Manifolds modelled over free modules over the double numbers ^{*}

P. M. GADEA (Madrid), J. GRIFONE (Toulouse) and
J. MUÑOZ MASQUÉ (Madrid)

Abstract

Manifolds over the algebra of double numbers, which include the case of manifolds equipped with a pair of equidimensional supplementary foliations, are studied. To this end, $\mathbb{B}$ -holomorphic functions and $\mathbb{B}$ -analytic functions on $\mathbb{B}^n$, where $\mathbb{B}$ denotes the algebra of double numbers, are defined and studied.

1 Introduction

Let A be a finite-dimensional commutative $\mathbb{R}$ -algebra. The notion of an A -holomorphic function and the relationship between A -holomorphy and A -analyticity are well known ([6, 12, 18, 19]). In the particular case of quadratic algebras, there are three essentially different 2-dimensional $\mathbb{R}$ -algebras, namely $\mathbb{A} = \mathbb{R}[x]/(x^2)$, $\mathbb{B} = \mathbb{R}[x]/(x^2 - 1)$, and $\mathbb{C} = \mathbb{R}[x]/(x^2 + 1)$.

Certainly, the geometry based on $\mathbb{C}$ as the ground algebra is the richest of the three ones and its development is enormous, but the other two cases are also related to interesting geometrical properties of manifolds. The geometry based on the ring $\mathbb{A}$ is related to tangent structures but we do not deal with it here. Similarly, $\mathbb{B}$ -manifolds (*i.e.*, manifolds endowed with a structure sheaf locally isomorphic to the sheaf of germs of $\mathbb{B}$ -holomorphic functions over a domain in $\mathbb{B}^n$) are related to geometrical properties of two equidimensional supplementary integrable distributions on a given manifold.

A manifold equipped with two equidimensional supplementary foliations can be understood as a sort of complex manifold over the ring $\mathbb{B}$, what we call it in the present paper a $\mathbb{B}$ -manifold. These manifolds appear in the specialized literature with different names and in different contexts. Because of the properties

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of the ground ring $\mathbb{B}$ and the usually associated compatible neutral metric, $\mathbb{B}$ -manifolds have applications to elliptic geometry ([16]), Anosov diffeomorphisms and manifolds of negative curvature ([7]), nonsymmetric theory of gravitation ([14]), pseudo-Riemannian space forms ([5]), etc. Moreover, symmetric [10] (or more generally homogeneous [8]) $\mathbb{B}$ -manifolds are diffeomorphic to cotangent bundles of certain Riemannian symmetric (resp. homogeneous) spaces and from this, some more applications follow: Causal manifolds and chronogeometry ([15]), dipolarizations ([2, 9]), quantizable coadjoint orbits ([11]), representations of semisimple groups ([3, 4]), etc.

We define the different classes of functions on $\mathbb{B}$ -manifolds, *i.e.*, $\mathbb{B}$ -differentiable, $\mathbb{B}$ -holomorphic, and $\mathbb{B}$ -analytic functions, and we analyze in detail their relationships.

2 $\mathbb{B}$ -holomorphic functions

Let $\mathbb{B} = \{z = x + jy : x, y \in \mathbb{R}, j^2 = 1\}$ be the algebra of double numbers (see [1] and references therein). We shall denote by $\bar{z} = x - jy$ the conjugate of the element $z = x + jy \in \mathbb{B}$.

Definition 2.1. Let U be an open subset of $\mathbb{B}$. A function $F : U \rightarrow \mathbb{B}$ is said to be D_1 -differentiable at $z_0 \in U$ if there exists $\lambda \in \mathbb{B}$ such that

$$(D_1) \quad \lim_{z \rightarrow z_0} \frac{|F(z) - F(z_0) - \lambda(z - z_0)|}{|z - z_0|} = 0,$$

where $|z|$ is the Euclidean norm of the point $z = x + jy \equiv (x, y) \in \mathbb{R}^2$. A function $F : U \rightarrow \mathbb{B}$ is said to be D_2 -differentiable at z_0 if the following limit exists in $\mathbb{B}$:

$$(D_2) \quad \lim_{\substack{z \rightarrow z_0 \\ z - z_0 \in \mathbb{B}^*}} \frac{F(z) - F(z_0)}{z - z_0},$$

where $\mathbb{B}^* = \{z \in \mathbb{B} : x^2 \neq y^2\}$ is the group of invertible elements.

Examples 2.2. (1) A non-continuous function may be D_2 -differentiable at a point: let $U = \mathbb{B}$ and F be the function defined by $F(z) = 0$ if $|x| \neq |y|$ or $x = y = 0$, and $F(z) = 1$ if $|x| = |y| \neq 0$. F is discontinuous at the origin because F vanishes at that point while it takes the value 1 at diagonal points arbitrarily near to the origin. Since F vanishes on $\mathbb{B}^*$, it follows that the limit in (D_2) exists and it is equal to 0 for $z_0 = 0$. It thus becomes apparent that D_2 -differentiability at a point does not imply D_1 -differentiability at that point, since clearly the last one implies continuity at the point.

(2) D_1 -differentiability at a point does not imply D_2 -differentiability at that point either. Let us see a counterexample. Let $F(z) = x^2 - y^2 + 2jxy = z\bar{z} + z^2 - |z^2|$. Take $z_0 = 0$ and $\lambda = 0$ in (D_1) . Then

$$\frac{|F(z) - F(z_0) - \lambda(z - z_0)|}{|z - z_0|} = |z| \rightarrow 0,$$

and (D_1) is obviously satisfied. The limit in (D_2) , however, does not exist. In fact,

$$\frac{F(z) - F(z_0)}{z - z_0} = \frac{F(z)}{z} = \bar{z} + z - \frac{|z|^2}{z}.$$

Taking polar coordinates $x = \rho \cos \theta$, $y = \rho \sin \theta$, we deduce

$$\left| \frac{F(z)}{z} \right| = \rho \frac{\sqrt{1 - 2 \sin^2 2\theta \cos 2\theta}}{|\cos 2\theta|}.$$

Letting $\rho \rightarrow 0$ and keeping constant θ , the function $|F(z)/z|$ has limit equal to 0, but for the sequence $\{z_n = x_n + jy_n\}$ corresponding to the values $\rho_n = 1/n$, $\theta_n = (\pi/4) - (1/2n)$ (notice that these points are not on the diagonal), we have $\lim |F(z_n)/z_n| = 1$, as $n \rightarrow \infty$.

Lemma 2.3. *For every $z, w \in \mathbb{B}$ one has $|zw| \leq \sqrt{2}|z||w|$.*

The proof is immediate. Notice that the bound cannot be sharpened. It suffices to consider $z = w = 1 + j$.

Lemma 2.4. *Let U be an open subset of $\mathbb{B}$. If $F: U \rightarrow \mathbb{B}$ is D_2 -differentiable at $z_0 \in U$ and we have*

$$\lim_{\substack{z \rightarrow z_0 \\ z - z_0 \in \mathbb{B}^*}} \frac{F(z) - F(z_0)}{z - z_0} = \lambda,$$

then we have

$$(D'_1) \quad \lim_{\substack{z \rightarrow z_0 \\ z - z_0 \in \mathbb{B}^*}} \frac{|F(z) - F(z_0) - \lambda(z - z_0)|}{|z - z_0|} = 0.$$

Proof. Given $\varepsilon > 0$, there exists $\delta > 0$ such that $|z - z_0| < \delta$ and $z - z_0 \in \mathbb{B}^* \cap U$ imply $|(F(z) - F(z_0))/(z - z_0) - \lambda| < \varepsilon$. Then, applying Lemma 2.3 we obtain $|F(z) - F(z_0) - \lambda(z - z_0)|/|z - z_0| < \varepsilon\sqrt{2}$, and since ε is arbitrary we conclude. $\square$

Proposition 2.5. *Let U be an open subset of $\mathbb{B}$. If the function $F: U \rightarrow \mathbb{B}$ is continuous on a neighbourhood of $z_0 \in U$ and D_2 -differentiable at z_0 then F is D_1 -differentiable at z_0 .*

Proof. Let $U_\delta(z_0) = \{z \in \mathbb{B} : |z - z_0| < \delta\}$. Given $\varepsilon > 0$ there exists $\delta > 0$ such that $z \in \mathbb{B}^* \cap U_\delta(z_0)$ implies $|F(z) - F(z_0) - \lambda(z - z_0)| \leq \varepsilon|z - z_0|$ because of Lemma 2.4. By restricting δ if necessary we can suppose that F is continuous on $U_\delta(z_0)$ and since $\mathbb{B}^* \cap U_\delta(z_0)$ is dense in $U_\delta(z_0)$, taking limits in the previous inequality we deduce that $|F(z) - F(z_0) - \lambda(z - z_0)| \leq \varepsilon|z - z_0|$ for every $z \in U_\delta(z_0)$, and so F is D_1 -differentiable. $\square$

Remarks 2.6. (1) Notice that in this case the D_1 -derivative coincides with the D_2 -derivative.

(2) To prove that (D_1) implies (D_2) , the difficulty lies in the fact that $|z^{-1}|$ cannot be bounded in terms of $|z|^{-1}$; i.e., there is no bound of the form $|z||z^{-1}| \leq c$, since for instance given $z_\varepsilon = (1 + \varepsilon) + j$, $\varepsilon > 0$, we have $\lim_{\varepsilon \rightarrow 0} |z_\varepsilon| = \sqrt{2}$, but $\lim_{\varepsilon \rightarrow 0} |z_\varepsilon^{-1}| = \infty$. This is one of the differences between $\mathbb{B}$ and $\mathbb{C}$.

Proposition 2.7. *Let U be an open subset of $\mathbb{B}$. If a function $F: U \rightarrow \mathbb{B}$ is D_2 -differentiable at each point of U then F also is D_1 -differentiable at each point of U .*

Proof. In account of Proposition 2.5 it will suffice to see that F is continuous on U since U is an open subset and thus it is a neighbourhood of each of its points. Let $z_0 \in U$. Given $\varepsilon > 0$, from (D'_1) in Lemma 2.4 it follows that there exists $\delta > 0$ such that: (1) $U_\delta(z_0) = \{z \in \mathbb{B} : |z - z_0| < \delta\} \subset U$, (2) $z \in U_\delta(z_0)$, $z - z_0 \in \mathbb{B}^*$ implies $|F(z) - F(z_0)| < \varepsilon/2$. Now, let $z_1 \in U$ such that $|z_1 - z_0| < \delta/2$. If $z_1 - z_0 \in \mathbb{B}^*$, then from (2) above it follows that $|F(z_1) - F(z_0)| < \varepsilon$. Assume $z_1 - z_0 \notin \mathbb{B}^*$. Since F is D_2 -differentiable at z_1 , as above we deduce the existence of $\delta' > 0$ such that (3) $\delta' < \delta/2$, (4) $z \in U_{\delta'}(z_1)$, $z - z_1 \in \mathbb{B}^*$ implies $|F(z) - F(z_1)| < \varepsilon/2$. Let us choose a point z satisfying (4) above. Then we have $|z - z_0| \leq |z - z_1| + |z_1 - z_0| < \delta' + (\delta/2)$. Hence $|F(z_1) - F(z_0)| \leq |F(z_1) - F(z)| + |F(z) - F(z_0)| < \varepsilon$, thus concluding. $\square$

Proposition 2.8. *Let U be an open subset of $\mathbb{B}$. A function $F: U \rightarrow \mathbb{B}$ of class C^1 is D_1 -differentiable at $z_0 \in U$ if and only if F satisfies the condition $(\partial F/\partial x)(z_0) = j(\partial F/\partial y)(z_0)$.*

Proof. Put $F = P + jQ$. Condition (D_1) in Definition 2.1 implies that F is differentiable at z_0 and that its differential at that point consists in multiplying by a double number $F'(z_0) = a + bj$. That is

$$\begin{pmatrix} \frac{\partial P}{\partial x}(z_0) & \frac{\partial P}{\partial y}(z_0) \\ \frac{\partial Q}{\partial x}(z_0) & \frac{\partial Q}{\partial y}(z_0) \end{pmatrix} = \begin{pmatrix} a & b \\ b & a \end{pmatrix},$$

or equivalently

$$(2.1) \quad \frac{\partial P}{\partial x}(z_0) = \frac{\partial Q}{\partial y}(z_0), \quad \frac{\partial P}{\partial y}(z_0) = \frac{\partial Q}{\partial x}(z_0),$$

and it is immediate that both conditions are equivalent to the unique condition of the assertion. $\square$

Proposition 2.9. *Let $F: U \rightarrow \mathbb{B}$ be a function defined on an open connected subset $U \subset \mathbb{B}$. Assume that F is D_1 -differentiable at each point of U . Then, there exist differentiable functions $f: p^-(U) \rightarrow \mathbb{R}$ and $g: p^+(U) \rightarrow \mathbb{R}$, where $p^-, p^+: U \rightarrow \mathbb{R}$ are defined by $p^-(x, y) = x + y$, $p^+(x, y) = x - y$, such that for every $(x, y) \in U$, we have $F(x, y) = (1 + j)f(x + y) + (1 - j)g(x - y)$.*

Proof. Set $u = p^-(x, y)$, $v = p^+(x, y)$, $F(x, y) = P(x, y) + jQ(x, y)$. Then $\partial/\partial u = (1/2)(\partial/\partial x + \partial/\partial y)$, $\partial/\partial v = (1/2)(\partial/\partial x - \partial/\partial y)$, and from (2.1) we obtain that $(\partial/\partial v)(P + Q) = (\partial/\partial u)(P - Q) = 0$. Hence we have $P + Q = f(u)$ and $P - Q = g(v)$. $\square$

Theorem 2.10. *Let $F: U \rightarrow \mathbb{B}$ be a function defined on an open connected subset $U \subset \mathbb{B}$. Then F is D_1 -differentiable on U if and only if F is D_2 -differentiable on U , and the D_1 -derivative coincides with the D_2 -derivative.*

Proof. That D_2 -differentiability on U implies D_1 -differentiability on U has been proved in Proposition 2.7. Let us prove the converse. Assume $F: U \rightarrow \mathbb{B}$ is D_1 -differentiable on U . According to Proposition 2.9 we have $F(x, y) = (1 + j)f(x + y) + (1 - j)g(x - y)$. Hence, for $z - z_0 \in \mathbb{B}^*$ small enough we obtain

$$\frac{F(z) - F(z_0)}{z - z_0} = (1 + j)\frac{f(u) - f(u_0)}{u - u_0} + (1 - j)\frac{g(v) - g(v_0)}{v - v_0},$$

and the result follows taking into account that f and g are differentiable. $\square$

Definition 2.11. Let U be an open subset of $\mathbb{B}^n$. A function $F: U \rightarrow \mathbb{B}$ is said to be $\mathbb{B}$ -differentiable if the function $z \mapsto F(z_1, \dots, z_{i-1}, z, z_{i+1}, \dots, z_n)$, $1 \leq i \leq n$, is either D_1 -differentiable or D_2 -differentiable on its domain. Let U be an open subset of $\mathbb{B}^n$. A function $F: U \rightarrow \mathbb{B}$ is said to be $\mathbb{B}$ -holomorphic on U if it is of class C^∞ and $\mathbb{B}$ -differentiable.

Theorem 2.12. (i) *Let U be an open subset of $\mathbb{B}^n$. A function $F: U \rightarrow \mathbb{B}$ of class C^1 is said to be $\mathbb{B}$ -differentiable if and only if F satisfies the conditions $\partial F / \partial x_i = j \partial F / \partial y_i$, $1 \leq i \leq n$.*

(ii) *Let U be a connected open subset of $\mathbb{B}^n$. A function $F: U \rightarrow \mathbb{B}$ is $\mathbb{B}$ -differentiable on U if and only if there exist differentiable functions $f: p^-(U) \rightarrow \mathbb{R}$ and $g: p^+(U) \rightarrow \mathbb{R}$ such that $F(x, y) = (1 + j)f(x + y) + (1 - j)g(x - y)$ for all $(x, y) = (x_1, \dots, x_n, y_1, \dots, y_n) \in U$, where $z_i = x_i + jy_i$, $1 \leq i \leq n$, and $p^-, p^+: U \rightarrow \mathbb{R}^n$ are the maps $p^-(x, y) = x + y$, $p^+(x, y) = x - y$. Moreover, F is of class C^r on U if and only if f and g are of class C^r on the open subsets $p^-(\mathbb{R}^n)$ and $p^+(\mathbb{R}^n)$ of $\mathbb{R}^n$, respectively.*

Proof. (i) It follows from Definition 2.11 and the statement similar to Proposition 2.8 for several variables. (ii) The proof is similar to Proposition 2.9. $\square$

Notation 2.13. Let U be an open subset of $\mathbb{B}^n$. We define a basis of the derivations of the ring $C^\infty(U, \mathbb{B}) = C^\infty(U) \otimes_{\mathbb{R}} \mathbb{B}$ by the formulae: $\partial / \partial z_i = (1/2)(\partial / \partial x_i + j \partial / \partial y_i)$, $\partial / \partial \bar{z}_i = (1/2)(\partial / \partial x_i - j \partial / \partial y_i)$, $1 \leq i \leq n$. Notice that this is the basis dual to $\{dz_i, d\bar{z}_i\}$, $1 \leq i \leq n$.

Proposition 2.14. *A function $F \in C^\infty(U, \mathbb{B})$ is $\mathbb{B}$ -holomorphic if and only if $\partial F / \partial \bar{z}_i = 0$, $1 \leq i \leq n$; or, equivalently, $dF = \sum_{i=1}^n (\partial F / \partial z_i) dz_i$.*

Proof. The first part directly follows from the equality $\frac{\partial F}{\partial \bar{z}_i} = \frac{1}{2} \left(\frac{\partial F}{\partial x_i} - j \frac{\partial F}{\partial y_i} \right)$. As for the second one, it suffices to see that for $F \in C^\infty(U, \mathbb{B})$,

$$dF = \sum_{i=1}^n \left(\frac{\partial F}{\partial z_i} dz_i + \frac{\partial F}{\partial \bar{z}_i} d\bar{z}_i \right).$$

$\square$

Notation 2.15. Let C_U^∞ be the sheaf of germs of C^∞ functions on an open subset of $\mathbb{B}^n$, and let $\mathcal{B}^\infty(U)$ denote the ring of $\mathbb{B}$ -holomorphic functions defined on U . Assigning to each open subset $V \subset U$ the ring $\mathcal{B}^\infty(V)$, we obtain the sheaf of germs of $\mathbb{B}$ -holomorphic functions on U , denoted by $\mathcal{B}_U^\infty$, which is a subsheaf of $C_U^\infty \otimes_{\mathbb{R}} \mathbb{B}$.

3 $\mathbb{B}$ -analytic functions

Definition 3.1. Let U be an open subset of $\mathbb{B}^n$. A function $F: U \rightarrow \mathbb{B}$ is said to be $\mathbb{B}$ -analytic if for every point $z^0 = (z_1^0, \dots, z_n^0) \in U$ there exists a series $\sum_{\alpha \in \mathbb{N}^n} b_\alpha z_1^{\alpha_1} \dots z_n^{\alpha_n} \in \mathbb{B}[[z_1, \dots, z_n]]$ absolutely convergent in a polydisk $|z_i| \leq r_i, 1 \leq i \leq n$, such that

$$F(z) = \sum_{\alpha \in \mathbb{N}^n} b_\alpha (z_1 - z_1^0)^{\alpha_1} \dots (z_n - z_n^0)^{\alpha_n},$$

$$\forall z = (z_1, \dots, z_n) \in U, |z_i - z_i^0| < r_i, 1 \leq i \leq n.$$

Notation 3.2. The set of $\mathbb{B}$ -analytic functions on a open subset U of $\mathbb{B}^n$ will be denoted by $\mathcal{A}(U)$. From the elementary theory of power series it follows that $\mathcal{A}(U)$ is a subring of $C^\infty(U, \mathbb{B})$. Assigning to each open subset $V \subset U$ the ring $\mathcal{A}(V)$ we obtain the sheaf of germs of $\mathbb{B}$ -analytic functions on U , denoted by $\mathcal{A}_U$, which is a subsheaf of $C_U^\infty \otimes_{\mathbb{R}} \mathbb{B}$.

Theorem 3.3. *Let U be an open subset of $\mathbb{B}^n$. A $\mathbb{B}$ -holomorphic function $F: U \rightarrow \mathbb{B}$ is $\mathbb{B}$ -analytic if and only if it is of class C^ω . That is,*

$$\mathcal{A}(U) = \mathcal{B}^\infty(U) \cap C^\omega(U, \mathbb{B}).$$

Proof. This is a particular case of [6, Theorem I]. Nevertheless in the present case it is interesting for us to obtain the coefficients of the power series expansion of a $\mathbb{B}$ -holomorphic function F of class C^ω explicitly.

If $F(z) = \sum_{\alpha \in \mathbb{N}^n} b_\alpha (z_1 - z_1^0)^{\alpha_1} \dots (z_n - z_n^0)^{\alpha_n}$, the function F is $\mathbb{B}$ -holomorphic since $\partial z_h / \partial \bar{z}_i = 0$ for every $h, 1 \leq i \leq n$, and consequently,

$$\frac{\partial F}{\partial \bar{z}_i} = \sum_{h=1}^n \sum_{\alpha \in \mathbb{N}^n} b_\alpha \alpha_h z_1^{\alpha_1} \dots z_h^{\alpha_h-1} \dots z_n^{\alpha_n} \frac{\partial z_h}{\partial \bar{z}_i} = 0.$$

Moreover, F is of class C^ω , since taking $z_i = x_i + jy_i, 1 \leq i \leq n, F = P + jQ$, it follows that P and Q are written as convergent power series in $x_i - x_i^0, y_i - y_i^0, 1 \leq i \leq n$. Hence $\mathcal{A}(U) \subseteq \mathcal{B}^\infty(U) \cap C^\omega(U, \mathbb{B})$.

Conversely, suppose that F is $\mathbb{B}$ -holomorphic and of class C^ω .

We proceed by induction on n .

(i) $n = 1$. Since F is $\mathbb{B}$ -holomorphic we have

$$F(z) = (1+j)f(x+y) + (1-j)g(x-y)$$

and since F is of class C^ω , then f and g also are. Hence $f(t) = \sum_{n=0}^\infty \lambda_n t^n, g(t) = \sum_{n=0}^\infty \mu_n t^n, \lambda_n, \mu_n \in \mathbb{R}$. As $(1 \pm j)^n = 2^{n-1}(1 \pm j)$, we deduce

$$f(x+y) = \sum_{n=0}^\infty \lambda_n (x+y)^n = \sum_{n=0}^\infty \lambda_n \left(\frac{1+j}{2} z^n + \frac{1-j}{2} \bar{z}^n \right).$$

Similarly, $g(x - y) = \sum_{n=0}^{\infty} \mu_n(((1 - j)/2)z^n + ((1 + j)/2)\bar{z}^n)$. Thus,

$$F(z) = \sum_{n=0}^{\infty} b_n z^n, \text{ with } b_n = (\lambda_n + \mu_n) + (\lambda_n - \mu_n)j.$$

(ii) $n > 1$. As F is of class C^ω we can write

$$F(x_1, \dots, x_n, y_1, \dots, y_n) = \sum_{h,i=0}^{\infty} \varphi_{hi}(x_1, \dots, x_{n-1}, y_1, \dots, y_{n-1}) x_n^h y_n^i,$$

where φ_{hi} are functions of class C^ω in $2(n - 1)$ variables. Moreover, φ_{hi} also is $\mathbb{B}$ -holomorphic. Thus, by the induction hypothesis, φ_{hi} is $\mathbb{B}$ -analytic:

$$\varphi_{hi}(x_1, \dots, x_{n-1}, y_1, \dots, y_{n-1}) = \sum_{\alpha \in \mathbb{N}^{n-1}} b_\alpha^{h_i} z_1^{\alpha_1} \cdots z_{n-1}^{\alpha_{n-1}}, \quad b_\alpha^{h_i} \in \mathbb{B},$$

and for $i + h = k$ we have

$$\begin{aligned} F &= \sum_{h,i} \binom{i+h}{h} j^h \varphi_{0,i+h} x_n^h y_n^i = \sum_k j^k \varphi_{0k} \left(\sum_h \binom{k}{h} x_n^h (j y_n)^{k-h} \right) \\ &= \sum_k j^k \varphi_{0k} z_n^k. \end{aligned} \quad \square$$

4 $\mathbb{B}$ -manifolds

Definition 4.1. A *non-integrable $\mathbb{B}$ -manifold* of class C^∞ (resp. of class C^ω) is a differentiable manifold of class C^∞ (resp. of class C^ω) and a tensor field J of class C^∞ (resp. of class C^ω) and type $(1, 1)$ on M satisfying the two

following conditions: (i) $J^2 = \text{id}$, (ii) For each $x \in M$, the subspaces $T_x^-(M)$ and $T_x^+(M)$ of eigenvectors of the map $J_x: T_x(M) \rightarrow T_x(M)$ of eigenvalues -1 and $+1$, respectively, have the same dimension. A map $f: (M, J) \rightarrow (M', J')$ between non-integrable $\mathbb{B}$ -manifolds of class C^∞ (resp. of class C^ω) is said to be a *J -holomorphic mapping* (cf. [10, 13]) if for all $x \in M$ we have $f_* \circ J_x = J'_{f(x)} \circ f_*$. A *$\mathbb{B}$ -manifold* of class C^∞ (resp. of class C^ω) is a non-integrable $\mathbb{B}$ -manifold (M, J) of class C^∞ (resp. of class C^ω) such that the distributions defined by the subbundles $T^-(M)$ and $T^+(M)$ are involutive.

Remark 4.2. Since $T_x(M) = T_x^-(M) \oplus T_x^+(M)$ for all $x \in M$, it follows that $\dim M = 2n$, and since $\dim T_x^-(M) = \dim T_x^+(M)$, it is clear that such subspaces define two subbundles of $T(M)$ such that $T(M) = T^-(M) \oplus T^+(M)$. Conversely, if we have two supplementary vector subbundles of $T(M)$ of the same dimension, then there exists a unique non-integrable $\mathbb{B}$ -structure on M having such subbundles as eigenbundles.

Example 4.3. Let U be an open subset of $\mathbb{B}^n$. We define J by $J(\partial/\partial x_i) = \partial/\partial y_i$, $J(\partial/\partial y_i) = \partial/\partial x_i$, $1 \leq i \leq n$. It is easily seen that J is a $\mathbb{B}$ -structure on U , named the *first canonical $\mathbb{B}$ -structure on the open subset $U \subset \mathbb{B}^n$* . We define I by $I(\partial/\partial x_i) = \partial/\partial x_i$, $I(\partial/\partial y_i) = -\partial/\partial y_i$. This is called the *second canonical $\mathbb{B}$ -structure on the open subset $U \subset \mathbb{B}^n$* (see [10]). Both structures are distinct but isomorphic, as the map $\Psi: \mathbb{B}^n = \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{B}^n$, $\Psi(u, v) = \frac{1}{2}(u + v) + \frac{1}{2}j(u - v)$, transforms I into J ; that is, $\Psi_* \circ I = J \circ \Psi_*$. Also see the proof of Proposition 4.16.

Proposition 4.4. *Let U be an open subset of $\mathbb{B}^n$. A map $F = (F_1, \dots, F_m): U \rightarrow \mathbb{B}^m$ is a J -holomorphic mapping if and only if its components are $\mathbb{B}$ -holomorphic functions; i.e., $\partial F_h / \partial \bar{z}_i = 0$, $1 \leq h, i \leq n$.*

Proof. Denote by $z'_h = x'_h + jy'_h$, $1 \leq h \leq m$, the coordinates on $\mathbb{B}^m$. We have

$$(4.1) \quad F_* = \sum_{h=1}^m \left(dF_h \otimes \frac{\partial}{\partial z'_h} + d\bar{F}_h \otimes \frac{\partial}{\partial \bar{z}'_h} \right).$$

As $J'(\partial/\partial z'_h) = j\partial/\partial z'_h$, $J'(\partial/\partial \bar{z}'_h) = -j\partial/\partial \bar{z}'_h$, by applying the equality $F_* \circ J = J' \circ F_*$ to $\partial/\partial x_i, \partial/\partial y_i$, from (4.1) we obtain: (a) $\partial F_h / \partial y_i = j\partial F_h / \partial x_i$, ($\bar{a}$) $\partial \bar{F}_h / \partial y_i = -j\partial \bar{F}_h / \partial x_i$ (b) $\partial F_h / \partial x_i = j\partial F_h / \partial y_i$, ($\bar{b}$) $\partial \bar{F}_h / \partial x_i = -j\partial \bar{F}_h / \partial y_i$. Notice that ($\bar{a}$) follows from (a) (resp. ($\bar{b}$) from (b)) by taking conjugates. Moreover, (a) is obtained from (b) multiplying by j . Hence, the condition $F_* \circ J = J' \circ F_*$ is satisfied if and only if (b) is satisfied, thus concluding. $\square$

Proposition 4.5. *Let (M, J) be a non-integrable $\mathbb{B}$ -manifold of class C^∞ (resp. of class C^ω). A necessary and sufficient condition for the map $F: M \rightarrow \mathbb{B}$ to be J -holomorphic is that functions $f, g \in C^\infty(M)$ (resp. $f, g \in C^\omega(M)$) exist such that: (i) $F = (1+j)f + (1-j)g$, (ii) f and g are first integrals of $T^-(M)$ and $T^+(M)$, respectively; that is, $X^-(f) = 0$, $X^+(g) = 0$, for all $X^- \in \Gamma(T^-(M))$, $X^+ \in \Gamma(T^+(M))$.*

Proof. Put $F = P + jQ$ and let us denote by $z' = x' + jy'$ the coordinates in $\mathbb{B}$, so that $J'(\partial/\partial x') = \partial/\partial y'$ and $J'(\partial/\partial y') = \partial/\partial x'$. Hence, the condition $F_* \circ J = J' \circ F_*$ amounts to $F_*(X^+) = \lambda^+(\partial/\partial x' + \partial/\partial y')$ and $F_*(X^-) = \lambda^-(\partial/\partial x' - \partial/\partial y')$ or, equivalently, to $X^+((x' - y') \circ F) = X^+(P - Q) = 0$ and $X^-((x' + y') \circ F) = X^-(P + Q) = 0$, and it suffices to take $f = \frac{1}{2}(P + Q)$ and $g = \frac{1}{2}(P - Q)$. $\square$

Notation 4.6. Given a non-integrable $\mathbb{B}$ -manifold (M, J) of class C^∞ we denote by $\mathcal{B}^\infty(M)$ the $\mathbb{B}$ -algebra of J -holomorphic mappings from (M, J) into $\mathbb{B}$. Let C_M^∞ denote the sheaf of germs of C^∞ functions on M . Assigning to each open subset $U \subseteq M$ the ring $\mathcal{B}^\infty(U)$ we obtain the sheaf of germs of J -holomorphic functions on M , which we shall denote by $\mathcal{B}_M^\infty$ and it is a subsheaf of $C_M^\infty \otimes_{\mathbb{R}} \mathbb{B}$. This notation is consistent with 2.15, as if U is an open subset of $\mathbb{B}^n$ the ring $\mathcal{B}^\infty(U)$ coincides with the ring of $\mathbb{B}$ -holomorphic functions (see Theorem 4.8 below).

Remark 4.7. The ring $\mathcal{B}^\infty(M)$ measures the integrability of the distributions defined by $T^-(M)$ and $T^+(M)$. In fact, if $\mathcal{D} \subseteq \mathcal{D}^{(1)} \subseteq \dots \subseteq \mathcal{D}^{(k)} \subseteq \dots$ is Tanaka's derived system associated to $\mathcal{D}$ (see [17, §1.1]), then $\mathcal{D}$ is said to be *completely non-integrable* if for every $x \in M$ there exists an integer k such that the values of the vector fields of $\mathcal{D}^{(k)}$ at x span $T_x(M)$. It is easily proved by induction that the first integrals of $\mathcal{D}$ are also annihilated by all the fields of $\mathcal{D}^{(k)}$. Consequently, if $\mathcal{D}$ is completely non-integrable and M is connected, the only first integrals of $\mathcal{D}$ are the constant functions. Hence, from Proposition 4.5 it follows that if the distributions defined by $T^-(M)$ and $T^+(M)$ are completely

non-integrable and M is connected, then the only J -holomorphic functions of M are the elements of $\mathbb{B}$. At the opposite end we have the case where both distributions are completely integrable, which we study below.

Theorem 4.8. *A non-integrable $\mathbb{B}$ -manifold (M, J) of class C^∞ and dimension $2n$ is a $\mathbb{B}$ -manifold if and only if its sheaf of germs of J -holomorphic functions is locally isomorphic to the sheaf of germs of $\mathbb{B}$ -holomorphic functions of $\mathbb{B}^n$.*

Proof. Assume $\mathcal{B}_M^\infty$ and $\mathcal{B}_{\mathbb{B}^n}^\infty$ are locally isomorphic. That is, for each $x \in M$ there exists an open neighbourhood U_x of x and a homeomorphism $\varphi: U_x \rightarrow U$ with an open subset U of $\mathbb{B}^n$ such that for each open subset $V \subset U$, the map $\varphi^*(F) = F \circ \varphi$ establishes an isomorphism between $\mathcal{B}^\infty(V)$ and $\mathcal{B}^\infty(\varphi^{-1}V)$. Let us denote by $z'_i = x'_i + jy'_i$, $1 \leq i \leq n$, the coordinates on $\mathbb{B}^n$ and by J' its $\mathbb{B}$ -structure. Since $z'_i = ((1+j)/2)(x'_i + y'_i) + ((1-j)/2)(x'_i - y'_i)$, $1 \leq i \leq n$, it is clear that $z'_i \in \mathcal{B}^\infty(V)$ and thus $\varphi^*(x'_i) + j\varphi^*(y'_i) = \varphi^*(z'_i) \in \mathcal{B}^\infty(\varphi^{-1}V)$. In particular, $\varphi^*(x'_i) = x'_i \circ \varphi$ and $\varphi^*(y'_i) = y'_i \circ \varphi$ are of class C^∞ . Hence φ is differentiable. Applying Proposition 4.5 we conclude that a function $f \in C^\infty(V)$ (resp. $f \in C^\infty(\varphi^{-1}V)$) is a first integral of $T^-(V)$ (resp. of $T^-(\varphi^{-1}V)$) if and only if $(1+j)f \in \mathcal{B}^\infty(V)$ (resp. $(1+j)f \in \mathcal{B}^\infty(\varphi^{-1}V)$), and similarly for T^+ . So, if f is a first integral of $T^-(V)$ (resp. of $T^+(V)$) then $\varphi^*(f) = f \circ \varphi$ is a first integral of $T^-(\varphi^{-1}V)$ (resp. of $T^+(\varphi^{-1}V)$), and conversely. If $X^+ \in T_{x'}^+(U_x)$, we thus have $(\varphi_* X^+)(x'_i - y'_i) = X^+ \varphi^*(x'_i - y'_i) = 0$, and hence

$$\varphi_* X^+ \in \left\langle \frac{\partial}{\partial x'_1} + \frac{\partial}{\partial y'_1}, \dots, \frac{\partial}{\partial x'_n} + \frac{\partial}{\partial y'_n} \right\rangle_{\varphi(x')} = T_{\varphi(x')}^+(U).$$

That is, $\varphi_*(T_{x'}^+(U_x)) \subset T_{\varphi(x')}^+(U)$, and similarly $\varphi_*(T_{x'}^-(U_x)) \subset T_{\varphi(x')}^-(U)$, hence $\varphi_* \circ J = J' \circ \varphi_*$. Now, since φ is a differentiable homeomorphism between manifolds of the same dimension, we deduce from Sard's Theorem that the set U'_x of points $x' \in U_x$ for which $\varphi_*: T_{x'}(U_x) \rightarrow T_{\varphi(x')}(U)$ is an isomorphism, is a dense open subset. We thus conclude that φ establishes a diffeomorphism between U'_x and a dense open subset $U' \subset U$. Consequently, given $X, Y \in \Gamma(T^+(U_x))$ we can define the direct image $\varphi \cdot [X, Y]$ on U' . In account of the facts that $\varphi \cdot [X, Y] = [\varphi \cdot X, \varphi \cdot Y]$ and the distributions of J are involutive, we deduce from the above arguments that $\varphi_*[X, Y]_{x'} = \varphi_*J([X, Y]_{x'})$ for all $x' \in U'$ or, equivalently, $[X, Y]_{x'} \in T_{x'}^+(U_x)$ for all $x' \in U'_x$, and by continuity we conclude $[X, Y] \in \Gamma(T^+(U_x))$. One can proceed in a similar way to prove that the distribution defined by $T^-(M)$ is involutive.

Conversely, suppose that the distributions defined by $T^-(M)$ and $T^+(M)$ are involutive. In this case it will suffice to prove that each point $x \in M$ has a connected open coordinate neighbourhood $(U_x; t_1, \dots, t_n, u_1, \dots, u_n)$ such that one has $T^+(U_x) = \langle \partial/\partial t_1, \dots, \partial/\partial t_n \rangle$, $T^-(U_x) = \langle \partial/\partial u_1, \dots, \partial/\partial u_n \rangle$, because in this case it is clear that the map $\varphi: U_x \rightarrow \mathbb{B}^n$, uniquely determined by the formulae $x'_i(\varphi(z)) = \frac{1}{2}(t_i(z) + u_i(z))$, $y'_i(\varphi(z)) = \frac{1}{2}(t_i(z) - u_i(z))$, $1 \leq i \leq n$, $z \in U_x$, establishes a homeomorphism between U_x and an open subset U of $\mathbb{B}^n$,

and for each open subset $V \subseteq U$ one has

$$\begin{aligned} \mathcal{B}^\infty(V) = \{ & F(x', y') = (1+j)f'(x' + y') + (1-j)g'(x' - y') \\ & : (x', y') \in V, f' \in C^\infty(p^-V), g' \in C^\infty(p^+V) \}, \end{aligned}$$

$p^-, p^+ : V \rightarrow \mathbb{R}^n$ being $p^-(x', y') = x' + y'$, $p^+(x', y') = x' - y'$. So φ^* establishes an isomorphism between $\mathcal{B}^\infty(V)$ and the subring of functions of the form $(F' \circ \varphi)(z) = (1+j)f'(t_1(z), \dots, t_n(z)) + (1-j)g'(u_1(z), \dots, u_n(z))$, $z \in \varphi^{-1}V$. That is, $F' \circ \varphi = (1+j)(f' \circ (t_1, \dots, t_n)) + (1-j)(g' \circ (u_1, \dots, u_n))$. This subring, by virtue of the choice of coordinates, coincides with $\mathcal{B}^\infty(\varphi^{-1}V)$ for all connected subset V , since the functions $f' \circ (t_1, \dots, t_n)$ are the first integrals of $T^-(\varphi^{-1}V)$, and the functions $g' \circ (u_1, \dots, u_n)$ are the first integrals of $T^+(\varphi^{-1}V)$. The general case (*i.e.*, when V is not connected) is reached by considering each connected component and making the direct product. Let us see how $(U_x; t_1, \dots, t_n, u_1, \dots, u_n)$ is constructed. Since $T^+(M)$ is involutive, by virtue of Frobenius Theorem, each $x \in M$ admits a coordinate open neighbourhood $(U_x; t_i)$, $1 \leq i \leq 2n$, such that $T^+(U_x) = \langle \partial/\partial t_1, \dots, \partial/\partial t_n \rangle$. Making a linear change of coordinates we can also assume that $(\partial/\partial t_{n+1})_x, \dots, (\partial/\partial t_{2n})_x$ is a basis of $T_x^-(U_x)$. In fact, let $X_i = \sum_{h=1}^{2n} \lambda_{hi} (\partial/\partial t_h)_x$, $1 \leq i \leq n$, be a basis of $T_x^-(U_x)$. As $T_x(U_x) = T_x^-(U_x) \oplus T_x^+(U_x)$, the vectors $(\partial/\partial t_1)_x, \dots, (\partial/\partial t_n)_x, X_1, \dots, X_n$ are a basis of $T_x(U_x)$. Hence the matrix

$$\begin{pmatrix} I_n & \Lambda_1 \\ 0 & \Lambda_2 \end{pmatrix},$$

where Λ_1, Λ_2 are the $n \times n$ matrices (λ_{ij}) , $(\lambda_{n+i,j})$, respectively and I_n stands for the $n \times n$ identity matrix, is of rank $2n$. Hence $\text{rk } \Lambda_2 = n$. We can thus define a coordinate system $\bar{t}_j = t_j + \sum_k a_{jk} t_{n+k}$, $\bar{t}_{n+j} = t_{n+j}$, $1 \leq j \leq n$, with $(a_{jk}) = -\Lambda_1 \Lambda_2^{-1}$ (note that the Jacobian of the transformation is equal to 1). Then, $\partial/\partial t_i = \partial/\partial \bar{t}_i$ and $X_i(t_j) = 0$ for $1 \leq i \leq n$, $1 \leq j \leq n$, thus proving our claim.

Moreover, let $\pi : U_x \rightarrow \mathbb{R}^n$ be the submersion $\pi(z) = (t_{n+1}(z), \dots, t_{2n}(z))$. Then, $T^+(U_x)$ is the vertical bundle of π , thus having an exact sequence of vector bundles over U_x : $0 \rightarrow T^+(U_x) \rightarrow T(U_x) \xrightarrow{\pi_*} \pi^*T(\mathbb{R}^n) \rightarrow 0$. Hence the restriction of π_* to $T^-(U_x)$ establishes an isomorphism with $\pi^*T(\mathbb{R}^n)$. Let $(s_1, \dots, s_n)$ denote the Cartesian coordinates of $\mathbb{R}^n$, so that $\pi^*(s_i) = s_i \circ \pi = t_{i+n}$, $1 \leq i \leq n$. Since $\pi_*|_{T^-(U_x)}$ is an isomorphism, there exist unique vector fields S_i on U_x such that: (1) $S_i(z) \in T_z^-(U_x)$ for all $z \in U_x$, (2) $\pi_*(S_i) = \partial/\partial s_i$, $1 \leq i \leq n$, and it is clear that the expression of S_i must be $S_i = \partial/\partial t_{i+n} + \sum_{k=1}^n f_{ik} \partial/\partial t_k$, for certain functions $f_{ik} \in C^\infty(U_x)$ vanishing at x as $(\partial/\partial t_{n+i})_x$, $1 \leq i \leq n$, span $T_x^-(U_x)$. Since $T^-(U_x)$ is also involutive, we have $[S_h, S_i] = \sum_{k=1}^n g_{hi}^k S_k$, $1 \leq h, i \leq n$, with $g_{hi}^k \in C^\infty(U_x)$. Moreover $g_{hi}^k = [S_h, S_i](t_{k+n}) = 0$ as $S_i(t_{k+n}) = \delta_{ik}$. Hence $[S_h, S_i] = 0$, and there exists a coordinate system $(u_1, \dots, u_{2n})$ such that $S_i = \partial/\partial u_i$, $1 \leq i \leq n$. We have

$$\frac{\partial(t_1, \dots, t_n; u_1, \dots, u_n)}{\partial(t_1, \dots, t_n; t_{n+1}, \dots, t_{2n})} = \begin{vmatrix} I_n & 0 \\ \partial u_i / \partial t_j & \delta_{ij} - \sum_k f_{jk} (\partial u_i / \partial t_k) \end{vmatrix}.$$

Since $f_{jk}(x) = 0$, $(t_1, \dots, t_n, u_1, \dots, u_n)$ is a coordinate system on a neighbourhood possibly smaller than U_x , and in this new system we still have $S_i = \partial/\partial u_i$, $1 \leq i \leq n$, thus concluding. $\square$

Remark 4.9. As the above proof shows, each $x \in M$ admits a coordinate system $(t_1, \dots, t_n, u_1, \dots, u_n)$ on a neighbourhood U_x , such that the foliations $T^-(M)$, $T^+(M)$ are defined on U_x by $t_1 = \dots = t_n = 0$, $u_1 = \dots = u_n = 0$, respectively (see [10, Proposition 1.2]).

Remark 4.10. Let (M, J) and (M', J') be $\mathbb{B}$ -manifolds of class C^∞ . A differentiable map $f: M \rightarrow M'$ is J -holomorphic if and only if for every open subset U' of M' and every function $F' \in \mathcal{B}^\infty(U')$, the composite map $F' \circ f$ belongs to $\mathcal{B}^\infty(f^{-1}U')$, as follows from Theorem 4.8 and Proposition 4.4.

Remark 4.11. Let (M, J) and (M, J') be $\mathbb{B}$ -manifolds of class C^∞ . Then $J = J'$ if and only if $\mathcal{B}_{M,J}^\infty = \mathcal{B}_{M,J'}^\infty$; that is, the sheaf of germs of $\mathbb{B}$ -holomorphic functions completely determines the non-integrable $\mathbb{B}$ -structure of the manifold. In fact, $\mathcal{B}_{M,J}^\infty = \mathcal{B}_{M,J'}^\infty$ if and only if $\text{id}: M \rightarrow M$ is J -holomorphic, if and only if $J = J'$.

Corollary 4.12. *A $2n$ -dimensional non-integrable $\mathbb{B}$ -manifold (M, J) of class C^∞ is a $\mathbb{B}$ -manifold if and only if it admits an open covering $\{U_\alpha\}_{\alpha \in A}$ such that for each α there exists an isomorphism $\varphi_\alpha: U_\alpha \rightarrow \mathbb{B}^n$ of non-integrable $\mathbb{B}$ -manifolds between U_α and an open subset of $\mathbb{B}^n$.*

Proof. If $\{(U_\alpha, \varphi_\alpha)\}$ exists, then (M, J) is a $\mathbb{B}$ -manifold, since $\mathbb{B}^n$ is a $\mathbb{B}$ -manifold and this property is preserved by isomorphisms of non-integrable $\mathbb{B}$ -manifolds. If (M, J) is a $\mathbb{B}$ -manifold, it suffices to observe that the map $\varphi_\alpha: U_\alpha \rightarrow \mathbb{B}^n$ (as in Theorem 4.8) is a J -isomorphism, by virtue of Remark 4.10. $\square$

Corollary 4.13. *Let $\{U_\alpha\}_{\alpha \in A}$ be an open covering of a topological space M and let $\varphi_\alpha: U_\alpha \rightarrow \mathbb{B}^n$ be a homeomorphism between U_α and an open subset of $\mathbb{B}^n$ such that the maps $\varphi_\beta \circ \varphi_\alpha^{-1}: \varphi_\alpha(U_\alpha \cap U_\beta) \rightarrow \varphi_\beta(U_\alpha \cap U_\beta)$ are $\mathbb{B}$ -holomorphic. In these conditions, there exists a unique structure of $\mathbb{B}$ -manifold J of class C^∞ on M such that, for each $\alpha \in A$, φ_α establishes an isomorphism between $\mathcal{B}_M^\infty|_{U_\alpha}$ and $\mathcal{B}_M^\infty|_{\varphi_\alpha(U_\alpha)}$.*

Proof. The uniqueness follows from Corollary 4.11. As for the existence, we define J on M by $J = (\varphi_\alpha^{-1})_* \circ J' \circ \varphi_{\alpha*}$, J' being the canonical $\mathbb{B}$ -structure of $\mathbb{B}^n$, and the definition makes sense because $(\varphi_\beta \circ \varphi_\alpha^{-1})_*$ commutes with J' , since $\varphi_\beta \circ \varphi_\alpha^{-1}$ is $\mathbb{B}$ -holomorphic. $\square$

Definition 4.14. A $\mathbb{B}$ -analytic manifold is a $2n$ -dimensional topological manifold M endowed with a subsheaf of $\mathbb{B}$ -algebras $\mathcal{A}_M \subset C_M^0 \otimes_{\mathbb{R}} \mathbb{B}$ which is locally isomorphic to $\mathcal{A}_{\mathbb{B}^n}$ (see Definition 3.1 and Notation 3.2).

Corollary 4.15. *Each $\mathbb{B}$ -analytic manifold structure on M induces a $\mathbb{B}$ -manifold structure on M .*

Proof. According to Definition 4.14, there exists an open covering $M = \bigcup U_\alpha$, and homeomorphisms $\phi_\alpha: U_\alpha \rightarrow U'_\alpha \subset \mathbb{B}^n$, such that for every open subset $V \subset U_\alpha$, the mapping $f \mapsto f \circ \phi^{-1}$ is an isomorphism $\mathcal{A}(V) \rightarrow \mathcal{A}(\phi^{-1}V)$. In particular, we have an isomorphism $\mathcal{A}(U_\alpha \cap U_\beta) \rightarrow \mathcal{A}(\phi_\alpha(U_\alpha \cap U_\beta))$, thus showing (by virtue of Theorem 3.3) that $\phi_\alpha: U_\alpha \cap U_\beta \rightarrow \phi_\alpha(U_\alpha \cap U_\beta)$ is $\mathbb{B}$ -holomorphic. Hence the composite map $\phi_\beta \circ \phi_\alpha^{-1}: \phi_\alpha(U_\alpha \cap U_\beta) \rightarrow \phi_\beta(U_\alpha \cap U_\beta)$ also is. $\square$

Proposition 4.16. *Let X, Y be two C^∞ differentiable manifolds of the same dimension n . In $M = X \times Y$ there exists a unique structure of $\mathbb{B}$ -manifold such that for each pair of open subsets $U \subset X$, $V \subset Y$, the ring of $\mathbb{B}$ -holomorphic functions of M defined on $U \times V$ is*

$$\{(1+j)f \circ p^- + (1-j)g \circ p^+ : f \in C^\infty(U), g \in C^\infty(V)\},$$

where $p^-: M = X \times Y \rightarrow X$ and $p^+: M = X \times Y \rightarrow Y$ are the canonical projections.

Proof. The uniqueness of the structure is deduced from the fact that the open subsets of the form $U \times V$ are a basis of the usual topology of M . Indeed, let $\mathcal{B}_M^\infty$ be the sheaf of germs of $\mathbb{B}$ -holomorphic functions of a $\mathbb{B}$ -structure on M , and let W be an open subset of M . One can cover W by product open subsets $U_\alpha \times V_\alpha$, $\alpha \in A$, so that from the definition of a sheaf it follows that $F \in \Gamma(W, \mathcal{B}_M^\infty)$ if and only if $F|_{U_\alpha \times V_\alpha} \in \Gamma(U_\alpha \times V_\alpha, \mathcal{B}_M^\infty)$ for all $\alpha \in A$. Hence, two $\mathbb{B}$ -structures on M satisfying the condition of the assertion necessarily coincide.

To establish the existence of $\mathcal{B}_M^\infty$ we proceed in the following way. With the above notations, we define $\Gamma(W, \mathcal{B}_M^\infty)$ as the set of continuous functions $F: W \rightarrow \mathbb{B}$ such that $F = \{(1+j)f \circ p^- + (1-j)g \circ p^+\}$ for any functions $f \in C^\infty(p^-W)$, $g \in C^\infty(p^+W)$. Notice that F is actually defined on $p^-(W) \times p^+(W)$. It only rests to verify that the so defined sheaf $\mathcal{B}_M^\infty$ is locally isomorphic to the sheaf $\mathcal{B}_{\mathbb{B}^n}^\infty$ of germs of $\mathbb{B}$ -holomorphic functions of the canonical $\mathbb{B}$ -structure of $\mathbb{B}^n$ (see Example 4.3). Let $\varphi: U \rightarrow \mathbb{R}^n$ and $\psi: V \rightarrow \mathbb{R}^n$ be two diffeomorphisms of U and V with two open subsets of $\mathbb{R}^n$. It is clear that the differentiable map $\Psi: U \times V \rightarrow \mathbb{B}^n$ defined by $\Psi(u, v) = \frac{1}{2}(\varphi(u) + \psi(v)) + \frac{1}{2}j(\varphi(u) - \psi(v))$, establishes a diffeomorphism between $U \times V$ and an open subset of $\mathbb{B}^n$. We shall now prove that the map $F \mapsto F \circ \Psi$ of the ring of $\mathbb{B}$ -holomorphic functions of $\mathbb{B}^n$ defined in $\Psi(U \times V)$ into $\Gamma(U \times V, \mathcal{B}_M^\infty)$ establishes an isomorphism of the first ring with $\Gamma(U \times V, \mathcal{B}_M^\infty)$. In fact, if $F: \Psi(U \times V) \rightarrow \mathbb{B}$ is $\mathbb{B}$ -holomorphic we then have $F(x, y) = (1+j)f'(x+y) + (1-j)g'(x-y)$ for all $(x, y) \in \Psi(U \times V)$, from which

$$\begin{aligned} (F \circ \Psi)(u, v) &= F\left(\frac{1}{2}(\varphi(u) + \psi(v)), \frac{1}{2}(\varphi(u) - \psi(v))\right) \\ &= (1+j)(f' \circ \varphi)(u) + (1-j)(g' \circ \psi)(v), \end{aligned}$$

and so $F \circ \Psi \in \Gamma(U \times V, \mathcal{B}_M^\infty)$. Conversely, if $G \in \Gamma(U \times V, \mathcal{B}_M^\infty)$, we have $G(u, v) = (1+j)f(u) + (1-j)g(v)$ for all $(u, v) \in U \times V$, and writing $\Psi^{-1}(x, y) = (u, v)$

(that is, $x = \frac{1}{2}(\varphi(u) + \psi(v))$, $y = \frac{1}{2}(\varphi(u) - \psi(v))$), we obtain $(G \circ \Psi^{-1})(x, y) = G(u, v) = (1 + j)(f \circ \varphi^{-1})(x + y) + (1 - j)(g \circ \psi^{-1})(x - y)$. Hence, $G \circ \Psi^{-1}$ is a $\mathbb{B}$ -holomorphic function of $\mathbb{B}^n$ defined on $\Psi(U \times V)$. $\square$

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CSIC
SERRANO 144
28006 MADRID
SPAIN
E-MAIL: PMGADEA@IEC.CSIC.ES
JAIME@IEC.CSIC.ES

LABORATOIRE EMILE PICARD C.N.R.S.
UNIVERSITÉ PAUL SABATIER
118 ROUTE DE NARBONNE
31062 TOULOUSE, CEDEX (FRANCE)
E-MAIL: GRIFONE@PICARD.UPS-TLSE.FR