bernoulli equation alone, and reads:

Problem D. *Find a 2π-periodic, analytic function $f = \theta + i\tau$ defined on the lower half-plane that satisfies*

$$q\frac{d\theta}{d\alpha} + \sinh H\theta + ge^{-H\theta}I(\theta) - \kappa e^{-H\theta} = 0. \tag{18}$$

on the boundary and tends to zero at infinity.

We prove that, in this case, there exist solutions of Problem D that exhibit splash singularities for all sufficiently small values of g. Moreover, we show that any such solution defines a solution of (2)–(7). For that matter, assume we found an f as required in Problem D. Then we obtain a candidate solution z setting

$$\frac{dz}{dw} := e^{if}$$

(cf. section 2.2). Note that periodicity of f and the Cauchy integral theorem imply z has the form

$$z : \mathbb{C}_- \to \Omega$$
$$z(w) = w + h(w),$$

where h is 2π-periodic (in the sense discussed in the Introduction). The boundary condition at infinity then implies that h tends to some finite limit as $\operatorname{Im} w \to -\infty$. This mapping defines a solution of (2)–(7), if it can be inverted. In this case, its inverse $w = \phi + i\psi$ defines the complex potential and the complex conjugate of its derivative dw/dz the velocity. If fact, we have the following result:

Lemma 9. *Let $z : \mathbb{C}_- \to \Omega$ be a solution of the capillary-gravity wave problem (see Problem D) such that the curve*

$$\mathcal{S} = \{z(\alpha) \,|\, \alpha \in \mathbb{R}\}$$

either does not self-intersect or has a splash singularity. Then z is invertible and its inverse

$$w = \phi + i\psi : \Omega \to \mathbb{C}_-$$

is such that $v = \nabla\phi$ satisfies (2)–(7).

Proof. When the curve does not have any self-intersections, the result is proved in [4], so let us assume that $\mathcal{S}$ is a splash curve.

We can restrict our attention to $\Omega \cap \{|\operatorname{Re} z| < \pi\}$, which we will still denote by Ω for the ease of notation. Let $P : \Omega \to \mathbb{C}$ denote the conformal mapping

$$P(z) = \sqrt{a - e^{-iz}},$$

where the branch cut of the complex square root was taken along the negative real axis. The constant $a \geq 0$ is chosen in such a way that the branch cut and $(0,0)$ lie in the vacuum region, with $(0,0)$ lying inside the bubble in the case of splash singularity.

The exponential function maps the interface $\mathcal{S}$ to a closed curve and the vertical lines $(\pm\pi, y)$ to the positive real axis, with $-\infty$ being mapped to $(0,0)$. By symmetry, the point where the splash singularity occurs has to be on the branch cut and neighboring points will therefore be mapped to different regions by the square root. More precisely, if z_0 is the splash point and B is a small ball centered at z_0, $P(B \cap \Omega)$ will consist of two connected components with disjoint closures.

Let us set

$$\tilde{\mathcal{S}} := P(\mathcal{S}).$$

The transformed interface $\tilde{\mathcal{S}}$ is a closed Jordan curve in the right half-plane touching the imaginary axis at exactly two points. Denoting by $\tilde{\Omega}$ the bounded connected component of $\mathbb{C} \setminus \tilde{\mathcal{S}}$, we see that the restriction

$$P : \Omega \to \tilde{\Omega} \setminus [\sqrt{a}, \infty)$$

is a conformal bijection. Note, however, that

$$\lim_{z \to -\infty} \frac{dP}{dz} = 0,$$

so a priori P cannot be extended to a conformal mapping on $\tilde{\Omega}$.

To avoid this difficulty, we map the half-strip

$$S := \mathbb{C}_- \cap \{|\operatorname{Re} w| < \pi\}.$$

to the unit disc $\mathbb{D}$ cut along the negative real axis via

$$w : \mathbb{D} \setminus (-1, 0] \to S$$
$$\zeta \mapsto i \log \zeta.$$

Note that derivative of w with respect to ζ becomes unbounded as $\zeta \to 0$. However, the composed mapping

$$P \circ z \circ w : \mathbb{D} \setminus (-1, 0] \to \tilde{\Omega} \setminus [\sqrt{a}, \infty)$$

can be extended to a mapping analytic on all of $\mathbb{D}$. In fact, by periodicity it is continuous over the cut and maps 0 to $\sqrt{a}$. Moreover, an elementary computation using the precise expression for z shows that the derivative of the composed mapping tends to a nonzero constant. In particular, we have a mapping, which is analytic on $\mathbb{D}$, continuous on $\overline{\mathbb{D}}$ and injective on $\partial\mathbb{D}$. This implies it actually has to be a conformal bijection $\mathbb{D} \to \tilde{\Omega}$ by [21, Corollary 2.10]. □

We are now ready to prove the main result of this section. Arguing as in the previous section, this result readily yields Theorem 2.

Proposition 10. *There exists a $g_0 > 0$, such that for every $|g| \leq g_0$ there exists an H^3 solution of the capillary-gravity wave problem (Problem D) such that the corresponding interface exhibits a splash singularity.*

16

Proof. Since we are only concerned with solutions to

$$G_1(g, \kappa, A; \theta) = q(A)\frac{d\theta}{d\alpha} + \sinh H\theta + ge^{-H\theta}I(\theta) - \kappa e^{-H\theta} = 0,$$

Remark 7 ensures that there are solutions to this equation in a vicinity of the point $(0, 0, A_0; \theta_0)$, where we set $\theta_0 := \theta_{A_0}$. Omitting the dependence on κ for notational simplicity, for small enough g_0 and δ we obtain a unique smooth mapping Θ which maps $|g| \leq g_0$, $|A - A_0| \leq \delta$ to a neighborhood of $\theta_0 \in H^3_{\mathrm{odd}}$ and solves the equation

$$G_1(g, A; \Theta(g, A)) = 0.$$

By uniqueness, this mapping essentially coincides with mapping found in Theorem 6 on their common domain of definition.

It is well-known that the interface of the Crapper solution with parameter A_0 has a splash point, while for values of A slightly smaller (resp. larger) than A_0 the curve does not self-intersect (resp. intersects transversally at exactly two points). By continuity and transversality, and shrinking the neighborhoods if necessary, we can ensure that the corresponding curve $z(\alpha)$ remains injective for $\Theta(g, A_0 - \delta)$ and any $|g| \leq g_0$, while it self-intersects transversally and at exactly two points for $\Theta(g, A_0 + \delta)$ and all $|g| \leq g_0$. In fact, for all (g, A) in this small neighbourhood of $(0, A_0)$, $\Theta(g, A)$ looks like θ_A in the sense that it is a concave function on $[-\pi, 0]$ and, by symmetry, a convex function on $[0, \pi]$. In particular, by continuity, for each $|g| \leq g_0$, there exists some A in the interval $|A - A_0| < \delta$ such that the curve corresponding to $\Theta(g, A)$ exhibits a splash singularity. Although the connection between the Crapper solutions with $A > A_0$ and the Euler equations is uncertain, Lemma 9 ensures that it is indeed a solution of the Problem D, so the theorem follows. $\square$

Acknowledgements

References

[1] AKERS B. F., AMBROSE D.M., AND WRIGHT J.D., Gravity perturbed Crapper waves, Proc. R. Soc. A, 470 (2013) 2161.

[2] AMBROSE D.M., STRAUSS W.A., AND WRIGHT J.D., Global bifurcation theory for periodic traveling interfacial gravity-capilary waves, arXiv:1411.5569.

[3] Baker G., Meiron D., and Orszag S., Generalized vortex methods for free-surface flow problems, J. Fluid Mech. 123 (1982) 477–501.

[4] Buffoni B., Dancer E. N., and Toland J.F. , The regularity and local bifurcation of steady periodic water-waves, Arch. Rational Mech. Anal. 152 (2000) 207–240.

[5] Castro A., Cordoba D., Fefferman C., Gancedo F., and Gomez-Serrano J., Finite time singularities for water waves with surface tension, J. Math. Phys. 53 (2012), 115622.

[6] Castro A., Cordoba D., Fefferman C., Gancedo F., and Gomez-Serrano J. Finite time singularities for the free boundary incompressible Euler equations. Ann. of Math. 178 (2013) 1061–1134.

[7] Chae D., and Constantin P., Remarks on a Liouville-type theorem for Beltrami flows, arXiv:1407.7303, 2014.

[8] Choffrut A., and Sverak V., Local structure of the set of steady-state solutions to the 2D incompressible Euler equations, Geom. Funct. Anal. 22 (2012) 136–201.

[9] Choffrut A., and Székelyhidi L., Weak solutions to the stationary incompressible Euler equations, SIAM J. Math. Anal. 46 (2014) 4060–4074

[10] Cordoba A., Cordoba D., and Gancedo F., Interface evolution: the Hele-Shaw and Muskat problem, Ann. of Math., 173 (2011) 477–542.

[11] Coutand D., and Shkoller S., On the impossibility of Finite-time splash singularities for vortex sheets. arXiv:1407.1479v1, 2014.

[12] Coutand D., and Shkoller S., On the finite-time splash and splat singularities for the 3-D free-surface Euler equations. Comm. Math. Phys., 325 (2014) 143–183.

[13] Crapper G.D., An exact solution for progressive capillary waves of arbitrary amplitude, J. Fluid Mech. 2 (1957) 532–540.

[14] de Boeck P., Existence of of capillary-gravity waves that are perturbations of Crappers waves, arXiv:1404.6189v1, 2014.

[15] Enciso A., and Peralta-Salas D., Knots and links in steady solutions of the Euler equation, Ann. of Math. 175 (2012) 345–367.

[16] Enciso A., and Peralta-Salas D., Existence of knotted vortex tubes in steady Euler flows, Acta Math., in press.

[17] Fefferman C., Ionescu A.D., and Lie V., On the absence of "splash" singularities in the case of two-fluids interfaces. arXiv:1312.2917v2, 2013.

[18] KHESIN B., KUKSIN S., AND PERALTA-SALAS D., KAM theory and the 3D Euler equation, Adv. Math., in press.

[19] NADIRASHVILI N., Liouville theorem for Beltrami flow, Geom. Funct. Anal., in press.

[20] OKAMOTO H., AND SHOJI M., *The mathematical theory of permanent progressive water waves,* World Scientific Publishing, Singapore, 2001.

[21] POMMERENKE C., *Boundary behaviour of conformal maps,* Springer Verlag, Berlin, 1992.

[22] SHARGORODSKY E., AND TOLAND J.F., Complex methods for Bernoulli Free-Boundary Problems In-The-Large, Preprint, 2013.