Homogeneous Lorentzian Structures on the Oscillator groups *

P.M. Gadea and J.A. Oubiña †

Abstract

We obtain all the homogeneous pseudo-Riemannian structures on the oscillator groups equipped with a family of left-invariant Lorentzian metrics. Moreover, in the 4-dimensional case we determine all the corresponding reductive decompositions and groups of isometries.

1 Introduction

In [7] Medina proved that the oscillator groups are, except for direct extensions with Euclidean groups, the only non-commutative simply connected solvable Lie groups which admit a bi-invariant Lorentzian metric. From the bi-invariance of the metric, it turns out that the corresponding pseudo-Riemannian spaces are symmetric. These groups also appear in various types of problems which arise from mathematical physics. For instance, the Lie algebra of the 4-dimensional oscillator group is associated to the harmonic oscillator problem (see Streater [11], where the group is so named because of this property) and, on the other hand, this Lorentzian symmetric space-time has been found to be a special case of solutions of the Einstein-Yang-Mills equations (see Levichev [5]).

It is a well-known fact that under certain topological conditions, a pseudo-Riemannian symmetric space is characterized by the vanishing of the covariant derivative of the curvature. In the homogeneous Riemannian case, Ambrose and Singer [1] extended that characterization. They proved that a connected, simply connected and complete Riemannian manifold (M, g) is homogeneous if and only if there exists a $(1, 2)$ tensor field S on M (called a homogeneous Riemannian structure) satisfying certain properties (see (2.1)). In [2] we have extended the Ambrose-Singer characterization to the case of pseudo-Riemannian manifolds, introducing homogeneous pseudo-Riemannian structures. We proved that a connected, simply connected and geodesically complete pseudo-Riemannian manifold (M, g) admits a homogeneous pseudo-Riemannian structure if and only if it is a reductive homogeneous pseudo-Riemannian space. Furthermore, in [3]

*MS Classification (1991): 53C30, 53C50, 22E35. Key Words: Homogeneous structures, pseudo-Riemannian manifolds, oscillator groups.

†Partially supported by DGES (Spain) under Project PB95-0124 and by Xunta de Galicia under Project XUGA 20703B98.

we have obtained a classification of homogeneous pseudo-Riemannian structures into eight classes similar to the Tricerri-Vanhecke classification [12] for the Riemannian case. If the signature of the metric is $(k, n-k)$, those classes are defined by the subspaces of certain space $\mathcal{S}_1 \oplus \mathcal{S}_2 \oplus \mathcal{S}_3$ which are invariant under the action of the pseudo-orthogonal group $O_k(n)$. The trivial class corresponds to the symmetric pseudo-Riemannian spaces.

In this paper, we consider a family of Lorentzian left-invariant metrics on the oscillator groups which generalize those ones introduced by Levichev [6] in his study of causal homogeneous Lorentzian 4-manifolds. All the corresponding pseudo-Riemannian spaces except one are not symmetric, and our purpose is to study their homogeneity by means of their homogeneous pseudo-Riemannian structures. The contents of this paper are as follows. In §2 we recall some results about homogeneous pseudo-Riemannian structures. In §3 we give the formulas for the Levi-Civita connections and curvatures of a family of left-invariant Lorentzian metrics on the $(2m+2)$-dimensional oscillator group $G(\lambda_1, \ldots, \lambda_m)$, $\lambda_1, \ldots, \lambda_m \in \mathbb{R}^+$. In §4 we obtain the general expressions for the homogeneous pseudo-Riemannian structures on these Lorentzian manifolds. Finally, in §5 we determine all the reductive decompositions associated to each homogeneous Lorentzian structure in the nonsymmetric 4-dimensional cases and we obtain all the corresponding groups of isometries.

2 Homogeneous pseudo-Riemannian structures

Let (M, g) be a connected C^∞ pseudo-Riemannian manifold of dimension n and signature $(k, n-k)$. Let ∇ be the Levi-Civita connection of g and R the curvature tensor field, for which we adopt the conventions $R_{XY}Z = \nabla_{[X,Y]}Z - \nabla_X \nabla_Y Z + \nabla_Y \nabla_X Z$, $R_{XYZW} = g(R_{XY}Z, W)$, for $X, Y, Z, W \in \mathfrak{X}(M)$.

A *homogeneous pseudo-Riemannian structure* on (M, g) is [2] a tensor field S of type $(1,2)$ on M such that the connection $\widetilde{\nabla} = \nabla - S$ satisfies

$$\widetilde{\nabla} g = 0, \quad \widetilde{\nabla} R = 0, \quad \widetilde{\nabla} S = 0. \tag{2.1}$$

If g is a Lorentzian metric $(k = 1)$, we say that S is a *homogeneous Lorentzian structure*. In [2] we have proved that if (M, g) is connected, simply connected and geodesically complete then it admits a homogeneous pseudo-Riemannian structure if and only if it is a reductive homogeneous pseudo-Riemannian manifold.

Let V be a real vector space endowed with an inner product $\langle\, , \rangle$ of signature $(k, n-k)$. The space $(V, \langle\, , \rangle)$ is the model for each tangent space T_xM, $x \in M$, of a reductive homogeneous pseudo-Riemannian manifold of signature $(k, n-k)$. Consider the vector space $\mathcal{S}(V)$ of tensors of type $(0,3)$ on $(V, \langle\, , \rangle)$ satisfying the same symmetries as those of a homogeneous pseudo-Riemannian structure S, that is, $\mathcal{S}(V) = \{S \in \otimes^3 V^* : S_{XYZ} = -S_{XZY},\ X, Y, Z \in V\}$, where $S_{XYZ} = \langle S_X Y, Z\rangle$. The inner product of V induces in a natural way an inner product in $\mathcal{S}(V)$, given by $\langle S, S'\rangle = \sum_{i,j,k=1}^n \varepsilon_i \varepsilon_j \varepsilon_k\, S_{e_i e_j e_k} S'_{e_i e_j e_k}$, where $\{e_i\}$ is an orthonormal basis of V, $\langle e_i, e_i\rangle = \varepsilon_i$, $\varepsilon_i = -1$ if $1 \le i \le k$, $\varepsilon_i = 1$ if $k+1 \le i \le n$. In [3], we have established the decomposition of $\mathcal{S}(V)$ into invariant and irreducible subspaces under the action of the pseudo-orthogonal group $O_k(n)$ given by $(aS)_{XYZ} = S_{a^{-1}X\,a^{-1}Y\,a^{-1}Z}$, $a \in O_k(n)$. If $c_{12}\colon \mathcal{S}(V) \to V^*$ is the map defined by

$$c_{12}(S)(Z) = \sum_{i=1}^{n} \varepsilon_i S_{e_i e_i Z}, \qquad Z \in V, \tag{2.2}$$

where $\{e_i\}$ is an orthonormal basis of V as above, we have

Theorem 2.1. *If* $\dim V \ge 3$, *then* $\mathcal{S}(V)$ *decomposes into the orthogonal direct sum of subspaces which are invariant and irreducible under the action of* $O_k(n)$, $\mathcal{S}(V) = \mathcal{S}_1(V) \oplus \mathcal{S}_2(V) \oplus \mathcal{S}_3(V)$, *where*

$$\begin{aligned}
\mathcal{S}_1(V) &= \{S \in \mathcal{S}(V) : S_{XYZ} = \langle X, Y\rangle\omega(Z) - \langle X, Z\rangle\omega(Y),\ \omega \in V^*\},\\
\mathcal{S}_2(V) &= \{S \in \mathcal{S}(V) : \mathfrak{S}_{XYZ} S_{XYZ} = 0,\ c_{12}(S) = 0\},\\
\mathcal{S}_3(V) &= \{S \in \mathcal{S}(V) : S_{XYZ} + S_{YXZ} = 0\}.\\
\mathcal{S}_1(V) \oplus \mathcal{S}_2(V) &= \{S \in \mathcal{S}(V) : \mathfrak{S}_{XYZ} S_{XYZ} = 0\},\\
\mathcal{S}_1(V) \oplus \mathcal{S}_3(V) &= \{S \in \mathcal{S}(V) : S_{XYZ} + S_{YXZ} = 2\langle X, Y\rangle\omega(Z)\\
&\qquad - \langle X, Z\rangle\omega(Y) - \langle Y, Z\rangle\omega(X),\ \omega \in V^*\},\\
\mathcal{S}_2(V) \oplus \mathcal{S}_3(V) &= \{S \in \mathcal{S}(V) : c_{12}(S) = 0\}.
\end{aligned}$$

Moreover, $\dim \mathcal{S}_1(V) = n$, $\dim \mathcal{S}_2(V) = n(n^2-4)/3$, $\dim \mathcal{S}_3(V) = n(n-1)(n-2)/6$ *and* $\dim \mathcal{S}(V) = n^2(n-1)/2$. *If* $\dim V = 2$ *then* $\mathcal{S}(V) = \mathcal{S}_1(V)$.

We say that the homogeneous pseudo-Riemannian structure S on (M, g) is of *type* $\{0\}$, $\mathcal{S}_i$ $(i = 1, 2, 3)$ or $\mathcal{S}_i \oplus \mathcal{S}_j$ $(1 \le i < j \le 3)$ if, for each point $x \in M$, $S(x) \in \mathcal{S}(T_xM)$ belongs to $\{0\}$, $\mathcal{S}_i(T_xM)$ or $(\mathcal{S}_i \oplus \mathcal{S}_j)(T_xM)$, respectively.

3 The oscillator groups

Let $\lambda_1, \dots, \lambda_m$ be m positive real numbers and $\lambda = (\lambda_1, \dots, \lambda_m)$. The *oscillator algebra* $\mathfrak{g}_m(\lambda) = \mathfrak{g}(\lambda_1, \dots, \lambda_m)$ is defined as the real Lie algebra with $2m+2$ generators $P, X_1, \dots, X_m, Y_1, \dots, Y_m, Q$, with nonzero brackets (see [7, 8, 10])

$$[X_j, Y_j] = P, \quad [Q, X_j] = \lambda_j Y_j, \quad [Q, Y_j] = -\lambda_j X_j, \qquad 1 \le j \le m.$$

That is, $\mathfrak{g}_m(\lambda)$ is the semidirect product of the Heisenberg algebra $\mathfrak{h}_m$ generated by $P, X_1, \dots, X_m, Y_1, \dots, Y_m$, and the line generated by Q, under the homomorphism $\mathrm{ad}_{|\mathfrak{h}_m}\colon \langle Q\rangle \to \mathrm{Der}\,(\mathfrak{h}_m)$. It is a solvable non-nilpotent Lie algebra and the connected simply connected Lie group whose Lie algebra is $\mathfrak{g}_m(\lambda)$ is the *oscillator group* $G_m(\lambda) = G(\lambda_1, \dots, \lambda_m)$.

If we identify the $(2m+1)$-dimensional Heisenberg group H_m with the manifold $\mathbb{R} \times \mathbb{C}^m$ equipped with the product

$$(p, z_1, \dots, z_m)(p', z_1', \dots, z_m') = (p + p' + \tfrac{1}{2}\sum_{j=1}^{m} \mathrm{Im}(\bar{z}_j z_j'),\ z_1 + z_1', \dots, z_m + z_m'),$$

then the oscillator group $G_m(\lambda)$ may be described as the semidirect product $H_m \times_\alpha \mathbb{R}$, where the action $\alpha : H_m \times \mathbb{R} \to H_m$ is given by $\alpha_q(p, z_1, \ldots, z_m) = (p,\, e^{i\lambda_1 q} z_1, \ldots,\, e^{i\lambda_m q} z_m)$, $q \in \mathbb{R}$. Thus, the group operation in $G_m(\lambda)$ is

$$
\begin{aligned}
&(p, z_1, \ldots, z_m, q)\,(p', z'_1, \ldots, z'_m, q') = \\
&\qquad (p + p' + \tfrac{1}{2} \sum_{j=1}^{m} \operatorname{Im}(\bar{z}_j e^{i\lambda_j q} z'_j),\ z_1 + e^{i\lambda_1 q} z'_1, \ldots,\ z_m + e^{i\lambda_m q} z'_m,\ q + q').
\end{aligned}
$$

We consider on $G_m(\lambda)$ the family of left-invariant Lorentzian metrics g_ε, $-1 < \varepsilon < 1$, with nonvanishing inner products $\langle\, ,\, \rangle_\varepsilon$ on $\mathfrak{g}_m(\lambda)$ given by

$$
\langle P, P\rangle_\varepsilon = \langle Q, Q\rangle_\varepsilon = \varepsilon, \quad \langle P, Q\rangle_\varepsilon = 1, \quad \langle X_i, X_j\rangle_\varepsilon = \langle Y_i, Y_j\rangle_\varepsilon = \delta_{ij}. \tag{3.1}
$$

If $\varepsilon = 0$ and $\lambda_i = 1$ for each $i = 1, \ldots, m$, the corresponding Lorentzian metric is also right-invariant. In the other cases, g_ε is not bi-invariant.

The Levi-Civita connection is given by $2\langle \nabla_U V, W\rangle_\varepsilon = \langle [U, V], W\rangle_\varepsilon - \langle [V, W], U\rangle_\varepsilon + \langle [W, U], V\rangle_\varepsilon$ for all $U, V, W \in \mathfrak{g}_m(\lambda)$. So, we obtain that the not always null covariant derivatives between generators are

$$
\begin{aligned}
&\nabla_P X_j = -\tfrac{\varepsilon}{2}\, Y_j = \nabla_{X_j} P, && \nabla_{X_j} Q = -\tfrac{1}{2}\, Y_j, && \nabla_Q X_j = (\lambda_j - \tfrac{1}{2}) Y_j, \\
&\nabla_P Y_j = \tfrac{\varepsilon}{2}\, X_j = \nabla_{Y_j} P, && \nabla_{Y_j} Q = \tfrac{1}{2}\, X_j, && \nabla_Q Y_j = -(\lambda_j - \tfrac{1}{2}) X_j, \\
&\nabla_{X_j} Y_k = \tfrac{1}{2}\, \delta_{jk} P = -\nabla_{Y_k} X_j,
\end{aligned}
$$

and the not always null components of the curvature tensor field are given by

$$
\begin{aligned}
&R_{PX_j} P = \tfrac{\varepsilon^2}{4}\, X_j, && R_{PY_j} P = \tfrac{\varepsilon^2}{4}\, Y_j, \\
&R_{PX_j} X_k = -\tfrac{\varepsilon}{4}\, \delta_{jk}\, P, && R_{PY_j} Y_k = -\tfrac{\varepsilon}{4}\, \delta_{jk}\, P, \\
&R_{PX_j} Q = \tfrac{\varepsilon}{4}\, X_j, && R_{PY_j} Q = \tfrac{\varepsilon}{4}\, Y_j, \\
&R_{X_j Q} P = -\tfrac{\varepsilon}{4}\, X_j, && R_{Y_j Q} P = -\tfrac{\varepsilon}{4}\, Y_j, \\
&R_{X_j Q} X_k = \tfrac{1}{4}\, \delta_{jk} P, && R_{Y_j Q} Y_k = \tfrac{1}{4}\, \delta_{jk} P, \\
&R_{X_j Q} Q = -\tfrac{1}{4}\, X_j, && R_{Y_j Q} Q = -\tfrac{1}{4}\, Y_j, \\
&R_{X_i X_j} Y_k = \tfrac{\varepsilon}{4}\, (\delta_{jk} Y_i - \delta_{ik} Y_j), && R_{Y_i Y_j} X_k = \tfrac{\varepsilon}{4}\, (\delta_{jk} X_i - \delta_{ik} X_j), \\
&R_{X_j Y_k} X_i = -\tfrac{\varepsilon}{2}\, (\delta_{jk} Y_i + \tfrac{1}{2}\, \delta_{ik} Y_j), \\
&R_{X_j Y_k} Y_i = \tfrac{\varepsilon}{2}\, (\delta_{jk} X_i + \tfrac{1}{2}\, \delta_{ij} X_k).
\end{aligned}
$$

4 Homogeneous Lorentzian structures on $G_m(\lambda)$

We shall determine the homogeneous Lorentzian structures on $G_m(\lambda)$ in terms of the basis $\{\eta, \alpha^1, \ldots, \alpha^m, \beta^1 \ldots, \beta^m, \xi\}$ dual to $\{P, X_1, \ldots, X_m, Y_1, \ldots, Y_m, Q\}$. If S is a homogeneous Lorentzian structure on $(G_m(\lambda), g_\varepsilon)$ and $\widetilde{\nabla} = \nabla - S$, then the condition $\widetilde{\nabla} g = 0$ in (2.1) is equivalent to $S_{WUV} + S_{WVU} = 0$ for all

$W, U, V \in \mathfrak{g}_m(\lambda)$. Moreover, $\widetilde{\nabla} R = 0$ is equivalent to the condition

$$\begin{aligned}\big(\nabla_W R\big)(V_1, V_2, V_3, V_4) &= -R(S_W V_1, V_2, V_3, V_4) - R(V_1, S_W V_2, V_3, V_4) \\ &\quad -R(V_1, V_2, S_W V_3, V_4) - R(V_1, V_2, V_3, S_W V_4),\end{aligned} \tag{4.1}$$

for all $W, V_1, V_2, V_3, V_4 \in \mathfrak{g}_m(\lambda)$. Substituting (V_1, V_2, V_3, V_4) by (P, X_j, P, Q), (X_j, Y_j, Y_j, Q), (P, Y_j, P, Q) and (X_j, Y_j, X_j, Q), we obtain, respectively,

$$\begin{aligned}&\varepsilon S_{WPX_j} + \varepsilon^2 S_{WX_jQ} = 0, \quad S_{WPX_j} - 3\varepsilon S_{WX_jQ} = 2\varepsilon \beta^j(W), \\ &\varepsilon S_{WPY_j} + \varepsilon^2 S_{WY_jQ} = 0, \quad S_{WPY_j} - 3\varepsilon S_{WY_jQ} = -2\varepsilon \alpha^j(W).\end{aligned}$$

From these equations we have

$$S_{WPX_j} = \frac{\varepsilon}{2}\beta^j(W), \qquad S_{WPY_j} = -\frac{\varepsilon}{2}\alpha^j(W), \tag{4.2}$$

$$\varepsilon\big(S_{WX_jQ} + \frac{1}{2}\beta^j(W)\big) = 0, \qquad \varepsilon\big(S_{WY_jQ} - \frac{1}{2}\alpha^j(W)\big) = 0. \tag{4.3}$$

Replacing (V_1, V_2, V_3, V_4) in (4.1) by (X_j, X_k, X_j, Y_j) and (X_i, X_j, Y_i, Y_k), with $j \neq k$, we obtain, respectively,

$$\varepsilon\big(S_{WX_jY_k} - S_{WX_kY_j}\big) = 0, \qquad \varepsilon\big(S_{WX_jX_k} - S_{WY_jY_k}\big) = 0. \tag{4.4}$$

Finally, replacing (V_1, V_2, V_3, V_4) in (4.1) by (X_j, Q, X_j, Q), we obtain

$$S_{WPQ} = 0. \tag{4.5}$$

It is easy to see that the condition $\widetilde{\nabla} R = 0$ in (2.1) is satisfied if and only if the equations (4.2), (4.3), (4.4) and (4.5) are satisfied for all $W \in \mathfrak{g}_m(\lambda)$. We put

$$\theta_{jk}(W) = S_{WX_jY_k}, \quad \mu_{jk}(W) = S_{WX_jX_k}, \quad \nu_{jk}(W) = S_{WY_jY_k}, \tag{4.6}$$

$$\rho_j(W) = S_{WX_jQ}, \qquad \sigma_j(W) = S_{WY_jQ}, \tag{4.7}$$

for $1 \leq j, k \leq m$. We have $\mu_{jk} = -\mu_{kj}$ and $\nu_{jk} = -\nu_{kj}$. Now, we shall determine the conditions for the 1-forms θ_{jk}, μ_{jk}, ν_{jk}, ρ_j and σ_j under which the condition $\widetilde{\nabla} S = 0$ in (2.1) is satisfied.

By (4.2), (4.5), (4.6) and (4.7), the connection $\widetilde{\nabla} = \nabla - S$ is given by

$$\begin{aligned}&\widetilde{\nabla}_Z P = 0, \qquad \widetilde{\nabla}_Z Q = \textstyle\sum_i \big(\rho_i + \frac{1}{2}\beta^i\big)(Z) X_i + \sum_i \big(\sigma_i - \frac{1}{2}\alpha^i\big)(Z) Y_i, \\ &\widetilde{\nabla}_Z X_j = -\big(\rho_j + \frac{1}{2}\beta^j\big)(Z) P + \textstyle\sum_i \mu_{ij}(Z) X_i - \sum_i \theta_{ji}(Z) Y_i + \big((\lambda_j - \frac{1}{2})\xi - \frac{\varepsilon}{2}\eta\big)(Z) Y_j, \\ &\widetilde{\nabla}_Z Y_j = -\big(\sigma_j - \frac{1}{2}\alpha^j\big)(Z) P + \textstyle\sum_i \theta_{ij}(Z) X_i + \big(\frac{\varepsilon}{2}\eta - (\lambda_j - \frac{1}{2})\xi\big)(Z) X_j + \sum_i \nu_{ij}(Z) Y_i,\end{aligned}$$

for every $Z \in \mathfrak{g}_m(\lambda)$. Then, replacing (V_1, V_2) in the equation $\big(\widetilde{\nabla}_Z S\big)(W, V_1, V_2) = 0$ by (X_j, Y_k), (X_j, X_k), (Y_j, Y_k), (X_j, Q) and (Y_j, Q) we obtain, respectively, being $\Lambda_j = \lambda_j - \frac{1}{2}$,

$$\widetilde{\nabla}\theta_{jk} = \textstyle\sum_i \big(\theta_{ik} \wedge \mu_{ji} + \nu_{ik} \wedge \theta_{ji}\big) + \Lambda_j \xi \otimes \nu_{jk} - \Lambda_k \xi \otimes \mu_{jk}, \tag{4.8}$$

$$\widetilde{\nabla}\mu_{jk} = \textstyle\sum_i \big(\mu_{ik} \wedge \mu_{ji} + \theta_{ji} \wedge \theta_{ki}\big) + \Lambda_k \xi \otimes \theta_{jk} - \Lambda_j \xi \otimes \theta_{kj}, \tag{4.9}$$

$$\widetilde{\nabla}\nu_{jk} = \textstyle\sum_i \big(\nu_{ik} \wedge \nu_{ji} + \theta_{ij} \wedge \theta_{ik}\big) + \Lambda_k \xi \otimes \theta_{kj} - \Lambda_j \xi \otimes \theta_{jk}, \tag{4.10}$$

$$\widetilde{\nabla}\rho_j = \textstyle\sum_i \big(\rho_i \wedge \mu_{ji} + \sigma_i \wedge \theta_{ji}\big) + \frac{1}{2}\sum_i \big(\beta^i \otimes \mu_{ji} - \alpha^i \otimes \theta_{ji}\big) + (\Lambda_j \xi - \frac{\varepsilon}{2}\eta) \otimes \sigma_j, \tag{4.11}$$

$$\widetilde{\nabla}\sigma_j = \textstyle\sum_i \big(\sigma_i \wedge \nu_{ji} - \rho_i \wedge \theta_{ij}\big) + \frac{1}{2}\sum_i \big(\alpha^i \otimes \nu_{ij} - \beta^i \otimes \theta_{ij}\big) - (\Lambda_j \xi - \frac{\varepsilon}{2}\eta) \otimes \rho_j. \tag{4.12}$$

In the case of the bi-invariant metric ($\varepsilon = 0$ and $\lambda_i = 1$ for each $i = 1, \dots, m$), the oscillator group is a Lorentzian symmetric space and the tensor field $S = 0$ is a homogeneous Lorentzian structure on $(G_m(\lambda), g_0)$. Moreover, from (4.2), (4.5), (4.6) and (4.7), we deduce

Theorem 4.1. *All the homogeneous Lorentzian structures on the oscillator group $G_m(\lambda)$ with the left-invariant Lorentzian metric g_0 are given by*

$$S = \sum_{i=1}^{m} \big(\rho_i \otimes (\alpha^i \wedge \xi) + \sigma_i \otimes (\beta^i \wedge \xi)\big) + \sum_{j,k=1}^{m} \theta_{jk} \otimes (\alpha^j \wedge \beta^k)$$
$$+ \sum_{j<k} \big(\mu_{jk} \otimes (\alpha^j \wedge \alpha^k) + \nu_{jk} \otimes (\beta^j \wedge \beta^k)\big),$$

where θ_{jk}, μ_{jk}, ν_{jk}, ρ_j, σ_j $(1 \le j, k \le m)$, are left-invariant 1-forms on $G_m(\lambda)$ satisfying $\mu_{jk} = -\mu_{kj}$, $\nu_{jk} = -\nu_{kj}$ and the equations (4.8), (4.9), (4.10), (4.11) *and* (4.12) *with $\varepsilon = 0$.*

In particular, putting $\theta_{jk} = \mu_{jk} = \nu_{jk} = \rho_j = \sigma_j = 0$ in the above theorem, we obtain that $S = 0$ is a homogeneous Lorentzian structure on $(G_m(\lambda), g_0)$ and hence we have

Corollary 4.2. *For each $\lambda = (\lambda_1, \dots, \lambda_m)$, $(G_m(\lambda), g_0)$ is a Lorentzian symmetric space.*

If $\varepsilon \neq 0$, equations (4.3) and (4.4) are equivalent respectively to

$$\rho_j = -\beta^j/2, \qquad \sigma_j = \alpha^j/2, \tag{4.13}$$

$$\theta_{jk} = \theta_{kj}, \qquad \mu_{jk} = \nu_{jk}. \tag{4.14}$$

By using (4.2), (4.5), (4.6), (4.7), (4.8), (4.9), (4.13) and (4.14), we obtain

Theorem 4.3. *All the homogeneous Lorentzian structures on the oscillator group $G_m(\lambda)$ with the left-invariant Lorentzian metric g_ε defined by* (3.1), *$\varepsilon \neq 0$,* *are given by*

$$S=\frac{\varepsilon}{2}\sum_{i=1}^{m}\big(\beta^i\otimes(\eta\wedge\alpha^i)-\alpha^i\otimes(\eta\wedge\beta^i)\big)+\frac{1}{2}\sum_{i=1}^{m}\big(\alpha^i\otimes(\beta^i\wedge\xi)-\beta^i\otimes(\alpha^i\wedge\xi)\big)$$
$$+\sum_{j,k=1}^{m}\theta_{jk}\otimes(\alpha^j\wedge\beta^k)+\sum_{j<k}\mu_{jk}\otimes(\alpha^j\wedge\alpha^k+\beta^j\wedge\beta^k),$$

where θ_{jk} *and* μ_{jk} $(1\leq j,k\leq m)$ *are left-invariant* 1*-forms on* $G_m(\lambda)$ *satisfying* $\theta_{jk}=\theta_{kj}$, $\mu_{jk}=-\mu_{kj}$, *and, being* $\widetilde{\nabla}=\nabla-S$,

$$\widetilde{\nabla}\theta_{jk}=\sum_i\big(\theta_{ik}\wedge\mu_{ji}+\mu_{ik}\wedge\theta_{ji}\big)+(\lambda_j-\lambda_k)\xi\otimes\mu_{jk},$$
$$\widetilde{\nabla}\mu_{jk}=\sum_i\big(\mu_{ik}\wedge\mu_{ji}+\theta_{ji}\wedge\theta_{ki}\big)+(\lambda_k-\lambda_j)\xi\otimes\theta_{jk}.$$

Remark 4.4. If a connected pseudo-Riemannian manifold admits a nonzero homogeneous pseudo-Riemannian structure of type $\mathcal{S}_1$ then it must have constant curvature (see [3] and [9]). Thus $(G_m(\lambda),g_\varepsilon)$, $\varepsilon\neq 0$, does not admit any homogeneous Lorentzian structure of type $\mathcal{S}_1$. The Lorentzian symmetric space $(G_m(\lambda),g_0)$ does not admit *nonzero* homogeneous Lorentzian structures of type $\mathcal{S}_1$ either, since $(G_m(\lambda),g_0)$ has not constant curvature.

An orthonormal basis of $(\mathfrak{g}_m(\lambda),\langle\,,\,\rangle_\varepsilon)$ is $\{(2-2\varepsilon)^{-1/2}(P-Q),X_1,\dots,X_m,$ $Y_1,\dots,Y_m,(2+2\varepsilon)^{-1/2}(P+Q)\}$. Suppose $\varepsilon\neq 0$. By (2.2),

$$c_{12}(S)(Z)=-\frac{1}{2-2\varepsilon}S_{P-Q,P-Q,Z}+\sum_{j=1}^{m}(S_{X_jX_jZ}+S_{Y_jY_jZ})+\frac{1}{2+2\varepsilon}S_{P+Q,P+Q,Z}$$
$$=\sum_{j,k=1}^{m}\big(\mu_{jk}(X_j)\alpha^k(Z)+\theta_{jk}(X_j)\beta^k(Z)-\theta_{jk}(Y_j)\alpha^k(Z)+\mu_{jk}(Y_j)\beta^k(Z)\big).$$

for all $Z\in\mathfrak{g}_m(\lambda)$. We have (compare with [4, Prop. 2.1] for Heisenberg groups).

Proposition 4.5. *A homogeneous Lorentzian structure on* $(G_m(\lambda),g_\varepsilon)$, $\varepsilon\neq 0$, *is of type* $\mathcal{S}_2\oplus\mathcal{S}_3$ *if and only if*

$$\sum_{j=1}^{m}\big(\mu_{jk}(X_j)-\theta_{jk}(Y_j)\big)=\sum_{j=1}^{m}\big(\theta_{jk}(X_j)+\mu_{jk}(Y_j)\big)=0,\qquad 1\leq k\leq m.$$

5 Reductive decompositions and groups of isometries of the 4-dimensional oscillator group

For each $\lambda\in\mathbb{R}$, we can consider the Lie algebra $\mathfrak{g}_1(\lambda)$ with generators P,X,Y,Q, and structure equations $[X,Y]=P$, $[Q,X]=\lambda Y$, $[Q,Y]=-\lambda X$.

In particular, $\mathfrak{g}_1(0)$ is the direct product of the 3-dimensional Heisenberg algebra and $\mathbb{R}$. If $\lambda \neq 0$, then $\mathfrak{g}_1(\lambda)$ is isomorphic to $\mathfrak{g} = \mathfrak{g}_1(1)$ and the corresponding Lie group $G_1(\lambda)$ is isomorphic to $G = G_1(1)$.

Let $\{\eta, \alpha, \beta, \xi\}$ be the basis dual to $\{P, X, Y, Q\}$. We have

Theorem 5.1. *All the homogeneous Lorentzian structures on the* 4*-dimensional oscillator group* (G, g_ε)*,* $-1 < \varepsilon < 1$*,* $\varepsilon \neq 0$*, are given by*

$$S = \frac{\varepsilon}{2}\beta \otimes (\eta \wedge \alpha) - \frac{\varepsilon}{2}\alpha \otimes (\eta \wedge \beta) - \frac{1}{2}\beta \otimes (\alpha \wedge \xi) + \frac{1}{2}\alpha \otimes (\beta \wedge \xi) + \theta \otimes (\alpha \wedge \beta), \tag{5.1}$$

where $\theta = a\,\eta + b\,\xi$*,* $a, b \in \mathbb{R}$*.*

Proof. By Theorem 4.3, all the Lorentzian homogeneous structures on G are given by (5.1), where θ is a 1-form on G satisfying $\widetilde{\nabla}\theta = 0$. In this case,

$$\widetilde{\nabla}_Z\theta = \theta(X)\big(- \theta(Z) + \frac{1}{2}\xi(Z) - \frac{\varepsilon}{2}\eta(Z)\big)\beta + \theta(Y)\big(\theta(Z) + \frac{\varepsilon}{2}\eta(Z) - \frac{1}{2}\xi(Z)\big)\alpha.$$

Replacing Z by X and Y, the condition $\widetilde{\nabla}\theta = 0$ implies $\theta(X) = 0$ and $\theta(Y) = 0$, respectively. Then $\theta = a\,\eta + b\,\xi$, $a, b \in \mathbb{R}$. Conversely, if $\theta = a\,\eta + b\,\xi$ then $\widetilde{\nabla}\theta = 0$, which proves the theorem. $\square$

The nonvanishing components of the $(1,2)$ tensor field corresponding to the tensor field S in (5.1) are given by

$$\begin{array}{llll} S_XP = -\frac{\varepsilon}{2}Y, & S_PX = aY, & S_PY = -aX, & S_XQ = -\frac{1}{2}Y, \\ S_YP = \frac{\varepsilon}{2}X, & S_YX = -\frac{1}{2}P, & S_XY = \frac{1}{2}P, & S_YQ = \frac{1}{2}X, \\ & S_QX = bY, & S_QY = -bX. & \end{array}$$

From Proposition 4.5, the definitions of the classes in §2, and the characterization of connected simply connected pseudo-Riemannian naturally reductive spaces in [3], we deduce

Proposition 5.2. *For every* $a, b \in \mathbb{R}$*, the homogeneous Lorentzian structure* $S = S_{(a,b)}$ *on the* 4*-dimensional oscillator group* (G, g_ε)*,* $\varepsilon \neq 0$*, given by* (5.1) *is of type* $\mathcal{S}_2 \oplus \mathcal{S}_3$*. Moreover,* $S_{(a,b)}$ *is of type* $\mathcal{S}_2$ *if and only if* $a = -\varepsilon$ *and* $b = -1$*, and of type* $\mathcal{S}_3$ *if and only if* $a = \varepsilon/2$ *and* $b = 1/2$*. In particular,* (G, g_ε) *is a naturally reductive Lorentzian space.*

The metric g_ε is geodesically complete (see [6]) and thus every homogeneous Lorentzian structure $S_{(a,b)}$ on (G, g_ε) has a corresponding group of isometries $\tilde{G}_{(a,b)}$ acting transitively and effectively on G, and an associated reductive decomposition $\tilde{\mathfrak{g}}_{(a,b)} \equiv \tilde{\mathfrak{h}}_{(a,b)} \oplus \mathfrak{g}$, where $\tilde{\mathfrak{h}}_{(a,b)}$ is the Lie algebra (isomorphic to the holonomy algebra of the connection $\widetilde{\nabla}_{(a,b)} = \nabla - S_{(a,b)}$) generated by the curvature operators $(\tilde{R}_{(a,b)})_{ZW} \in \mathfrak{so}_1(\mathfrak{g})$, $Z, W \in \mathfrak{g}$. Here, $\mathfrak{so}_1(\mathfrak{g})$ is the algebra (isomorphic to $\mathfrak{so}_1(4)$) of skew-symmetric endomorphisms of $(\mathfrak{g}, \langle\ ,\ \rangle_\varepsilon)$. The structure of Lie algebra of $\tilde{\mathfrak{g}}_{(a,b)}$ is given by

$$[A, A'] = AA' - A'A,\ \ A, A' \in \tilde{\mathfrak{h}}_{(a,b)}, \qquad [A, Z] = A(Z),\ \ A \in \tilde{\mathfrak{h}}_{(a,b)}, Z \in \mathfrak{g},$$
$$[Z, W] = (\tilde{R}_{(a,b)})_{ZW} + (S_{(a,b)})_ZW - (S_{(a,b)})_WZ,\ \ Z, W \in \mathfrak{g}.$$

With respect to the basis $\{P, X, Y, Q\}$ of $\mathfrak{g}$, the connection $\widetilde{\nabla} = \widetilde{\nabla}_{(a,b)}$ is given by

$$\widetilde{\nabla}_P X = -(\tfrac{\varepsilon}{2} + a)Y,\ \widetilde{\nabla}_P Y = (\tfrac{\varepsilon}{2} + a)X,\ \widetilde{\nabla}_Q X = (\tfrac{1}{2} - b)Y,\ \widetilde{\nabla}_Q Y = (b - \tfrac{1}{2})X,$$

with the rest vanishing. Hence, the only nonvanishing component of the curvature tensor field is $\tilde{R}_{XY} = (\frac{\varepsilon}{2} + a)(\beta \otimes X - \alpha \otimes Y)$. First, we shall suppose that $a = -\varepsilon/2$. In this case, the holonomy algebra of $\widetilde{\nabla}$ is trivial and the reductive decomposition associated to the homogeneous Lorentzian structure given by (5.1) is $\tilde{\mathfrak{g}} = \{0\} \oplus \mathfrak{g}$ with structure equations $[X, Y] = P$, $[Q, X] = (b + \frac{1}{2})Y$, $[Q, Y] = -(b + \frac{1}{2})X$. Then $\tilde{\mathfrak{g}}_{(-\frac{\varepsilon}{2}, b)} = \mathfrak{g}_1(b + \frac{1}{2})$ and we have

Theorem 5.3. *Let $S = S_{(a,b)}$ be the homogeneous Lorentzian structure on the 4-dimensional oscillator group $G = G_1(1)$ defined by (5.1) and $a = -\varepsilon/2$. For $b = -1/2$ the corresponding group of isometries $\tilde{G}_{(-\frac{\varepsilon}{2}, -\frac{1}{2})}$ is the direct product $G_1(0)$ of the 3-dimensional Heisenberg group and $\mathbb{R}$ and for $b \neq -1/2$ it is the oscillator group $G_1(b + \frac{1}{2})$; in particular, if $b = 1/2$ then the group of isometries is G itself. For each $b \in \mathbb{R}$, $\tilde{G}_{(-\frac{\varepsilon}{2}, b)} = G_1(b + \frac{1}{2})$ acts simply transitively on the left on G, for $p, q, p', q' \in \mathbb{R}$, $z, z' \in \mathbb{C}$, by*

$$(p, z, q) \cdot (p', z', q') = (p + p' + \tfrac{1}{2}\operatorname{Im}(\bar{z}e^{iq(b+\frac{1}{2})}z'),\ z + e^{iq(b+\frac{1}{2})}z',\ q + q').$$

Now, suppose that $a \neq -\varepsilon/2$. Then $U = \tilde{R}_{XY} = (\frac{\varepsilon}{2} + a)(X \otimes \beta - Y \otimes \alpha)$ generates the holonomy algebra $\tilde{\mathfrak{h}}_{(a,b)}$ of $\widetilde{\nabla}_{(a,b)}$ and the reductive decomposition associated to the homogeneous Lorentzian structure $S_{(a,b)}$ is $\tilde{\mathfrak{g}}_{(a,b)} \equiv \tilde{\mathfrak{h}}_{(a,b)} \oplus \mathfrak{g} = \langle\{U, P, X, Y, Q\}\rangle$ with nonvanishing brackets

$$\begin{array}{lll}
[U, X] = -(\frac{\varepsilon}{2} + a)Y, & [X, Y] = U + P, & [P, Y] = -(\frac{\varepsilon}{2} + a)X, \\
[P, X] = (\frac{\varepsilon}{2} + a)Y, & [U, Y] = (\frac{\varepsilon}{2} + a)X, & [Q, Y] = -(b + \frac{1}{2})X, \\
[Q, X] = (b + \frac{1}{2})Y. & &
\end{array}$$

If we put $T = U + P$ then with respect to the basis $\{T, X, Y, Q, U\}$ of $\tilde{\mathfrak{g}}_{(a,b)}$ the nonvanishing brackets are $[X, Y] = T$, $[Q, X] = (b + \frac{1}{2})Y$, $[Q, Y] = -(b + \frac{1}{2})X$, $[U, X] = -(\frac{\varepsilon}{2} + a)Y$, $[U, Y] = (\frac{\varepsilon}{2} + a)X$. If $b = -1/2$ then $\tilde{\mathfrak{g}}_{(a,b)}$ is the direct product of the oscillator algebra $\mathfrak{g}_1(-(\frac{\varepsilon}{2} + a))$ generated by $\{T, X, Y, U\}$ and the line generated by Q. If $b \neq -\frac{1}{2}$ then $\tilde{\mathfrak{g}}_{(a,b)}$ is the semidirect product of the oscillator algebra $\mathfrak{g}_1(b + \frac{1}{2})$ generated by $\{T, X, Y, Q\}$ and the line generated by U under the homomorphism $\operatorname{ad}_{|\mathfrak{g}_1(b+\frac{1}{2})} \colon \langle U \rangle \to \operatorname{Der}(\mathfrak{g}_1(b + \frac{1}{2}))$. In both cases, $\tilde{\mathfrak{g}}_{(a,b)}$ may also be considered a semidirect product of the 3-dimensional Heisenberg algebra generated by $\{T, X, Y\}$ and the plane generated by $\{Q, U\}$. The corresponding connected simply connected Lie group is the semidirect product $H_1 \times_\gamma \mathbb{R}^2$, where γ is the action of the additive group $\mathbb{R}^2$ on the 3-dimensional Heisenberg group H_1, given by $\gamma_{(q,u)}(t, z) = \big(t,\ e^{i((b+\frac{1}{2})q - (\frac{\varepsilon}{2}+a)u)}z\big)$. If the manifold $\hat{G}_{(a,b)} = \mathbb{C} \times \mathbb{R}^3$ is equipped with the group operation such that the bijection $(z, p, q, u) \in \hat{G}_{(a,b)} \mapsto ((p, z), (q, u - p)) \in H_1 \times_\gamma \mathbb{R}^2$ is a group isomorphism, then

$\hat{G}_{(a,b)}$ acts transitively and almost effectively in a natural way on G as a group of isometries. The normal subgroup of elements of $\hat{G}_{(a,b)}$ which act as the identity transformation on G is the discrete subspace $N = \{(0,0,0,\frac{4\pi k}{\varepsilon+2a}) : k \in \mathbb{Z}\}$, and the quotient group $\tilde{G}_{(a,b)} = \hat{G}_{(a,b)}/N$ acts transitively and effectively on G. The group operation of $\tilde{G}_{(a,b)} \equiv \mathbb{C} \times \mathbb{R}^2 \times \mathbb{S}^1$ is given by

$$
\begin{aligned}
(z,p,q,e^{iu})(z',p',q',e^{iu'}) &= \big(z + \exp\big(i((\tfrac{\varepsilon}{2}+a)p + (b+\tfrac{1}{2})q + u)\big)z', \\
&\quad p + p' + \frac{1}{2}\operatorname{Im}\big(\bar{z}\exp\big(i((\tfrac{\varepsilon}{2}+a)p + (b+\tfrac{1}{2})q + u)\big)z'\big),\ q + q', \\
&\quad \exp\big(i(u + u' - \frac{\varepsilon+2a}{4}\operatorname{Im}(\bar{z}\exp\big(i((\tfrac{\varepsilon}{2}+a)p + (b+\tfrac{1}{2})q + u)\big)z'))\big)\big), \qquad (5.2)
\end{aligned}
$$

and we conclude

Theorem 5.4. *Let $S = S_{(a,b)}$ be the homogeneous Lorentzian structure on the 4-dimensional oscillator group $G = G_1(1)$ defined by* (5.1) *and $a \neq -\varepsilon/2$. The corresponding group of isometries is $\tilde{G}_{(a,b)} = \mathbb{C} \times \mathbb{R}^2 \times \mathbb{S}^1$ with the operation defined by* (5.2)*, which acts transitively and effectively on G by*

$$
\begin{aligned}
(z,p,q,e^{iu}) \cdot (p',z',q') &= \big(p + p' + \frac{1}{2}\operatorname{Im}\big(\bar{z}\exp\big(i((\tfrac{\varepsilon}{2}+a)p + (b+\tfrac{1}{2})q + u)\big)z'\big), \\
z + \exp\big(i((\tfrac{\varepsilon}{2}+a)p + (b+\tfrac{1}{2})q + u)\big)z',\ q + q'\big), &\quad z, z' \in \mathbb{C},\ p,q,p',q',u \in \mathbb{R}.
\end{aligned}
$$

References

[1] W. Ambrose and I.M. Singer, On homogeneous Riemannian manifolds, Duke Math. J. **25** (1958) 647–669.

[2] P.M. Gadea and J.A. Oubiña, Homogeneous pseudo-Riemannian structures and homogeneous almost para-Hermitian structures, Houston J. Math. **18** (1992) 449–465.

[3] P.M. Gadea and J.A. Oubiña, Reductive homogeneous pseudo-Riemannian manifolds, Monatsh. Math. **124** (1997) 17–34.

[4] J.C. González and D. Chinea, Quasi-Sasakian homogeneous structures on the generalized Heisenberg group $H(p,1)$, Proc. Amer. Math. Soc. **105** (1989) 173–184.

[5] A.V. Levichev, Chronogeometry of an electromagnetic wave given by a biinvariant metric on the oscillator group, Siberian Math. J. **27** (1986) 237–245.

[6] A.V. Levichev, Methods of investigation of the causal structure of homogeneous Lorentz manifolds, Siberian Math. J. **31** (1990) 395–408.

[7] A. Medina, Groupes de Lie munis de métriques bi-invariantes, Tôhoku Math. J. **37** (1985) 405–421.

[8] A. Medina and P. Revoy, Les groupes oscillateurs et leurs réseaux, Manuscripta Math. **52** (1985) 81–95.

[9] A. Montesinos Amilibia, Degenerate homogeneous pseudo-Riemannian structures of class $\mathcal{S}_1$ on pseudo–Riemannian manifolds, preprint.

[10] D. Müller and F. Ricci, On the Laplace-Beltrami operator on the oscillator group, J. Reine Angew. Math. **390** (1988) 193–207.

[11] R.F. Streater, The representations of the oscillator group, Comm. Math. Phys. **4** (1967) 217–236.

[12] F. Tricerri and L. Vanhecke, Homogeneous Structures on Riemannian Manifolds, London Math. Soc. Lect. Notes Ser. **83**, Cambridge Univ. Press, Cambridge, 1983.

Anschrift der autoren:

P. M. Gadea
IMAFF
CSIC
Serrano 144
28006 Madrid
Spain
e-mail: pmgadea@iec.csic.es

J. A. Oubiña
Dep. de Xeometría e Topoloxía
Facultade de Matemáticas
Universidade de Santiago de Compostela
15706 Santiago de Compostela
Spain
e-mail: oubina@zmat.usc.es