A logic for reasoning about the probability of fuzzy events

Tommaso Flaminio[a,*], Lluís Godo[b]

[a]*Dip. di Matematica e Scienze Informatiche, University of Siena, 53100 Siena, Italy*
[b]*Artificial Intelligence Research Institute (IIIA), CSIC, 089193 Bellaterra, Spain*

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Abstract

In this paper we present the logic $FP(Ł_n, Ł)$ which allows to reason about the probability of fuzzy events formalized by means of the notion of state in a MV-algebra. This logic is defined starting from a basic idea exposed by Hájek [Metamathematics of Fuzzy Logic, Kluwer, Dordrecht, 1998]. Two kinds of semantics have been introduced, namely the class of *weak* and *strong* probabilistic models. The main result of this paper is a completeness theorem for the logic $FP(Ł_n, Ł)$ w.r.t. both weak and strong models. We also present two extensions of $FP(Ł_n, Ł)$: the first one is the logic $FP(Ł_n, RPL)$, obtained by expanding the $FP(Ł_n, Ł)$-language with truth-constants for the rationals in [0, 1], while the second extension is the logic $FCP(Ł_n, ŁΠ\frac{1}{2})$ allowing to reason about conditional states.
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1. Introduction

A fuzzy-logical treatment for the probability of classical (crisp) events has been widely studied in the last years. In particular, starting from the basic ideas exposed by Hájek, Godo and Esteva in [14] and then later refined by Hájek in [13], simple (i.e. unconditional) and conditional probability can be studied by using various kind of modal-fuzzy logics (see [6,8–10,18]). The very basic idea allowing a treatment of simple probability inside a fuzzy-logical setting consists of interpreting the probability of an (classical) proposition φ as the truth value of a modal proposition $P(\varphi)$ which reads φ *is probable*.

Taking Łukasiewicz logic Ł as base logic, this is done by first enlarging the language of Ł by means of a unary (fuzzy) modality P for *probably*, and defining two kinds of formulas: classical Boolean formulas $\varphi, \psi, \ldots$ (which are definable in Ł) and modal formulas: for each Boolean formula φ, $P(\varphi)$ is an atomic modal formula and, moreover, such a class of modal formulas, MF, is taken closed under the connectives of Łukasiewicz logic. And then by defining a set of axioms and an inference rule reflecting those of a probability measure, namely:

(FP1) $P(\neg\varphi \lor \psi) \rightarrow (P(\varphi) \rightarrow P(\psi))$,
(FP2) $P(\neg\varphi) \equiv \neg P(\varphi)$,
(FP3) $P(\varphi \lor \psi) \equiv ((P(\varphi) \rightarrow P(\varphi \land \psi)) \rightarrow P(\psi))$,

and the necessitation Rule: from φ deduce $P(\varphi)$ for any Boolean formula φ.

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* Corresponding author.
E-mail addresses: flaminio@unisi.it (T. Flaminio), godo@iiia.csic.es (L. Godo).

The resulting logic, $FP(Ł)$, is sound and (finite) strong complete [13] with respect to the intended probabilistic semantics given by the class of *probabilistic Kripke models*. These are structures $\mathcal{M} = \langle W, \mu, e \rangle$ where W is a non-empty set, $e : W \times BF \to \{0, 1\}$ (where BF denotes the set of Boolean formulas) is such that, for all $w \in W$, $e(w, \cdot)$ is a Boolean evaluation of non-modal formulas, and μ is a finitely additive probability measure on a Boolean subalgebra $\Omega \subseteq 2^W$ such that, for every Boolean formula φ, the set $[\varphi]_W = \{w \in W : e(w, \varphi) = 1\}$ is μ-measurable, i.e. $[\varphi]_W \in \Omega$ and hence $\mu([\varphi]_W)$ is defined. Then, the truth-evaluation of a formula $P\varphi$ in a model $\mathcal{M}$ is given by $\| P(\varphi) \|_{\mathcal{M}} = \mu([\varphi]_W)$ and it is extended to compound (modal) formulas using Łukasiewicz logic connectives. The completeness result for $FP(Ł)$ states that a (modal) formula Φ follows from a finite set of (modal) formulas Γ (using the axioms and rules of $FP(Ł)$) iff $\| \Phi \|_{\mathcal{M}} = 1$ in any probabilistic Kripke model $\mathcal{M}$ that evaluates all formulas in Γ with value 1. The same results holds $FP(RPL)$, that is, if instead of Ł we use as base logic RPL, the expansion of Ł with rational truth-constants. Thus both $FP(Ł)$ and $FP(RPL)$ are adequate for a treatment of simple probability.

An extension of the notion of probability to the framework of fuzzy sets was earlier defined by Zadeh in order to represent and reasoning about sentences like *the probability that the traffic in Rome will be chaotic tomorrow is 0.7*. Clearly, the modeling of this kind of knowledge cannot be done using the classical approach to probability since, given the un-sharp nature of events like *chaotic traffic*, the structure of such fuzzy events cannot be considered to be a Boolean algebra any longer. The study of finitely additive measures in the context of MV-algebras, structures more general than Boolean algebras, was started by Mundici in [20] and further developed by Mundici and Riečan in [21], as well as by Kroupa [17].

Therefore, a fuzzy logical approach to reason about the probability of fuzzy events is, in our opinion, a natural generalization of the previous works which can bring an important improvement to their expressive power and, moreover, it can be also useful from the point of view of applications. In logical terms, this can be approached by assuming that the logic of events is a (suitable) many-valued logic and by defining and axiomatizing appropriate probability-like measures on top of the many-valued propositions.

In [13] Hájek already proposed a logic built up over the Łukasiewicz predicate calculus Ł∀ allowing a treatment of (simple) probability of fuzzy events. To model this kind of probability, Hájek introduced in Ł∀ a generalized fuzzy quantifier standing for *most* together with a set of characteristic axioms and denoted his logic by Ł∀ $\int$. In his monograph Hájek also proposed two (Kripke-style) probabilistic semantics for this logic, called *weak* and *strong*. A variant of these two kinds of models will be introduced in detail later on, but roughly speaking they can be described as follows:

- A weak probabilistic model for Ł∀ $\int$ evaluates a modal formula $P(\varphi)$ by means of a finitely additive measure (or *state*) defined over the MV-algebra of provably equivalent Łukasiewicz formulas (see [16–21] for a detailed definition of state over an MV-algebra).
- A strong probabilistic model for Ł∀ $\int$ consists of a probability distribution σ over the set of all the evaluations of the events (remember that an event is now a formula of the Łukasiewicz calculus). Then the truth value of a modal formula $P(\varphi)$ is defined as the *integral* of the fuzzy-set of all the evaluations of φ under the measure σ.

Hájek shows this logic is Pavelka-style complete w.r.t. weak probabilistic models, but the issue of completeness w.r.t. the strong semantics still remains as an open problem.

In this paper, instead of considering a predicate calculus, we want to remain at a propositional level and investigate probabilistic completeness in the usual sense by using the same approach as in the above $FP(Ł)$ logic, but considering fuzzy events instead of Boolean events. In particular we will use the finite-valued Łukasiewicz logics $Ł_n$ (for any $n > 2$) in order to treat fuzzy events and we will consider for them weak and strong models adapted to our case. We will use $FP(Ł_n, Ł)$ to denote such a logic. This notation, although it differs from Hájek's original notation, allows us to point out both, the logic of events (the first argument) and the logic which is used in order to reason about modal formulas $P(\varphi)$ (the second argument).

The reasons why we have decided to start with modeling fuzzy events as formulas of a finitely valued Łukasiewicz logic $Ł_n$ and not of the infinitely valued Łukasiewicz logic (such a logic would be denoted by $FP(Ł, Ł)$) are essentially the following:

(a) Finitely valued Łukasiewicz logics $Ł_n$ are natural generalizations of classical Boolean logic and extensions of the infinitely valued Łukasiewicz logic Ł, with good logical and algebraic properties, hence a good compromise.
(b) Studying $FP(Ł_n, Ł)$ could make the study of $FP(Ł, Ł)$ easier which, indeed, seems quite problematic. The idea is, in fact, to treat the logic $FP(Ł, Ł)$ as a *limit case* of $FP(Ł_n, Ł)$ when n tends to infinity. Just remember that

this is what happens in Ł: a formula is a theorem of Ł iff it is a theorem of $Ł_n$ for all $n \in \mathbb{N}$ (see [2] for more details).

In this paper we will prove that, for each $n \in \mathbb{N}$, $FP(Ł_n, Ł)$ is (finite) strong complete w.r.t. both classes of weak and strong models.

This paper is organized as follows. In Section 2 we recall some known logical and algebraic notions and results which will be used throughout this paper. In Section 3 we define the logic $FP(Ł_n, Ł)$ and we also introduce the classes of weak and strong probabilistic models, while in Section 4 we prove that $FP(Ł_n, Ł)$ is complete (in the usual sense) w.r.t. these both classes of models. In Section 5 we present two extensions of $FP(Ł_n, Ł)$: the logic $FP(Ł_n, RPL)$ obtained by expanding the $FP(Ł_n, Ł)$-language with rational truth-constants, and the logic $FCP(Ł_n, Ł\Pi\frac{1}{2})$ allowing to reason about conditional probabilities (states). Finally, Section 6 contains some concluding remarks and the outline of our future work.

2. Preliminaries

2.1. Łukasiewicz logics

In this first part we introduce some (well) known notions about finitely valued and infinitely valued Łukasiewicz logics which will be used throughout the rest of this paper.

For both logics we will consider a language consisting of a countable set of propositional variables $V = \{p_1, p_2, \ldots\}$, two binary connectives $\&$ and $\to$ and the truth-constant $\bar{0}$. Further connectives are defined as follows:

$\neg\varphi$ stands for $\varphi \to \bar{0}$,
$\varphi \oplus \psi$ stands for $\neg(\neg\varphi \& \neg\psi)$,
$\varphi \equiv \psi$ stands for $(\varphi \to \psi) \& (\psi \to \varphi)$,
$\varphi \wedge \psi$ stands for $\varphi \& (\varphi \to \psi)$,
$\varphi \vee \psi$ stands for $((\varphi \to \psi) \to \psi) \wedge ((\psi \to \varphi) \to \varphi)$.

The set of formulas built from with this language will be denoted $Fm(V)$.

Definition 2.1 (*cf. Cignoli et al. [2], Grigolia [12], Hájek [13]*). The infinitely valued Łukasiewicz logic Ł is defined by the following axioms:

(Ł1) $\varphi \to (\psi \to \varphi)$,
(Ł2) $(\varphi \to \psi) \to ((\psi \to \chi) \to (\varphi \to \chi))$,
(Ł3) $((\varphi \to \bar{0}) \to (\psi \to \bar{0})) \to (\psi \to \varphi)$,
(Ł4) $((\varphi \to \psi) \to \psi) \to ((\psi \to \varphi) \to \varphi)$.

and the only deduction rule is *modus ponens*: from φ and $\varphi \to \psi$ deduce ψ.

For each $n \in \mathbb{N}$, the $(n + 1)$-valued Łukasiewicz logic $Ł_n$ is the schematic extension of Ł with the following two axiom schemata:

(Ł5) $(n - 1)\varphi \equiv n\varphi$,
(Ł6) $(k\varphi^{k-1})^n \equiv n\varphi^k$ for each integer $k = 2, \ldots, n - 2$ that does not divide $n - 1$,

where $n\varphi$ is an abbreviation for $\varphi \oplus \cdots \oplus \varphi$ (n-times) and φ^k is an abbreviation for $\varphi \& \ldots \& \varphi$ (k-times).

The notions of *theorem* and *proof* are defined as usual. As for notation, given a theory (i.e. a set of formulas) Γ and a formula φ, we will write $\Gamma \vdash_Ł \varphi$ (resp., $\Gamma \vdash_{Ł_n} \varphi$) to say that φ is derivable from Γ in Ł (resp., in $Ł_n$).

The algebraic counterpart for the logic Ł is the variety of MV-algebras while the algebraic counterpart for the logics $Ł_n$ are the MV-subvarieties MV_n defined next, where we use the same notation for the algebraic operation as for the corresponding logical connectives. We note that MV-algebras are traditionally defined taking $\oplus$ and $\neg$ as primitive operations instead of $\&$ and $\to$. In that case the corresponding algebras are called Wajsberg-algebras, and they are definitionally equivalent to MV-algebras.

Definition 2.2. An MV-algebra is an algebra $\mathcal{A} = \langle A, \oplus, \neg, 0 \rangle$ of type $\langle 2, 1, 0 \rangle$ such that $\langle A, \oplus, 0 \rangle$ is a commutative monoid satisfying the following equations:

$x \oplus \neg 0 = \neg 0$,
$\neg \neg x = x$,
$\neg(\neg x \oplus y) \oplus y = \neg(\neg y \oplus x) \oplus x$.

An MV_n algebra is an MV-algebra further satisfying the equations:

$nx = (n-1)x$,
$(kx^{k-1})^n = nx^k$, for every natural $k = 2, \ldots, n-2$ not dividing $n-1$,

where again nx is an abbreviation for $x \oplus \cdots \oplus x$ (n-times) and x^k is an abbreviation for $x \& \ldots \& x$ (k-times), with $x \& y$ standing for $\neg(\neg x \oplus \neg y)$.

If $\sim_{\text{Ł}}$ and $\sim_{\text{Ł}_n}$ denote the relations of provable equivalence over Ł and Ł_n, respectively, the Lindenbaum algebras $Fm(\mathcal{L})/\!\!\sim_{\text{Ł}}$ and $Fm(\mathcal{L})/\!\!\sim_{\text{Ł}_n}$ are examples of (non-linearly ordered) MV and MV_n-algebras, respectively.

A prototypical example of a linearly ordered MV-algebra is the algebra over the real unit interval, so-called *standard*, $[0, 1]_{\mathrm{MV}} = \langle [0, 1], \oplus, \neg, 0 \rangle$, where for each $x, y \in [0, 1]$, $x \oplus y = \min\{1, x + y\}$ and $\neg x = 1 - x$. Replacing the real unit interval $[0, 1]$ by the set $S_n = \{0, 1/n, \ldots, (n-1)/n, 1\}$, and defining the operations $\oplus$ and $\neg$ as the restriction of those of $[0, 1]_{\mathrm{MV}}$ to $\{0, 1/n, \ldots, (n-1)/n, 1\}$, the obtained structure is the *standard* MV_n-algebra $\mathcal{S}_n$.

An *evaluation* of formulas into $[0, 1]_{\mathrm{MV}}$ (or $[0, 1]_{\mathrm{MV}}$-evaluation) is a map e from the set V of propositional variables into $[0, 1]$ which is extended to all formulas by induction as follows: $e(\overline{0}) = 0$, $e(\varphi \oplus \psi) = \max(1, e(\varphi) + e(\psi))$ and $e(\neg \varphi) = \mathbf{1} - e(\varphi)$. A formula φ is a *tautology* over $[0, 1]_{\mathrm{MV}}$ if $e(\varphi) = 1$ for any $[0, 1]_{\mathrm{MV}}$-evaluation e. Moreover, φ is a logical consequence of a set of formulas Γ over $[0, 1]_{\mathrm{MV}}$, written $\Gamma \models_{[0,1]_{\mathrm{MV}}} \varphi$, if $e(\varphi) = 1$ for every $[0, 1]_{\mathrm{MV}}$-evaluation e such that $e(\psi) = 1$ for all $\psi \in \Gamma$. The notions of S_n-evaluation, and tautology and logical consequence over S_n are defined analogously.

Infinitely valued Łukasiewicz logic Ł is known to be finite strongly complete w.r.t. the standard MV-algebra $[0, 1]_{\mathrm{MV}}$ (cf. [13]). This means that, given a finite theory Γ and a formula φ, $\Gamma \vdash_{\text{Ł}} \varphi$ iff $\Gamma \models_{[0,1]_{\mathrm{MV}}} \varphi$. On the other hand, $(n+1)$-valued Łukasiewicz logic Ł_n is known to be strong complete, i.e. it holds that $\Gamma \vdash_{\text{Ł}_n} \varphi$ iff $\Gamma \models_{S_n} \varphi$ for arbitrary theories Γ (cf. [2]).

Remark 2.3. Notice that strong completeness for Ł does not hold, there are infinite theories Γ and formulas φ such that φ is a logical consequence of Γ but $\Gamma \not\vdash_{\text{Ł}} \varphi$. A nice example is available in [13].

2.2. Probability on MV-algebras

The classical notion of (finitely additive) probability measure on Boolean algebras was generalized in [20] by the notion of *state on MV-algebras*.

Definition 2.4 (*Mundici [20]*). By a state of an MV-algebra $\mathcal{A} = \langle A, \oplus, \neg, 0 \rangle$ we mean a function $s : A \to [0, 1]$ satisfying:

 (i) $s(0) = 0$,
 (ii) $s(\neg x) = 1 - s(x)$,
(iii) if $x \& y = 0$, then $s(x \oplus y) = s(x) + s(y)$.

In [20] it is shown that a state s on an MV-algebra $\mathcal{A}$ further satisfies the two following properties for all $x, y \in A$:

(iv) $s(x \oplus y) = s(x) + s(y) - s(x \& y)$,
 (v) if $x \leqslant y$ then $s(x) \leqslant s(y)$.

Interesting examples of MV-algebras are the so-called Łukasiewicz *clans* of functions. Given a non-empty set X, consider the set of functions $[0, 1]^X$. A (Łukasiewicz) *clan* over X is a subset $\mathcal{C} \subseteq [0, 1]^X$ such that

 1. if $f, g \in \mathcal{C}$ then $f \oplus g \in \mathcal{C}$,
 2. if $f \in \mathcal{C}$ then $\neg f \in \mathcal{C}$,

where the operations $\oplus$ and $\neg$ are the point-wise extensions of the operations in the standard MV-algebra $[0, 1]_{\mathrm{MV}}$. A clan $\mathcal{T}$ over X is called a (Łukasiewicz) *tribe* when it is closed with respect to a countable (pointwise) application of the $\oplus$ operation, i.e. if the following condition:

if $\{f_n \mid n \in \mathbb{N}\} \subseteq \mathcal{T}$ then $\bigoplus_{n \in \mathbb{N}} f_n \in \mathcal{T}$

holds. In particular $[0, 1]^X$ and $(S_n)^X$ with the above operations are examples of Lukasiewicz tribes.

In [23], Zadeh introduced the following notion of probability on fuzzy sets. A fuzzy subset of X can be considered just as a function $\mu \in [0, 1]^X$. Then, given a probability measure $p : 2^X \to [0, 1]$ on X, the probability of μ is defined as

$$p^*(\mu) = \sum_{x \in X} \mu(x) \cdot p(x),$$

where we have written $p(x)$ for $p(\{x\})$. Indeed, p^* is an example of state over the tribe $[0, 1]^X$. The restriction of p^* over the S_n-valued fuzzy sets is also an example of state over $(S_n)^X$.

3. The logic $FP(Ł_n, Ł)$ and its semantics

In this section we will define the modal-fuzzy logic $FP(Ł_n, Ł)$. Moreover, the classes of weak and strong probabilistic Kripke models will be introduced.

Definition 3.1. The language of the logic $FP(Ł_n, Ł)$ is built over a countable set of propositional variables $V = \{p_1, p_2, \ldots\}$, the truth-constant $\overline{0}$, the connectives of Łukasiewicz logic, namely $\&$, $\to$, and a symbol P for the modality *probably*. Formulas of $FP(Ł_n, Ł)$ split into two classes:

- The set $Fm(V)$ of non-modal formulas: these will be formulas of $Ł_n$. Non-modal formulas will be denoted by lower case greek letters $\varphi, \psi \ldots$.
- The set $MFm(V)$ of modal formulas, built from atomic modal formulas $P(\varphi)$, with $\varphi \in Fm(V)$, using connectives $\&$, $\to$ and the truth-constant $\overline{0}$. We shall denote them by upper case greek letters $\Phi, \Psi \ldots$.

Axioms and rules of $FP(Ł_n, Ł)$ are as follows:

- Axioms of Ł for modal and non-modal formulas.
- Axioms (Ł5) and (Ł6) restricted to non-modal formulas.
- The following axiom schemata for the modality P:

  (FP1) $P(\neg\varphi) \equiv \neg P(\varphi)$,
  (FP2) $P(\varphi \to \psi) \to (P(\varphi) \to P(\psi))$,
  (FP3) $P(\varphi \oplus \psi) \equiv [(P(\varphi) \to P(\varphi \& \psi)) \to P(\psi)]$.
- The rule of modus ponens (for modal and non-modal formulas).
- The rule of necessitation: from φ derive $P\varphi$.

The notion of *proof* in $FP(Ł_n, Ł)$, denoted $\vdash_{FP}$, is defined as usual. For instance, given a modal theory Γ we will write $\Gamma \vdash_{FP} \Phi$ to denote that Φ is provable from Γ in $FP(Ł_n, Ł)$.

As anticipated in the introduction, for $FP(Ł_n, Ł)$ we consider two kinds of probabilistic Kripke models. The first kind of models is the class of *weak* probabilistic Kripke models which are defined as follows:

Definition 3.2. A weak probabilistic Kripke model (or weak model) for $FP(Ł_n, Ł)$ is a system $\mathcal{M} = \langle W, e, I \rangle$ where:

- W is a non-empty set whose elements are called *nodes*.
- $e : W \times V \to \{0, 1/n, \ldots, (n-1)/n, 1\}$ is such that, for each $w \in W$, $e(w, \cdot) : V \to \{0, 1/n, \ldots, (n-1)/n, 1\}$ is an evaluation of propositional variables which extends to a S_n-evaluation of (non-modal) formulas of $Fm(V)$ in the usual way.

- For each $\varphi \in Fm(V)$, define the *course of values* of φ as the function $\varphi_W^{\#} : W \to [0, 1]$ by putting $\varphi_W^{\#}(w) = e(w, \varphi)$. The set of courses of values $Fm_W = \{\varphi_W^{\#} \mid \varphi \in Fm(V)\}$ is a clan over W.
- I is a state over the clan Fm_W, i.e. $I : Fm_W \to [0, 1]$ satisfies:
  (i) $I(\overline{1}_W^{\#}) = 1$,
  (ii) $I(\neg \varphi_W^{\#}) = 1 - I(\varphi_W^{\#})$,
  (iii) $I(\varphi_W^{\#} \oplus \psi_W^{\#}) = I(\varphi_W^{\#}) + I(\psi_W^{\#}) - I(\varphi_W^{\#} \& \psi_W^{\#})$,

  where $\neg$, $\oplus$ and $\&$ here are taken as the point-wise extensions of the Łukasiewicz operations in $[0, 1]_{\text{MV}}$.

Given a weak probabilistic Kripke model $\mathcal{M}$ for $FP(Ł_n, Ł)$, a formula Φ and a $w \in W$, the truth value of Φ in $\mathcal{M}$ at the node w ($\|\Phi\|_{\mathcal{M},w}$) is inductively defined as follows:

- if Φ is a non-modal formula φ, then $\|\varphi\|_{\mathcal{M},w} = e(w, \varphi)$,
- if Φ is an atomic modal formula $P(\psi)$, then $\|P(\psi)\|_{\mathcal{M},w} = I(\psi_W^{\#})$,
- if Φ is a non-atomic modal formula, then its truth value is computed by evaluating its atomic modal sub-formulas, and then by using the truth functions associated to the Ł-connectives occurring in Φ.

Note that if Φ is a modal formula, then its truth value in a weak probabilistic Kripke model is independent from w, thus we will omit the subscript w. The notions of *model* and *validity of a formula in a theory* are defined as usual.

The second kind of models for $FP(Ł_n, Ł)$ is the class of *strong* probabilistic Kripke models which are defined as follows.

Definition 3.3. A strong probabilistic Kripke model (or strong model) for $FP(Ł_n, Ł)$ is a system $\mathcal{N} = \langle W, e, \sigma \rangle$, where W and e are defined as in the case of a weak probabilistic Kripke model (Definition 3.2) and σ is a probability distribution on W, i.e. $\sigma : W \to [0, 1]$ satisfies

$$\sum_{w \in W} \sigma(w) = 1.$$

Evaluations of formulas of $FP(Ł_n, Ł)$ in a strong probabilistic Kripke model $\mathcal{N}$ are defined as in the case of weak model except for the case of atomic modal formulas:

- If Φ is an atomic modal formula $P(\psi)$, then

$$\|P(\psi)\|_{\mathcal{N}} = \sum_{w \in W} e(w, \psi) \cdot \sigma(w).$$

Notice that the name *strong* is indeed justified by the fact that each strong probabilistic model $\mathcal{M} = \langle W, e, \sigma \rangle$ induces a weak probabilistic model $\mathcal{M}' = \langle W, e, I_\sigma \rangle$, where $I_\sigma : Fm_W \to [0, 1]$ is defined as

$$I_\sigma(\varphi_W^{\#}) = \sum_{w \in W} e(w, \varphi) \cdot \sigma(w),$$

which is equivalent in the sense that $\|\Phi\|_{\mathcal{M}} = \|\Phi\|_{\mathcal{M}'}$ for any modal Φ.

It is easy to show that $FP(Ł_n, Ł)$ is sound with respect to the classes of both weak and strong probabilistic models.

4. Weak and strong probabilistic completeness for $FP(Ł_n, Ł)$

This section will be devoted to the proof of the main theorems of this paper. Namely, we are going to prove that $FP(Ł_n, Ł)$ is complete w.r.t. the classes of both weak and strong probabilistic models.

To do this we will follow the technique used in [13]: for each modal formula Θ, let $\Theta^\star$ be obtained from Θ by replacing every occurrence of an atomic sub-formula of the form $P(\varphi)$ by a new propositional variable p_φ. We write $(P(\varphi))^\star = p_\varphi$ and, for each modal formulas Θ and Λ, we inductively define $(\Theta \circ \Lambda)^\star = \Theta^\star \circ \Lambda^\star$ (with $\circ \in \{\&, \to\}$) and $\overline{0}^\star = \overline{0}$.

Let now Γ be a modal theory of $FP(Ł_n, Ł)$. Analogously we can define $\Gamma^\star$ and $FP^\star$ as

$$\Gamma^\star = \{\Psi^\star \mid \Psi \in \Gamma\}$$

and

$$FP^\star = \{\Upsilon^\star \mid \Upsilon \text{ is an instance of } (FPi), \ i = 1, 2, 3\} \cup \{p_\varphi \mid Ł_n \vdash \varphi\},$$

respectively.

Using the same technique used in [13] it is not difficult to prove that, if Φ is any modal formula of $FP(Ł_n, Ł)$

$$\Gamma \ \nvdash_{FP} \Phi \text{ iff } \Gamma^\star \cup FP^\star \ \nvdash_Ł \Phi^\star. \tag{1}$$

Since the set $V^0 \subset V$ of propositional variables appearing in $\Gamma \cup \{\Phi\}$ is finite, without loss of generality we can assume to work with a finitely generated (over V^0) non-modal language $Fm(V^0)$.

Notice then that the Lindenbaum algebra

$$Fm(V^0)/\sim_n,$$

where $\sim_n$ denotes the relation of provable equivalence in $Ł_n$, is finite (see [2] for more details). This means that there are only finitely many different classes

$$[\varphi]_{\sim_n} = \{\psi \in Fm(V^0) \mid Ł_n \vdash \varphi \equiv \psi\},$$

For each $[\varphi]_{\sim_n}$ we can choose a representative of the class, we will denote it by $\varphi^\square$. Again notice that there are only finitely many $\varphi^\square$'s. Let us now adopt the following further translation:

- For each modal formula Φ, let $\Phi^\square$ be the formula resulting from the substitution of each propositional variable p_φ occurring in $\Phi^\star$ by $p_{\varphi^\square}$.
- If $\Phi = \Theta \circ \Lambda$ then $\Phi^\square = \Theta^\square \circ \Lambda^\square$ (with $\circ \in \{\&, \rightarrow\}$) and $\overline{0}^\square = \overline{0}$.

In accordance with such translation, we define $\Gamma^\square$ and $FP^\square$ as:

$$\Gamma^\square = \{\Psi^\square \mid \Psi^\star \in \Gamma^\star\}$$

and

$$FP^\square = \{\Upsilon^\square \mid \Upsilon \text{ is an instance of } (FPi), \ i = 1, 2, 3\} \cup \{p_{\varphi^\square} \mid Ł_n \vdash \varphi\}.$$

Now we can prove the following:

Lemma 4.1. $\Gamma^\star \cup FP^\star \vdash_Ł \Phi^\star$ iff $\Gamma^\square \cup FP^\square \vdash_Ł \Phi^\square$.

Proof. ($\Leftarrow$): Let $\Gamma^\square \cup FP^\square \vdash_Ł \Phi^\square$. Then, in order to get the claim we have to show that $\Gamma^\star \cup FP^\star \vdash_Ł \Phi^\star$ for each Φ such that its "boxed" translation is $\Phi^\square$. For instance, if $\Phi = P\psi$ then $\Phi^\square = p_{\psi^\square} = p_{\gamma^\square}$ for each $\gamma \in [\psi]_{\sim_n}$, therefore, if $\Gamma^\square \cup FP^\square \vdash_Ł p_{\varphi^\square}$ we have to show that $\Gamma^\star \cup FP^\star \vdash_Ł p_\gamma$ for each $\gamma \in [\varphi]_{\sim_n}$.

First of all let us prove the following:

Claim A. Let φ, ψ be $Ł_n$-formulas. Then, if $Ł_n \vdash \varphi \equiv \psi$, then $FP(Ł_n, Ł) \vdash P(\varphi) \equiv P(\psi)$ (and in particular $FP^\star \vdash_Ł p_\varphi \equiv p_\psi$).

Proof of Claim A. $Ł_n \vdash \varphi \equiv \psi$ means that $Ł_n \vdash (\varphi \rightarrow \psi) \& (\psi \rightarrow \varphi)$ and thus $Ł_n \vdash (\varphi \rightarrow \psi) \wedge (\psi \rightarrow \varphi)$. In particular, we have that $Ł_n \vdash \varphi \rightarrow \psi$ and $Ł_n \vdash \psi \rightarrow \varphi$. If $Ł_n \vdash \varphi \rightarrow \psi$, then $FP(Ł_n, Ł) \vdash P(\varphi \rightarrow \psi)$ and therefore (it follows from the monotonic property of P) $FP(Ł_n, Ł) \vdash P(\varphi) \rightarrow P(\psi)$. Analogously we can show that, if $Ł_n \vdash \psi \rightarrow \varphi$, then $FP(Ł_n, Ł) \vdash P(\psi) \rightarrow P(\varphi)$. Therefore, $FP(Ł_n, Ł) \vdash P(\varphi) \rightarrow P(\psi)$ and $FP(Ł_n, Ł) \vdash P(\psi) \rightarrow P(\varphi)$ and thus $FP(Ł_n, Ł) \vdash P(\varphi) \equiv P(\psi)$. $\square$

Let us now turn back to the proof of Lemma 4.1. Let Φ be a modal formula of $FP(\text{Ł}_n, \text{Ł})$ and let $P(\varphi_1), \ldots, P(\varphi_k)$ be all the atomic modal formulas occurring in Φ. If $\Gamma^\square \cup FP^\square \vdash_\text{Ł} \Phi^\square$, then, it easily follows from the above claim that $\Gamma^\star \cup FP^\star \vdash_\text{Ł} \Phi^\star$ where $\Phi^\star$ is any Ł-formula obtained by replacing each occurrence of a propositional variable p_{φ_i} with another one p_{ψ_i} such that $\psi_i \in [\varphi_i]_{\sim_n}$. In fact, if $\psi_i \in [\varphi_i]_{\sim_n}$, then $\text{Ł}_n \vdash \psi_i \equiv \varphi_i$ and therefore, from the above claim, $FP^\star \vdash_\text{Ł} p_{\psi_i} \equiv p_{\varphi_i}$. Thus p_{φ_i} can be substituted with p_{ψ_i} without loss of generality in the proof. Thus, in particular $\Gamma^\star \cup FP^\star \vdash_\text{Ł} \Phi^\star$ and this direction is complete.

($\Rightarrow$): In order to prove the other direction let us assume $\Gamma^\star \cup FP^\star \vdash_\text{Ł} \Phi^\star$ and let $\Psi_1^\star, \ldots, \Psi_k^\star$ be an Ł-proof of $\Phi^\star$ in $\Gamma^\star \cup FP^\star$. For each $1 \leqslant j \leqslant k$ replace $\Psi_j^\star$ with $\Psi_j^\square$, the representative of its equivalence class modulo $\sim_n$. Clearly $\Psi_1^\square, \ldots, \Psi_k^\square$ is an Ł-proof of (a formula logically equivalent to) $\Phi^\square$. In fact, if $\Psi_k^\star = \Phi^\star$, then $\Psi_k^\square \equiv \Phi^\square$. Moreover, for each $1 \leqslant i < k$ one of the following holds:

(i) $\Psi_i^\square$ is (logically equivalent to) an axiom of Ł,
(ii) $\Psi_i^\square \in \Gamma^\square \cup FP^\square$,
(iii) if $\Psi_t^\star$ is obtained by modus ponens from $\Psi_s^\star \to \Psi_t^\star$ and $\Psi_s^\star$, then we claim that $\Psi_t^\square$ is obtained by modus ponens from $\Psi_s^\square \to \Psi_t^\square$ and $\Psi_s^\square$. In fact we have just to note that $(\Psi_s \to \Psi_t)^\square = \Psi_s^\square \to \Psi_t^\square$ and thus the claim easily follows.

Moreover, since modus ponens is the only inference rule of Ł we have nothing to add. This concludes the proof of Lemma 4.1. $\square$

Theorem 4.2. *The logic $FP(\text{Ł}_n, \text{Ł})$ is sound and (finite) strongly complete with respect to the class of weak probabilistic Kripke models.*

Proof. As usual soundness is easy. In order to prove is completeness, let us assume that $\Gamma \cup \{\Phi\}$ a finite modal theory of $FP(\text{Ł}_n, \text{Ł})$ such that $\Gamma \not\vdash_{FP} \Phi$.

By using (1) and Lemma 4.1 we have the following:

$$\Gamma \not\vdash_{FP} \Phi \text{ iff } \Gamma^\square \cup FP^\square \not\vdash_\text{Ł} \Phi^\square. \tag{2}$$

The task is now to find a weak probabilistic Kripke model $\mathcal{M}$ of Γ such that $\|\Phi\|_\mathcal{M} < 1$. Notice that now $\Gamma^\square \cup FP^\square$ is a finite Ł-theory, therefore (2) and the finite strong standard completeness of Łukasiewicz logic ensure that, if $\Gamma \not\vdash_{FP} \Phi$, then there exists an Ł-evaluation v which is a model for $\Gamma^\square \cup FP^\square$, and such that $v(\Phi^\square) < 1$.

Let now $\mathcal{M}$ be the system $\mathcal{M} = \langle \Omega_n, e, I \rangle$, where Ω_n is a shorthand for $\Omega_{\text{Ł}_n}$, the class of all the Ł_n-evaluations over the formulas $Fm(V)$, $e : \Omega_n \times V \to \{0, 1/n, \ldots, n-1/n, 1\}$ is defined as $e(w, q) = w(q)$ if $q \in V^0$ and $e(w, q) = 0$ otherwise, and $I : Fm_{\Omega_n} \to [0, 1]$ is defined as $\mu(\varphi_{\Omega_n}^\#) = v(p_{\varphi^\square})$. For the sake of a lighter notation, we will write $\varphi^\#$ instead of $\varphi_{\Omega_n}^\#$. In order to prove that $\mathcal{M}$ is a weak probabilistic Kripke model for $FP(\text{Ł}_n, \text{Ł})$ we have just to show that the following properties for I hold:

(1) If $\text{Ł}_n \vdash \varphi \equiv \psi$, then $I(\varphi^\#) = I(\psi^\#)$: if $\text{Ł}_n \vdash \varphi \equiv \psi$, then $\varphi^\square = \psi^\square$ and thus the claim follows.
(2) If $\text{Ł}_n \vdash \varphi$, then $I(\varphi^\#) = 1$: if $\text{Ł}_n \vdash \varphi$, then $\varphi \in [\bar{1}]$ and thus the claim follows by the above property (1).
(3) $I(\neg\varphi^\#) = 1 - I(\varphi^\#)$. This instance of axiom *FP1*, $p_{(\neg\varphi)^\square} \equiv \neg p_{\varphi^\square}$, is in $FP^\square$, hence we have $I(\neg\varphi^\#) = I((\neg\varphi)^\#) = v(p_{(\neg\varphi)^\square}) = v(\neg p_{\varphi^\square}) = 1 - v(p_{\varphi^\square}) = 1 - I(\varphi^\#)$.
(4) $I(\varphi^\# \oplus \psi^\#) = I(\varphi^\#) + I(\psi^\#) - I(\varphi^\# \& \psi^\#)$. This instance of axiom *FP2*, $p_{(\varphi \oplus \psi)^\square} \equiv (p_{\varphi^\square} \to p_{\psi^\square}) \to p_{(\varphi \& \psi)^\square}$, is in $FP^\square$, hence $I(\varphi^\# \oplus \psi^\#) = I((\varphi \oplus \psi)^\#) = v(p_{(\varphi \oplus \psi)^\square}) = v(p_{\varphi^\square}) + v(p_{\psi^\square}) - v(p_{(\varphi \& \psi)^\square}) = I(\varphi^\#) + I(\psi^\#) - I(\varphi^\# \& \psi^\#)$.

Then $\mathcal{M}$ is a weak probabilistic Kripke model for $FP(\text{Ł}_n, \text{Ł})$. Moreover, it is trivial to observe that $\mathcal{M}$ is clearly a model for Γ, but $\|\Phi\|_\mathcal{M} < 1$. This ends the proof of the theorem. $\square$

Theorem 4.3. *The logic $FP(\text{Ł}_n, \text{Ł})$ is sound and (finite) strongly complete with respect to the class of strong probabilistic Kripke models.*

Proof. Let $\Gamma \cup \{\Phi\}$ be a finite subset of modal formulas of $FP(\text{Ł}_n, \text{Ł})$ and let us assume $\Gamma \nvdash_{FP} \Phi$. From the above theorem, there is a weak probabilistic Kripke model $\mathcal{M}$ for $FP(\text{Ł}_n, \text{Ł})$ such that $\|\Psi\|_{\mathcal{M}} = 1$ for each $\Psi \in \Gamma$, but $\|\Phi\|_{\mathcal{M}} < 1$. Moreover, without loss of generality we can assume $\mathcal{M} = \langle \Omega_n, e, I \rangle$ to be the weak model defined as in the previous proof. In particular recall that Ω_n is the set of all the Ł_n-evaluations over the propositional variables V, $e : W \times V \to \{0, 1/n, \ldots, n-1/n, 1\}$, for each $w \in \Omega_n$ and for each $q \in V$, is defined as $e(w, q) = w(q)$ if q is a propositional variable occurring in $\Gamma \cup \{\Phi\}$ and $e(w, q) = 0$ otherwise, and $I : Fm_{\Omega_n}(V) \to [0, 1]$ is defined as $I(\varphi^{\#}) = v(p_{\varphi^{\square}})$ ($\varphi^{\square}$ being defined as above). The state I defines a *probability* on formulas μ by putting $\mu(\varphi) = I(\varphi^{\#})$.

Now let us recall what Paris has shown in [22]. For each $i = 0, \ldots, n$, the McNaughton theorem ensures the existence of a formula $\psi_i(q)$, defined using just the propositional variable q, such that, for each Ł_n-evaluation w,

$$w(\psi_i(q)) = \begin{cases} 1 & \text{if } w(q) = i/n, \\ 0 & \text{otherwise.} \end{cases}$$

Now remember that we are working with the finite set $V^0 = \{q_1, \ldots, q_m\}$ of propositional variables, then for each $w \in W$, let χ_w be the following formula:

$$\chi_w = \psi_{i_1}(q_1) \wedge \psi_{i_2}(q_2) \wedge \cdots \wedge \psi_{i_m}(q_m),$$

where $w(q_r) = i_r/n$. It is easy to see that $w(\chi_w) = 1$ and $w(\chi_{w'}) = 0$ for each $w' \neq w$. Then, under such conditions, Paris shows that the probability μ is defined as

$$\mu(\varphi) = \sum_{w \in \Omega_n} \mu(\chi_w) \cdot w(\varphi)$$

(see [22] for more details).

Now we define $\mathcal{N} = \langle \Omega_n, e, \sigma \rangle$, where Ω_n and e are defined as in $\mathcal{M}$, and $\sigma : W \to [0, 1]$ is defined as $\sigma(w) = \mu(\chi_w)$. Then the following holds:

Claim B. (1) $\sum_{w \in W} \sigma(w) = 1$, i.e. $\mathcal{N}$ *is a strong probabilistic Kripke model for* $FP(\text{Ł}_n, \text{Ł})$
(2) *For each modal formula* Θ, $\|\Theta\|_{\mathcal{N}} = v(\Theta^{\square})$.

Proof of Claim B. (1) It is easy to check (see e.g. [22]) that $\text{Ł}_n \vDash \bigvee_{w \in W} \chi_w$, and since the χ_w's are Boolean, i.e. $\text{Ł}_n \vDash \chi_w \vee \neg \chi_w$, the weak and strong disjunction coincide over the χ_w's, hence we also have

$$\text{Ł}_n \vDash \bigoplus_{w \in W} \chi_w.$$

Moreover, since the χ_w's are also mutually contradictory, i.e. $\text{Ł}_n \vDash \chi_w \& \chi_{w'} \equiv \overline{0}$ for $w \neq w'$, we also have

$$\text{Ł}_n \vDash \neg \chi_w \equiv \bigoplus_{w' \neq w} \chi_{w'}.$$

Thus we get $FP(\text{Ł}_n, \text{Ł}) \vdash P(\bigoplus_{w \in W} \chi_w)$, and $FP(\text{Ł}_n, \text{Ł}) \vdash \neg P\chi_w \equiv P(\bigoplus_{w' \neq w} \chi_{w'})$. Now, by (FP3) and taking into account that $\chi_w \& \chi_{w'} \equiv \overline{0}$ for $w \neq w'$, we get $FP(\text{Ł}_n, \text{Ł}) \vdash \bigoplus_{w \in W} P\chi_w$ and $FP(\text{Ł}_n, \text{Ł}) \vdash \neg P\chi_w \equiv \bigoplus_{w' \neq w} P\chi_{w'}$, hence

$$FP^{\square} \vdash_{\text{Ł}} \bigoplus_{w \in W} p_{\chi_w^{\square}}$$

and

$$FP^{\square} \vdash_{\text{Ł}} \neg p_{\chi_w} \equiv \bigoplus_{w' \neq w} p_{\chi_{w'}^{\square}}.$$

Then, since v is a model of $FP^\square$, it follows:

$$1 = v\left(\bigoplus_{w \in W} p_{\chi_w^\square}\right) = \sum_{w \in W} v(p_{(\chi_w^\square)}) = \sum_{w \in W} \mu(\chi_w) = \sum_{w \in W} \sigma(w).$$

(2) By induction:

- $\Phi = P\varphi$,
  $\|\Phi\|_M = \sum_w \sigma(w) \cdot e(w, \varphi) = \sum_w \mu(\chi_w) \cdot w(\varphi) = \mu(\varphi) = v(p_{\varphi^\square}) = v(\Phi^\square)$,
  $\Phi = \Psi \circ \Theta$, for $\circ \in \{\&, \rightarrow\}$,
  $\|\Phi\|_M = \|\Psi \circ \Theta\|_M = \|\Psi\|_M \odot \|\Theta\|_M = v(\Psi^\square) \odot v(\Theta^\square) = v((\Psi \circ \Theta)^\square) = v(\Phi^\square)$,

where $\odot$ is the truth-function corresponding to $\circ$. $\quad\square$

Finally, it follows from the above claim that $\|\Psi\|_M = 1$ for each $\Psi \in \Gamma$, but $\|\Phi\|_M < 1$ as desired. This completes the proof. $\quad\square$

5. Extensions

In this section we comment on two extensions of the logics $FP(\text{Ł}_n, \text{Ł})$ regarding the logic used to reason with probability formulas. One is obtained replacing Ł by the so-called Rational Pavelka logic, RPL, the expansion of Ł with rational truth-constants. The resulting logic $FP(\text{Ł}_n, RPL)$ will allow us to reason with explicit (rational) probability values. The other is obtained replacing Ł by the stronger logic $\text{Ł}\Pi\frac{1}{2}$ in order to allow also reasoning with conditional probabilities of fuzzy events.

5.1. Extending the logic $FP(\text{Ł}_n, \text{Ł})$ with rational truth values

In this section we are going to extend the logic $FP(\text{Ł}_n, \text{Ł})$ with truth-constants for the rational values in $[0, 1]$. The idea underlying this extension is the same as used by Hájek in [13] in order to add the rational values to $FP(\text{Ł})$. In order to define such an extension we will replace the Łukasiewicz logic Ł in $FP(\text{Ł}_n, \text{Ł})$ by the so-called Rational Pavelka logic, *RPL* for short.

RPL is obtained by adding to the language of Ł a countable class of truth-constants, one constant $\bar{r}$ for every rational $r \in [0, 1]$. The axioms and rules of *RPL* are those of Ł plus the following schema for rational constants (the so called *bookkeeping axioms*), namely:

$$(\bar{r} \rightarrow \bar{s}) \equiv \overline{\min(1, 1 - r + s)},$$

$$(\bar{r}\&\bar{s}) \equiv \overline{\max(0, r + s - 1)},$$

for all $r, s \in [0, 1] \cap \mathbb{Q}$. Using *RPL* as base many-valued logic, we can define the following fuzzy-modal logic.

Definition 5.1. The language of the logic $FP(\text{Ł}_n, RPL)$ over a countable set of propositional variables V is obtained by adding to the language of $FP(\text{Ł}_n, \text{Ł})$ a truth-constant $\bar{r}$ for every rational $r \in [0, 1]$. Formulas of $FP(\text{Ł}_n, RPL)$ consist of

(i) The set $Fm(V)$ of non-modal formulas as in the case of $FP(\text{Ł}_n, \text{Ł})$ (see Definition 3.1).
(ii) The class of modal formulas $RMFm(V)$ is defined as the smallest set of formulas such that, $P(\varphi) \in RMFm(V)$ for each $\varphi \in Fm(V)$, $\bar{r} \in RMFm(V)$ for each $r \in [0, 1] \cap \mathbb{Q}$, and closed under the connectives of Łukasiewicz logic.

The axioms and rules of $FP(\text{Ł}_n, RPL)$ are those of $FP(\text{Ł}_n, \text{Ł})$ plus the previously defined book-keeping axioms for the truth-constants $\bar{r}$.

The notion of *proof*, still denoted $\vdash_{FP}$, is defined as usual.

Weak and Strong models for $FP(Ł_n, RPL)$ are defined as in the case of $FP(Ł_n, Ł)$ (see Definitions 3.2 and 3.3). The evaluation of $FP(Ł_n, RPL)$-formulas into a weak (strong) models is defined by adding to Definition 3.2 (Definition 3.3 resp.) a further condition for the truth-constants:

$$\|\bar{r}\|_{\mathcal{M}} = r.$$

for each $r \in [0, 1] \cap \mathbb{Q}$ in each weak (strong) probabilistic model $\mathcal{M}$.

Notice that the logic $FP(Ł_n, RPL)$ is quite more expressive that $FP(Ł_n, Ł)$. In fact it is now possible to deal with formulas like, for instance, $P(\varphi) \leftrightarrow \frac{1}{2}$ and $P(\psi) \rightarrow \frac{1}{3}$ whose intended interpretation is that *the probability of φ is $\frac{1}{2}$* and *the probability of ψ is at most $\frac{1}{3}$*, respectively, and so forth.

From Theorems 4.2 and 4.3, it is not difficult to prove that $FP(Ł_n, RPL)$ is also sound and (finite) strongly complete w.r.t. both the classes of weak and strong probabilistic models. In fact RPL is also finitely strong standard complete (see [13]). This means that, if $\Gamma \cup \{\varphi\}$ is a finite RPL-theory, then $\Gamma \vdash_{RPL} \varphi$ iff $e(\varphi) = 1$ for each RPL-evaluation e model of Γ.

Let now $\Gamma \cup \{\Phi\}$ be a modal theory over $FP(Ł_n, RPL)$ and let us assume $\Gamma \nvdash_{FP} \Phi$ (where $\vdash_{FP}$ is now in the sense of $FP(Ł_n, RPL)$). Let $\square$ be the translation mapping defined as in the proof of Theorem 4.2. Following the same line used in such a proof it is easy to prove that the following hold:

(1) $\Gamma \nvdash_{FP} \Phi$ iff $\Gamma^{\square} \cup FP^{\square} \nvdash_{RPL} \Phi^{\square}$,
(2) $\Gamma^{\square} \cup FP^{\square}$ is a finite RPL-theory.

Therefore, also in this case it is easy to define (modulo the finite strong standard completeness of RPL) a weak $FP(Ł_n, RPL)$-model $\mathcal{M}$ such that $\|\Psi\|_{\mathcal{M}} = 1$ for each $\Psi \in \Gamma$, but $\|\Phi\|_{\mathcal{M}} < 1$.

It is straightforward to adapt the proof of Theorem 4.3 also to $FP(Ł_n, RPL)$, therefore, $FP(Ł_n, RPL)$ is also finite strong complete w.r.t. the class of strong probabilistic models.

As usual, when we extend a logic by means of rational truth values it is possible to define the notions of *provability degree* and *truth degree* of a formula ψ over an *arbitrary* theory Γ. For $FP(Ł_n, RPL)$ they are defined as follows:

Definition 5.2. Let Γ be an $FP(Ł_n, RPL)$ modal theory and let Φ be a modal formula. Then, the *provability degree* of Φ over Γ is defined as

$$|\Phi|_{\Gamma} = \sup\{r \in [0, 1] \cap \mathbb{Q} \mid \Gamma \vdash_{FP} \bar{r} \rightarrow \Phi\},$$

and the *truth degree* of Φ over Γ is defined as

$$\|\Phi\|_{\Gamma} = \inf\{\|\Phi\|_{\mathcal{M}} \mid \mathcal{M} \text{ is a weak probabilistic model of } \Gamma\}.$$

Now we are going to show that $FP(Ł_n, RPL)$ is Pavelka-style complete. Just as a remark notice that, with respect to this kind of completeness, we are allowed to relax the hypothesis about the cardinality of the modal theory we are working with. In fact Γ is assumed to be an arbitrary (countable) theory, not necessarily finite. This is due to the fact that RPL is indeed strong Pavelka-style complete. Now the Pavelka-style completeness for $FP(Ł_n, RPL)$ reads as follows.

Theorem 5.3. *For each modal theory Γ over $FP(Ł_n, RPL)$ and each modal formula Φ,*

$$|\Phi|_{\Gamma} = \|\Phi\|_{\Gamma}.$$

The proof of this theorem is routine (see for instance [13, Theorem 8.4.9]).
We end this section with the following remark.

Remark 5.4. Note in the above proof that we have not required the modal theory Γ to be finite. This would allow us to also prove a Pavelka-syle completeness also for the logic $FP(Ł, RPL)$ using the same working methodology. This result can be seen as another step forward in the direction of proving the logic $FP(Ł, Ł)$ to be finite strong complete w.r.t. the classes of weak and/or strong probabilistic Kripke models.

5.2. Towards a logic of conditional probability for fuzzy events

While there seems to be an agreement on the notion of state as the proper generalization of probability on MV-algebras, the generalization of the notion of conditional probability on MV-algebras is a matter of discussion. Indeed, in the last years, different answers have been provided to the question "what is a conditional state?" (see [5,11,15]). Roughly speaking there have been two main approaches: in the first one, exploited in [5,11], and similarly to what happens e.g. in classical probability theory under de Finetti's interpretation, a conditional state is introduced as a primitive notion, that is, as a two-place function $s(\cdot \mid \cdot)$ satisfying some basic properties. For instance, Gerla generalizes in [11] a notion of conditional state previously proposed by Di Nola et al. in [5]. The definition is as follows, where $\mathbb{B}(\mathcal{A})$ denotes the Boolean skeleton of the MV-algebra $\mathcal{A}$, that is, $\mathbb{B}(\mathcal{A}) = \{x \in A \mid x \oplus x = x\}$, which is the largest sub-Boolean algebra of $\mathcal{A}$.

Definition 5.5 (*Gerla [11]*). A conditional state of an MV-algebra $\mathcal{A}$ is a function $s : A \times B \to [0, 1]$, where $B \subseteq A$ is an MV-bunch,[1] satisfying the following conditions:

(i) $s(\cdot \mid y)$ is a state on $\mathcal{A}$ for every $y \in B$,
(ii) $s(y \mid y) = 1$ for each $y \in B \cap \mathbb{B}(\mathcal{A})$,
(iii) $s(x \& y \mid z) = s(y \mid z) \cdot s(x \mid y \& z)$ for any $x \in A$, $y \in \mathbb{B}(\mathcal{A})$, $z \in B \cap \mathbb{B}(\mathcal{A})$ such that $y \& z \in B$,
(iv) $s(x \mid y) \cdot s(y \mid 1) = s(y \mid x) \cdot s(x \mid 1)$ for any $x, y \in B$.

The other approach has been essentially developed by Kroupa in [15] where the notion of conditional state has been introduced as definable from the notion of state on a MV-algebra enriched with a product operation. A structure $\mathcal{A}' = (A, \oplus, \odot, \neg, 0)$ is an MV-algebra with product (those algebras are also called PMV-algebras in [19]) if $\mathcal{A} = (A, \oplus, \neg, 0)$ is an MV-algebra and $\odot$ is a commutative and associative binary operation on A such that for all $a, b, c \in A$:

(1) $1 \odot a = a$,
(2) $a \odot (b \oplus c) = (a \odot b) \oplus (a \odot c)$.

Definition 5.6 (*Kroupa [15]*). Let $\mathcal{A}' = (A, \oplus, \odot, \neg, 0)$ be an MV-algebra with product and let s be a state on $\mathcal{A}$. Then, a non-negative real number $s(x \mid y)$ is a conditional state of x *given* y if $s(x \mid y)$ is any solution of the equation

$$s(y) \cdot s(x \mid y) = s(x \odot y).$$

It is clear that when $s(y) > 0$, a conditional state is simply defined as

$$s(x \mid y) = \frac{s(x \odot y)}{s(y)},$$

for all $x \in A$. In such a case, $s(\cdot \mid y)$ is a state on $\mathcal{A}$. It is worth noticing here that if we replace in the above definition $\odot$ by the MV-algebra conjunction $\&$, then $s(\cdot \mid y)$ might not be a state any longer.

In this section our aim is to provide some ideas about how to extend the approach developed in Sections 4 and 5 to come up with a (fuzzy) logical formalization of conditional probability over MV-events. Since Kroupa's approach needs to extend the algebra of events with a new product connective, and this brings technical difficulties from the logical point of view, we will:

- keep the modeling of fuzzy events as $Ł_n$ propositions,
- adopt Gerla and Di Nola et al.'s definition of conditional state as a primitive notion,
- adopt the logic $Ł\Pi\frac{1}{2}$ as base fuzzy logic to reason about conditional probabilities of $Ł_n$ events.

Recall that the $Ł\Pi\frac{1}{2}$ logic combines in a single framework both the connectives of Łukasiewicz logic ($\&, \to$) and the connectives of Product logic Π ($\&_\Pi, \to_\Pi$), as well as an additional truth-constant $\overline{\frac{1}{2}}$, see [3,4,7] for details. We will

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[1] For any MV-algebra $\mathcal{A} = \langle A, \oplus, \neg, 0, 1 \rangle$, $B \subseteq A$ is an *MV-bunch* if $1 \in B$, $0 \notin B$, and B is closed under $\oplus$. In [11], the following example has been presented: Let $\mathcal{A}$ be an MV-algebra and let s be a state on $\mathcal{A}$, then the set $B = \{x \in A \mid s(x) \neq 0\}$ is an MV-bunch. In this case B is also said to be the MV-bunch of s.

define below a logic $FCP(Ł_n, Ł\Pi\frac{1}{2})$ introducing a binary modality P to deal with conditional states which can be considered as an extension of the logic $FCP(Ł\Pi)$ developed in [18] for reasoning about conditional probability on classical (Boolean) events.

Indeed, formulas of $FCP(Ł_n, Ł\Pi\frac{1}{2})$ again split into two classes: (i) the set of non-modal formulas $Fm(V)$, which are formulas of $Ł_n$ like in $FP(Ł_n, Ł)$; and (ii) letting $Sat(V) = \{\varphi \in Fm(V) \mid Ł_n \nvdash \neg\varphi\}$, the set of modal formulas built from atomic modal formulas $P(\varphi \mid \psi)$, with $\varphi \in Fm(V)$ and $\psi \in Sat(V)$, using the connectives and constants of $Ł\Pi\frac{1}{2}$ ($\&, \rightarrow, \&_\Pi, \rightarrow_\Pi, \overline{0}, \overline{\frac{1}{2}}$). Axioms and rules of $FP(Ł_n, Ł)$ are as follows, where $\mathbb{B}(V) = \{\varphi \mid Ł_n \vdash \varphi \vee \neg\varphi\}$:

- Axioms and rules of $Ł_n$ for non-modal formulas, and axioms and rules of $Ł\Pi\frac{1}{2}$ for modal formulas.
- The following axioms for the modality P:

(CP1) $P(\varphi \rightarrow \chi \mid \psi) \rightarrow (P(\varphi \mid \psi) \rightarrow P(\chi \mid \psi))$,
(CP2) $P(\neg\varphi \mid \psi) \equiv \neg P(\varphi \mid \psi)$,
(CP3) $P(\varphi \oplus \chi \mid \psi) \equiv [(P(\varphi \mid \psi) \rightarrow P(\varphi\&\chi \mid \psi)) \rightarrow P(\chi \mid \psi)]$,
(CP4) $P(\psi \mid \psi)$, for each $\psi \in \mathbb{B}(V)$,
(CP5) $P(\varphi\&\chi \mid \psi) \equiv P(\chi \mid \psi)\&_\Pi P(\varphi \mid \chi\&\psi)$, for each $\chi, \psi \in \mathbb{B}(V)$,
(CP6) $P(\varphi \mid \psi)\&_\Pi P(\psi \mid \overline{1}) \equiv P(\psi \mid \varphi)\&_\Pi P(\varphi \mid \overline{1})$.

- The rule of necessitation: from φ derive $P(\varphi \mid \psi)$.
- The rule of substitution of equivalents: from $\varphi \equiv \psi$ derive $P(\chi \mid \varphi) \equiv P(\chi \mid \psi)$, for $\varphi, \psi \in \mathbb{B}(V)$.

The intended semantics for $FCP(Ł_n, Ł\Pi\frac{1}{2})$ is given by the class of *conditional probabilistic Kripke models* $\mathcal{M} = \langle W, e, \mu \rangle$, where:

(1) W is a set of worlds and $e : W \times Fm(V) \rightarrow S_n$ is such that $e(w, \cdot)$ is a $\mathcal{S}_n$-evaluation for each $w \in W$.
(2) Let $Fm_W = \{\varphi_W^\# \mid \varphi \in Fm(V)\}$ be the clan over W as defined in Definition 3.2. Then μ is a conditional state on $Fm_W \times (Fm_W \setminus \{\overline{0}_W^\#\})$.
(3) $e(w, P(\varphi \mid \psi)) = \mu(\varphi_W^\#, \psi_W^\#)$, if $\psi_W^\# \neq \overline{0}_W^\#$; otherwise let it be undefined.
(4) e is extended to modal formulas using $Ł\Pi\frac{1}{2}$ truth-functions when defined.

A formal proof of completeness of $FCP(Ł_n, Ł\Pi\frac{1}{2})$ with respect to this class of models is out of the scope of this paper, but it can be devised combining the techniques used in [18] and in Section 5.

6. Final remarks

In this paper we have presented a fuzzy modal approach to reasoning about the probability of fuzzy events. The very basic idea has been to treat a fuzzy event as a formula of the finitely valued Łukasiewicz logic $Ł_n$ instead of classical Boolean logic, and using the notion of state to capture the generalization of finitely additive probability measures on MV-algebras. In this setting we have introduced, for each $n \in \mathbb{N}$, the logic $FP(Ł_n, Ł)$ which has been proved to be complete with respect two classes of (Kripke-style) probabilistic models, namely the classes of weak and strong probabilistic models. Two further extensions of $FP(Ł_n, Ł)$ have been also sketched, namely the logic $FP(Ł_n, RPL)$ obtained by enlarging the language of $FP(Ł_n, Ł)$ by means of truth-constants for the rationals in $[0, 1]$ and $FCP(Ł_n, Ł\Pi\frac{1}{2})$ allowing to reason about conditional probability.

Our logical approach to the probability of fuzzy events hides indeed the more general one to generalize the results we have shown holding here (in particular the completeness results), also to a logic allowing to treat the probability of those events which can be modeled by formulas of the whole Łukasiewicz logic and not only every finite valued one. Such a logic (which we have named in the introduction $FP(Ł, Ł)$) is easily defined by replacing, in Definition 3.1, the schema ($Ł_n$) with the axioms of the whole Łukasiewicz logic. Also the notions of weak and strong models for $FP(Ł, Ł)$ can be easily generalized, but, unfortunately, we have not succeeded so far to prove completeness of $FP(Ł, Ł)$ with respect to either weak or strong probabilistic models. In such a direction however we have shown some partial results. In particular if the following equality

$$Th(FP(Ł, Ł)) = \bigcap_{n \in \mathbb{N}} Th(FP(Ł_n, Ł))$$

holds, where Th stands for theorems, then it would be easy to show completeness for $FP(Ł, Ł)$. This is an open problem.

Clearly our future work will be devoted either to solve this open problem or to find a new complete axiomatization for the logic $FP(Ł, Ł)$. Another important feature that remains to be investigated deals more directly with the notion of *fuzzy event*. Indeed we have based our investigation interpreting a probability over fuzzy events as a state on a MV-algebra. A more general study could be done by using well-known alternative fuzzy logics, different from Łukasiewicz logic, as logics for the events.

Acknowledgments

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