HOMOGENEOUS KÄHLER AND SASAKIAN STRUCTURES RELATED TO COMPLEX HYPERBOLIC SPACES

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Abstract We study homogeneous Kähler structures on a non-compact Hermitian symmetric space and their lifts to homogeneous Sasakian structures on the total space of a principal line bundle over it, and we analyze the case of the complex hyperbolic space.

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homogeneous almost contact Riemannian manifolds;
non-compact Hermitian symmetric spaces; Sasakian spaces

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1 Introduction

The general theory of homogeneous Kähler manifolds is well-known, as well as the relation between homogeneous symplectic and homogeneous contact manifolds (see e.g. Boothby and Wang [6], Díaz Miranda and Reventós [9], and Dorfmeister and Nakajima [10]).

As is also widely known, a connected, simply connected and complete Riemannian manifold is a symmetric space if and only if its curvature tensor field is parallel. Ambrose and Singer [2] extended this result to obtain a characterization of homogeneous Riemannian manifolds in terms of the existence of a tensor field S of type $(1, 2)$ on the manifold, called a homogeneous Riemannian structure (see Tricerri and Vanhecke [26], where a classification of such structures is also given), satisfying certain properties (see (2.1); if $S = 0$ one has the symmetric case). Moreover, Sekigawa [24] obtained the corresponding result for almost Hermitian manifolds, defining homogeneous almost Hermitian structures, which were classified by Abbena and Garbiero in [1] (among them the homogeneous Kähler structures). Its odd-dimensional version, the almost contact metric case, has been also studied (see, for instance, [8, 11, 14, 19]).

In Section 2, we give the basic results about homogeneous Riemannian and homogeneous Kähler structures. In particular we consider these structures on Hermitian symmetric spaces of non-compact type. Besides the trivial homogeneous structure $S = 0$ associated to the description of one such space as a symmetric space, other structures can be obtained associated to other descriptions as a homogeneous space and, in particular, to its description as a solvable Lie group given by an Iwasawa decomposition (§ 2.2).

We also give a construction of homogeneous Sasakian structures on the bundle space of a principal line bundle over a Hermitian symmetric space of non-compact type, endowed with a connection 1-form that is the contact form of a Sasakian structure on the total space (Proposition 2.5).

The complex hyperbolic space $\mathbb{CH}(n) = SU(n, 1)/S(U(n) \times U(1))$ with the Bergman metric is an irreducible Hermitian symmetric space of non-compact type, and, up to homotheties, is the simply-connected complete complex space form of negative curvature. It has been characterized in [12] in terms of the existence of certain type of homogeneous Kähler structure on it, and in [7] a Lie-theoretical description of its homogeneous structure of linear type is found. In Section 3 we study the homogeneous Kähler structures on $\mathbb{CH}(n)$ from other point of view, which in particular provide an infinite number of descriptions of $\mathbb{CH}(n)$ as non-isomorphic solvable Lie groups. Moreover, we consider the principal line bundle over $\mathbb{CH}(n)$ with its Sasakian structure given in a natural way from a connection form on the bundle, and we obtain the families of homogeneous Sasakian structures on its bundle space following our previous general construction. Summarizing, we get:

(a) All the homogeneous Kähler structures on $\mathbb{CH}(n) \equiv AN$. They are given in terms of some 1-forms related by a system of differential equations on the solvable Lie group AN (Theorem 3.1).

(b) The explicit description of a multi-parametric family of homogeneous Kähler structures on $\mathbb{CH}(n)$, given by using the generators of $\mathfrak{a} + \mathfrak{n}$ (Proposition 3.6), and the corresponding subgroups of the full isometry group $SU(n, 1)$ of AN (Theorem 3.7).

(c) The explicit description of a one-parametric family of homogeneous Sasakian structures on the bundle space of the line bundle $\bar{M} \rightarrow \mathbb{CH}(n)$, given in terms of the horizontal lifts of the generators of $\mathfrak{a} + \mathfrak{n}$ and the fundamental vector field ξ on $\bar{M}$ (Proposition 3.9), and their associated reductive decompositions (Propositions 3.11 and 3.12). One of them describes $\bar{M}$ as the complete simply connected φ -symmetric Sasakian space $\widetilde{SU}(n, 1)/SU(n)$, which is also a Sasakian space form.

On the other hand, complex hyperbolic space was the first target spacetime where Nishino's [21] alternative (i.e., neither necessarily hyper-Kähler nor quaternion-Kähler) $N = (4, 0)$ superstring theory proved to work. This model has some interesting features, among them, not to have (which is a trait common to heterotic σ -models) the incompatibility between the torsion tensor and quaternion-Kähler manifolds found by de Wit and van Nieuwenhuizen [27]. Another peculiarity is that in this case, one of the two scalars of the relevant global multiplet is promoted to coordinates on $\mathbb{CH}(n)$, while the other plays the role of a tangent vector under the holonomy group $S(U(n) \times U(1))$.

2 Homogeneous Riemannian Structures

Ambrose and Singer [2] proved that a connected, simply connected and complete Riemannian manifold is homogeneous if and only there exists a tensor field S of type $(1, 2)$ on M such that the connection $\tilde{\nabla} = \nabla - S$ satisfies the Eqs.

$$\tilde{\nabla}g = 0, \quad \tilde{\nabla}R = 0, \quad \tilde{\nabla}S = 0, \quad (2.1)$$

where ∇ is the Levi-Civita connection of g and R its curvature tensor field, for which we adopt the conventions $R_{XY}Z = \nabla_{[X,Y]}Z - \nabla_X\nabla_YZ + \nabla_Y\nabla_XZ$, $R_{XYZW} = g(R_{XY}Z, W)$. Such a tensor field S is called a homogeneous Riemannian structure ([26]). We also denote by S the associated tensor field of type $(0, 3)$ on M defined by $S_{XYZ} = g(S_XY, Z)$.

2.1 Homogeneous Kähler structures

An almost Hermitian manifold (M, g, J) is said to be a homogeneous almost Hermitian manifold if there exists a Lie group of holomorphic isometries which acts transitively and effectively on M . Sekigawa proved the following

Theorem 2.1. ([24]) *A connected, simply connected and complete almost Hermitian manifold (M, g, J) is homogeneous if and only if there is a tensor field S of type $(1, 2)$ on M which satisfies Eqs. (2.1) and $\tilde{\nabla}J = 0$.*

A tensor S satisfying the Eqs. (2.1) and $\tilde{\nabla}J = 0$ is called a homogeneous almost Hermitian structure. The almost Hermitian manifold (M, g, J) is Kähler if and only if J is integrable and the fundamental 2-form Ω on M , given by $\Omega(X, Y) = g(X, JY)$, is closed, or equivalently $\nabla J = 0$. In this case, a homogeneous almost Hermitian structure is also called a homogeneous Kähler structure, and we have

Proposition 2.2. *A homogeneous Riemannian structure S on a Kähler manifold (M, g, J) is a homogeneous Kähler structure if and only if $S \cdot J = 0$, or equivalently $S_{XYZ} = S_{XJYJZ}$ for all the vector fields X, Y, Z on M .*

Corollary 2.3. *A connected, simply connected and complete Kähler manifold (M, g, J) is a homogeneous Kähler manifold if and only if there exists a homogeneous Kähler structure on M .*

If $(M = G/H, g)$ is a homogeneous Riemannian manifold, where G is a connected Lie group acting transitively and effectively on M as a group of isometries and H is the isotropy group at a point $o \in M$, then the Lie algebra $\mathfrak{g}$ of G may be decomposed into a vector space direct sum $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$, where $\mathfrak{h}$ is the Lie algebra of H and $\mathfrak{m}$ is an $\text{Ad}(H)$ -invariant subspace of $\mathfrak{g}$. If G is connected and M is simply connected then H is connected, and the condition $\text{Ad}(H)\mathfrak{m} \subset \mathfrak{m}$ is equivalent to $[\mathfrak{h}, \mathfrak{m}] \subset \mathfrak{m}$. The vector space $\mathfrak{m}$ is identified with $T_o(M)$ by the isomorphism $X \in \mathfrak{m} \rightarrow X_o^* \in T_o(M)$, where X^* is the Killing vector field on M generated by the one-parameter subgroup $\{\exp tX\}$ of G acting on M . If $X \in \mathfrak{g} = \mathfrak{h} + \mathfrak{m}$, we write $X = X_{\mathfrak{h}} + X_{\mathfrak{m}}$, ($X_{\mathfrak{h}} \in \mathfrak{h}, X_{\mathfrak{m}} \in \mathfrak{m}$). The canonical connection $\tilde{\nabla}$ of $M = G/H$ (with regard to the reductive decomposition $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$) is determined by

$$(\tilde{\nabla}_{X^*}Y^*)_o = [X^*, Y^*]_o = -[X, Y]_o^* = -([X, Y]_{\mathfrak{m}})_o^*, \quad X, Y \in \mathfrak{m}, \quad (2.2)$$

and $S = \nabla - \tilde{\nabla}$ satisfies the Ambrose-Singer Eqs. (2.1), and it is the homogeneous Riemannian structure associated to the reductive decomposition $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$. If (M, g) is endowed with a compatible almost complex structure J invariant by G (so that $(M = G/H, g, J)$ is a homogeneous almost Hermitian manifold), restricting J to $T_o(M) \equiv \mathfrak{m}$, we obtain a

linear endomorphism J_o of $\mathfrak{m}$ such that $J_o^2 = -1$, and $J_o \operatorname{ad}_{\mathfrak{h}} = \operatorname{ad}_{\mathfrak{h}} J_o$. Moreover, J is integrable if and only if

$$[J_o X, J_o Y]_{\mathfrak{m}} - [X, Y]_{\mathfrak{m}} - J_o[X, J_o Y]_{\mathfrak{m}} - J_o[J_o X, Y]_{\mathfrak{m}} = 0$$

for all $X, Y \in \mathfrak{m}$ ([18], Ch. 10, Prop. 6.5).

Conversely, suppose that (M, g) is a connected, simply connected and complete Riemannian manifold, and let S be a homogeneous Riemannian structure on (M, g) . We put $\mathfrak{m} = T_o(M)$, where $o \in M$. If $\tilde{R}$ is the curvature tensor of the connection $\tilde{\nabla} = \nabla - S$, the holonomy algebra $\tilde{\mathfrak{h}}$ of $\tilde{\nabla}$ is the Lie subalgebra of the Lie algebra of antisymmetric endomorphisms $\mathfrak{so}(\mathfrak{m})$ of $(\mathfrak{m}, g_o)$ generated by the operators $\tilde{R}_{XY}$, where $X, Y \in \mathfrak{m}$. A Lie bracket is defined (Nomizu [20]) in the vector space direct sum $\tilde{\mathfrak{g}} = \tilde{\mathfrak{h}} + \mathfrak{m}$ by

$$\begin{aligned} [U, V] &= UV - VU, & U, V &\in \tilde{\mathfrak{h}}, \\ [U, X] &= U(X), & U &\in \tilde{\mathfrak{h}}, X \in \mathfrak{m}, \\ [X, Y] &= \tilde{R}_{XY} + S_X Y - S_Y X, & X, Y &\in \mathfrak{m}, \end{aligned} \tag{2.3}$$

and $\tilde{\mathfrak{g}} = \tilde{\mathfrak{h}} + \mathfrak{m}$ is the reductive decomposition corresponding to the homogeneous Riemannian structure S . Let $\tilde{G}$ be the connected simply connected Lie group whose Lie algebra is $\tilde{\mathfrak{g}}$ and $\tilde{H}$ the connected Lie subgroup of $\tilde{G}$ whose Lie algebra is $\tilde{\mathfrak{h}}$. Then $\tilde{G}$ acts transitively on M as a group of isometries and M is diffeomorphic to $\tilde{G}/\tilde{H}$. If Γ is the set of the elements of $\tilde{G}$ which act trivially on M , then Γ is a discrete normal subgroup of $\tilde{G}$, and the Lie group $G = \tilde{G}/\Gamma$ acts transitively and effectively on M as a group of isometries, with isotropy group $H = \tilde{H}/\Gamma$. Then M is diffeomorphic to G/H . Now, if J is a compatible almost complex structure on (M, g) and S is a homogeneous almost Hermitian structure, then the holonomy algebra $\tilde{\mathfrak{h}}$ is a subalgebra of the Lie algebra $\mathfrak{u}(\mathfrak{m}) = \{A \in \mathfrak{so}(\mathfrak{m}) : A \cdot J = 0\}$ of the unitary group, and $M \approx \tilde{G}/\tilde{H} \approx G/H$ is a homogeneous almost Hermitian manifold.

2.2 Hermitian symmetric spaces of non-compact type

Suppose that $(M = G/K, g, J)$ is a connected Hermitian symmetric space of non-compact type, where $G = I_0(M)$ is the identity component of the group of (holomorphic) isometries and K is a maximal compact subgroup of G . Then M is simply connected and the Hermitian structure is Kähler. We consider a Cartan decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ of the Lie algebra $\mathfrak{g}$ of G , and the Iwasawa decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{a} + \mathfrak{n}$, where $\mathfrak{k}$ is the Lie algebra of K , $\mathfrak{a} \subset \mathfrak{p}$ is a maximal $\mathbb{R}$ -diagonalizable subalgebra of $\mathfrak{g}$, and $\mathfrak{n}$ is a nilpotent subalgebra. Let A and N be the connected abelian and nilpotent Lie subgroups of G whose Lie algebras are $\mathfrak{a}$ and $\mathfrak{n}$, respectively. The solvable Lie group AN acts simply transitively on M , so M is isometric to AN equipped with the left-invariant Riemannian metric defined by the scalar product $\langle \cdot, \cdot \rangle$, induced on $\mathfrak{a} + \mathfrak{n} \cong \mathfrak{g}/\mathfrak{k} \cong \mathfrak{p}$ by a positive multiple of $B|_{\mathfrak{p} \times \mathfrak{p}}$, where B is the Killing form of $\mathfrak{g}$.

Now, let $\hat{G}$ be a connected closed Lie subgroup of G which acts transitively on M . The isotropy group of this action at $o = K \in M$ is $H = \hat{G} \cap K$. Then $M = G/K$ has also the description $M \equiv \hat{G}/H$, and $o \equiv H \in \hat{G}/H$. Let $\hat{\mathfrak{g}} = \mathfrak{h} + \mathfrak{m}$ be a reductive decomposition of the Lie algebra $\hat{\mathfrak{g}}$ of $\hat{G}$ corresponding to $M \equiv \hat{G}/H$.

We have the isomorphisms of vector spaces

$$\mathfrak{p} \cong \mathfrak{g}/\mathfrak{k} \cong \hat{\mathfrak{g}}/\mathfrak{h} \cong \mathfrak{m} \cong T_o(M) \cong \mathfrak{a} + \mathfrak{n},$$

with

$$\xi: \mathfrak{p} \xrightarrow{\cong} \mathfrak{m}, \quad \mu: \mathfrak{m} \xrightarrow{\cong} T_o(M), \quad \zeta: T_o(M) \xrightarrow{\cong} \mathfrak{a} + \mathfrak{n},$$

given by

$$\xi^{-1}(Z) = Z_{\mathfrak{p}}, \quad \mu(Z) = Z_o^*, \quad \zeta^{-1}(X) = X_o^*, \quad Z \in \mathfrak{m}, \quad X \in \mathfrak{a} + \mathfrak{n}.$$

For each $X \in \mathfrak{g}$, we have $(X_{\mathfrak{k}})_o^* = 0$ and $(\nabla(X_{\mathfrak{p}})^*)_o = 0$, and since the Levi-Civita connection ∇ has no torsion, for each $X, Y \in \mathfrak{g}$, we have

$$(\nabla_{X^*} Y^*)_o = (\nabla_{(X_{\mathfrak{p}})^*} (Y_{\mathfrak{k}})^*)_o = [(X_{\mathfrak{p}})^*, (Y_{\mathfrak{k}})^*]_o = -[X_{\mathfrak{p}}, Y_{\mathfrak{k}}]_o^*. \quad (2.4)$$

The reductive decomposition $\hat{\mathfrak{g}} = \mathfrak{h} + \mathfrak{m}$ defines the homogeneous Riemannian structure $S = \nabla - \tilde{\nabla}$, where $\tilde{\nabla}$ is the canonical connection of $M \equiv \hat{G}/H$ with respect to $\hat{\mathfrak{g}} = \mathfrak{h} + \mathfrak{m}$, which is $\hat{G}$ -invariant and uniquely determined by $(\tilde{\nabla}_{X^*} Y^*)_o = -[X, Y]_o^*$, for $X, Y \in \mathfrak{m}$ (2.2). The tensor field S is also uniquely determined by its value at o because $M \equiv \hat{G}/H$ and S is $\hat{G}$ -invariant. Since J is $\hat{G}$ -invariant, from [18], Ch. 10, Prop. 2.7, it follows that $\tilde{\nabla}J = 0$, and by Theorem 2.1, S is a homogeneous Kähler structure.

We have

$$(S_{X^*} Y^*)_o = (\nabla_{X^*} Y^*)_o + [X, Y]_o^* = \nabla_{Y_o^*} X_o^*, \quad X, Y \in \mathfrak{m}. \quad (2.5)$$

By (2.4) and (2.5), S is given by

$$S_{X_o^*} Y_o^* = [X_{\mathfrak{k}}, Y_{\mathfrak{p}}]_o^*, \quad X, Y \in \mathfrak{m}.$$

Then, for each $X, Y \in \mathfrak{a} + \mathfrak{n}$, we have

$$S_{X_o^*} Y_o^* = S_{\xi(X_{\mathfrak{p}})^*_o} \xi(Y_{\mathfrak{p}})^*_o = [(\xi(X_{\mathfrak{p}}))_{\mathfrak{k}}, Y_{\mathfrak{p}}]_o^*.$$

The complex structure J on $M = G/K$ is defined by an element E_J in the center of $\mathfrak{k}$, and it defines the complex structure $J \in \text{End}(\mathfrak{a} + \mathfrak{n})$ such that the following diagram is commutative, and $(\mathfrak{a} + \mathfrak{n}, \langle, \rangle, J)$ becomes a Hermitian vector space isomorphic to $(T_o(M), g_o, J_o)$.

$$\begin{array}{ccccccc} \mathfrak{p} & \xrightarrow{\xi} & \mathfrak{m} & \xrightarrow{\mu} & T_o(M) & \xrightarrow{\zeta} & \mathfrak{a} + \mathfrak{n} \\ \text{ad}_{E_J} \downarrow & & J_o \downarrow & & J_o \downarrow & & \downarrow J \\ \mathfrak{p} & \xrightarrow{\xi} & \mathfrak{m} & \xrightarrow{\mu} & T_o(M) & \xrightarrow{\zeta} & \mathfrak{a} + \mathfrak{n} \end{array}$$

Let A and N be the connected abelian and nilpotent Lie subgroups of G whose Lie algebras are $\mathfrak{a}$ and $\mathfrak{n}$, respectively. The solvable Lie group AN acts simply transitively on M . Then M is isometric to AN equipped with the left-invariant Riemannian metric defined by the scalar product induced on $\mathfrak{a} + \mathfrak{n} \cong \mathfrak{g}/\mathfrak{k} \cong \mathfrak{p}$ by a positive multiple of $B|_{\mathfrak{p} \times \mathfrak{p}}$, where B is the Killing form of $\mathfrak{g}$, so that AN equipped with the left-invariant almost complex structure defined by J is a Kähler manifold.

2.3 Homogeneous almost contact Riemannian manifolds

An almost contact structure on a $(2n + 1)$ -dimensional manifold $\bar{M}$ is a triple (φ, ξ, η) , where φ is a tensor field of type $(1, 1)$, ξ a vector field (called the characteristic vector field) and η a differential 1-form on $\bar{M}$ such that

$$\varphi^2 = -\text{id} + \eta \otimes \xi, \quad \eta(\xi) = 1.$$

Then $\varphi\xi = 0$, $\eta \circ \varphi = 0$, and φ has rank $2n$. If $\bar{g}$ is a Riemannian metric on $\bar{M}$ such that $\bar{g}(\varphi\tilde{X}, \varphi\tilde{Y}) = \bar{g}(\tilde{X}, \tilde{Y}) - \eta(\tilde{X})\eta(\tilde{Y})$ for all vector fields $\tilde{X}$ and $\tilde{Y}$ on $\bar{M}$ then $(\varphi, \xi, \eta, \bar{g})$ is said to be an almost contact metric structure on $\bar{M}$. In this case, $\bar{g}(\tilde{X}, \xi) = \eta(\tilde{X})$. The 2-form Φ on $\bar{M}$ defined by $\Phi(\tilde{X}, \tilde{Y}) = \bar{g}(\tilde{X}, \varphi\tilde{Y})$ is called the fundamental 2-form of the almost contact metric structure $(\varphi, \xi, \eta, \bar{g})$. If $d\eta(\tilde{X}, \tilde{Y}) = \tilde{X}\eta(\tilde{Y}) - \tilde{Y}\eta(\tilde{X}) - \eta([\tilde{X}, \tilde{Y}]) = 2\Phi(\tilde{X}, \tilde{Y})$, then $(\varphi, \xi, \eta, \bar{g})$ is called a contact metric (or contact Riemannian) structure; in particular, $\eta \wedge (d\eta)^n \neq 0$, that is, η is a contact form on $\bar{M}$. If

$$(D_{\tilde{X}}\varphi)\tilde{Y} = \bar{g}(\tilde{X}, \tilde{Y})\xi - \eta(\tilde{Y})\tilde{X}, \quad (2.6)$$

where D is the Levi-Civita connection of $\bar{g}$, then $(\varphi, \xi, \eta, \bar{g})$ is called a Sasakian structure, and the manifold $\bar{M}$ with such a structure is a Sasakian manifold. Sasakian manifolds can also be characterized as normal contact metric manifolds and they are in some sense odd-dimensional analogues of Kähler manifolds (see Blair [3, 4]).

If $(\varphi, \xi, \eta, \bar{g})$ is an almost contact metric structure on $\bar{M}$ and $(\bar{M} = \bar{G}/H, \bar{g})$ is a homogeneous Riemannian manifold such that φ is invariant under the action of the connected Lie group $\bar{G}$ (and hence so are ξ and η) then $(\bar{M}, \varphi, \xi, \eta, \bar{g})$ is called a homogeneous almost contact Riemannian manifold ([8, 14, 19]). Let $\bar{R}$ be the curvature tensor field of the Levi-Civita connection D of $\bar{g}$. Let S be a homogeneous Riemannian structure on $\bar{M}$, that is $\tilde{D}\bar{g} = 0$, $\tilde{D}\bar{R} = 0$ and $\tilde{D}S = 0$, where $\tilde{D} = D - S$. If S satisfies the additional condition $\tilde{D}\varphi = 0$ (and hence $\tilde{D}\xi = 0$ and $\tilde{D}\eta = 0$), then S is called a homogeneous almost contact metric structure on $(\bar{M}, \varphi, \xi, \eta, \bar{g})$. From the results of Kiričenko in [17] on homogeneous Riemannian spaces with invariant tensor structure, it follows

Theorem 2.4. *A connected, simply connected and complete almost contact metric manifold $(\bar{M}, \varphi, \xi, \eta, \bar{g})$ is a homogeneous almost contact Riemannian manifold if and only if there exists a homogeneous almost contact metric structure on $\bar{M}$.*

A homogeneous almost contact metric structure on a Sasakian manifold will be also called a homogeneous Sasakian structure.

2.4 Principal 1-bundles over almost Hermitian manifolds

Let (M, g, J) be an almost Hermitian manifold and let $\bar{M}$ be the bundle space of a principal 1-bundle over M . Let η be a connection (form) on the principal bundle $\pi: \bar{M} \rightarrow M$, and let ξ be the fundamental vector field on $\bar{M}$ defined by the element 1 of the Lie algebra $\mathbb{R}$ of the structure group of the bundle. Then $\eta(\xi) = 1$. For each vector field X on M , we denote by X^H the horizontal lift of X with respect to η . If $\tilde{X}$ is a vector field on $\bar{M}$, its vertical part is $\eta(\tilde{X})\xi$. Then, for any vector fields X and Y on M , we have

$$[X^H, Y^H] = [X, Y]^H + \eta([X^H, Y^H])\xi.$$

Moreover, $[X^H, \xi] = 0$, because X^H is invariant under the action of the structural group. We define a tensor field φ of type $(1, 1)$ and a Riemannian metric $\bar{g}$ on $\bar{M}$ by

$$\varphi X^H = (JX)^H, \quad \varphi \xi = 0, \quad \bar{g} = \pi^* g + \eta \otimes \eta, \quad (2.7)$$

where X and Y are vector fields on M . Clearly, $(\varphi, \xi, \eta, \bar{g})$ is an almost contact metric structure on $\bar{M}$, and we have $\bar{g}(X^H, Y^H) = g(X, Y) \circ \pi$, and $\bar{g}(X^H, \xi) = 0$. Let Φ be its 2-fundamental form. If Ω is the fundamental 2-form of the almost Hermitian manifold (M, g, J) , then $\pi^* \Omega = \Phi$.

If ∇ and D are the the Levi-Civita connections of g and $\bar{g}$, respectively, then (Ogiue [22])

$$D_{X^H} Y^H = (\nabla_X Y)^H + \frac{1}{2} \eta([X^H, Y^H]) \xi = (\nabla_X Y)^H - \frac{1}{2} d\eta(X^H, Y^H) \xi,$$

and $D_{X^H} \xi = D_\xi X^H = -\varphi X^H$. Now, if $2\Phi = d\eta$, Eq. (2.6) is satisfied as one can easily see by replacing $(\tilde{X}, \tilde{Y})$ by (X^H, Y^H) , (X^H, ξ) , and (ξ, Y^H) , respectively. Then, if the almost contact metric structure $(\varphi, \xi, \eta, \bar{g})$ is a contact structure, it is also Sasakian.

Suppose now that the structural group of the principal 1-bundle $\pi: \bar{M} \rightarrow M$ is $\mathbb{R}$ and that the base manifold is a $2n$ -dimensional connected Hermitian symmetric space of non-compact type $(M = G/K, g, J)$, so that M is isometric to the solvable Lie group AN as in § 2.2. Then M is holomorphically diffeomorphic to a bounded symmetric domain, i.e., to a simply connected open subset of $\mathbb{C}^n$ such that each point is an isolated fixed point of an involutive holomorphic diffeomorphism of itself ([15], Ch. VIII, Th. 7.1). Since $\pi: \bar{M} \rightarrow M$ is a principal line bundle over the paracompact manifold M , then it admits a global section ([18], Ch. I, Th. 5.7), so there exists a diffeomorphism $\bar{M} \rightarrow M \times \mathbb{R}$, and the bundle space $\bar{M}$ may be identified with $AN \times \mathbb{R}$, with π being the projection on AN . On the other hand, since the fundamental 2-form Ω associated to the Kähler structure (g, J) is closed, $\Omega = d\zeta$ for some real analytic 1-form ζ on AN . We consider the connection form $\eta = 2\pi^* \zeta + dt$ on $\bar{M}$, where t is the coordinate of $\mathbb{R}$. The vertical vector field ξ with $\eta(\xi) = 1$ can be identified with $\frac{d}{dt}$, and we consider φ and $\bar{g}$ given by (2.7). Then $2\Phi = 2\pi^* \Omega = 2\pi^* d\zeta = d\eta$, and hence $(\varphi, \xi, \eta, \bar{g})$ is a Sasakian structure on $\bar{M}$.

If $\bar{S}$ is a homogeneous almost contact metric structure on $\bar{M}$, and $\tilde{D} = D - \bar{S}$, then $\tilde{D}\xi = 0$, and hence $\bar{S}_{X^H} \xi = D_{X^H} \xi = -\varphi X^H$. We have

Proposition 2.5. *Let $(M = G/K, g, J)$ be a connected Hermitian symmetric space of non-compact type. Let $\pi: \bar{M} \rightarrow M$ be a principal line bundle with connection form η such that the almost contact metric structure $(\varphi, \xi, \eta, \bar{g})$ on $\bar{M}$ defined by (2.7) is Sasakian.*

(a) *If S is a homogeneous Kähler structure on M then the tensor field $\bar{S}$ on $\bar{M}$ defined by*

$$\bar{S}_{X^H} Y^H = (S_X Y)^H - \bar{g}(X^H, \varphi Y^H) \xi, \quad \bar{S}_{X^H} \xi = -\varphi X^H = \bar{S}_\xi X^H, \quad \bar{S}_\xi \xi = 0,$$

for all vector fields X and Y on M , is a homogeneous Sasakian structure on $\bar{M}$.

(b) *$\{S^t : t \in \mathbb{R}\}$ defined by*

$$S_{X^H}^t Y^H = -\bar{g}(X^H, \varphi Y^H) \xi, \quad S_{X^H}^t \xi = -\varphi X^H, \quad S_\xi^t X^H = -t\varphi X^H, \quad S_\xi^t \xi = 0,$$

is a family of homogeneous Sasakian structures on $\bar{M}$.

Proof. (a) If $\tilde{D} = D - \bar{S}$, then since $\bar{S}_{X^H Y^H Z^H} = \bar{g}((S_X Y)^H, Z^H) = g(S_X Y, Z) \circ \pi = -g(Y, S_X Z) \circ \pi = -\bar{g}(Y^H, (S_X Z)^H) = -\bar{S}_{X^H Z^H Y^H}$, and $\bar{S}_{X^H Y^H \xi} = -\bar{S}_{X^H \xi Y^H}$, the condition $\tilde{D}\bar{g} = 0$ is satisfied. On the other hand, if $\tilde{\nabla} = \nabla - S$ we have

$$\tilde{D}_{X^H} Y^H = (\tilde{\nabla}_X Y)^H, \quad \tilde{D}_{X^H} \xi = \tilde{D}_\xi X^H = 0. \quad (2.8)$$

We can identify $M = G/K$ with the solvable Lie group AN in an Iwasawa decomposition $G = KAN$ and consider the Lie algebra $\mathfrak{a} + \mathfrak{n}$ of AN . If $\tilde{U}, \tilde{V}, \tilde{X}, \tilde{Y}, \tilde{Z}$ are horizontal lifts of elements of $\mathfrak{a} + \mathfrak{n}$ or some of them are the vertical vector field ξ , then

$$(\tilde{D}_{\tilde{U}} \bar{R})_{\tilde{X}\tilde{Y}\tilde{Z}\tilde{V}} = -\bar{R}_{\tilde{X}\tilde{Y}\tilde{Z}\tilde{D}_{\tilde{U}}\tilde{V}} + \bar{R}_{\tilde{X}\tilde{Y}\tilde{V}\tilde{D}_{\tilde{U}}\tilde{Z}} - \bar{R}_{\tilde{Z}\tilde{V}\tilde{X}\tilde{D}_{\tilde{U}}\tilde{Y}} + \bar{R}_{\tilde{Z}\tilde{V}\tilde{Y}\tilde{D}_{\tilde{U}}\tilde{X}}, \quad (2.9)$$

since $\tilde{U}(\bar{R}_{\tilde{X}\tilde{Y}\tilde{Z}\tilde{V}}) = 0$. Now, if $X, Y, Z, V \in \mathfrak{a} + \mathfrak{n}$, then

$$\begin{aligned} \bar{R}_{X^H Y^H Z^H V^H} &= (R_{XYZV} - 2g(X, JY)g(Z, JV) \\ &\quad + g(X, JV)g(Y, JZ) - g(X, JZ)g(Y, JV)) \circ \pi, \\ \bar{R}_{X^H Y^H Z^H \xi} &= -\bar{g}([X, Y]^H, \varphi Z^H) \\ &\quad + \bar{g}((\nabla_X Z)^H, \varphi Y^H) - \bar{g}((\nabla_Y Z)^H, \varphi X^H), \\ \bar{R}_{X^H \xi Z^H \xi} &= \bar{g}(D_{X^H} \xi, D_{Z^H} \xi). \end{aligned} \quad (2.10)$$

By using (2.8) and (2.10), the conditions $\tilde{\nabla} R = 0$ and $\tilde{\nabla} J = 0$ for the homogeneous Kähler structure S on M , and the formula $\bar{R}_{\tilde{X}\tilde{Y}} \xi = \eta(\tilde{X})\tilde{Y} - \eta(\tilde{Y})\tilde{X}$ for the Sasakian manifold $(\bar{M}, \varphi, \xi, \eta, \bar{g})$ ([4], Prop. 7.3), one obtains from (2.9) that $\tilde{D}\bar{R} = 0$. Now, $(\tilde{D}_{\tilde{U}} \bar{S})_{X^H Y^H} = ((\tilde{\nabla}_U S)_X Y)^H$, $(\tilde{D}_{\tilde{U}} \bar{S})_{X^H \xi} = -((\tilde{\nabla}_U J)X)^H$, and $\tilde{D}_\xi \bar{S} = 0$, then $\tilde{D}\bar{S} = 0$. Moreover, $(\tilde{D}_{X^H} \varphi)Y^H = ((\tilde{\nabla}_X J)Y)^H$ and $(\tilde{D}_{X^H} \varphi)\xi = 0$, then $\tilde{D}\varphi = 0$, and $\bar{S}$ is a homogeneous Sasakian structure on M .

(b) If $t = 1$ the corresponding tensor S^1 coincides with $\bar{S}$ in (a) for $S = 0$. For arbitrary t , if $\tilde{D}^t = D - S^t$ we have $\tilde{D}_\xi^t X^H = (t-1)(JX)^H$, and we get $\tilde{D}^t \bar{g} = 0$, $\tilde{D}^t \bar{R} = 0$, $\tilde{D}^t \bar{S}^t = 0$, $\tilde{D}^t \varphi = 0$. $\square$

3 The Complex Hyperbolic Space $\mathbb{CH}(n)$

3.1 $\mathbb{CH}(n)$ as a solvable Lie group

The complex hyperbolic space $\mathbb{CH}(n)$, which may be identified with the unit ball in $\mathbb{C}^n$ endowed with the hyperbolic metric of constant holomorphic sectional curvature -4 , may also be viewed as the irreducible Hermitian symmetric space of non-compact type $SU(n, 1)/S(U(n) \times U(1))$.

The Lie algebra $\mathfrak{su}(n, 1)$ of $SU(n, 1)$ can be described as the subalgebra of $\mathfrak{sl}(n+1, \mathbb{C})$ of all matrices of the form

$$X = \begin{pmatrix} Z & P^T \\ \bar{P} & ic \end{pmatrix}, \quad (3.1)$$

where $Z \in \mathfrak{u}(n)$, $c \in \mathbb{R}$, and $P = (p_1, \dots, p_n) \in \mathbb{C}^n$. The involution τ of $\mathfrak{su}(n, 1)$ given by $\tau(X) = -\bar{X}^T$ defines the Cartan decomposition $\mathfrak{su}(n, 1) = \mathfrak{k} + \mathfrak{p}$, where

$$\mathfrak{k} = \left\{ \begin{pmatrix} Z & 0 \\ 0 & ic \end{pmatrix} : \text{tr } Z + ic = 0 \right\} \cong \mathfrak{s}(\mathfrak{u}(n) \oplus \mathfrak{u}(1)), \quad \mathfrak{p} = \left\{ \begin{pmatrix} 0 & P^T \\ \bar{P} & 0 \end{pmatrix} \right\}.$$

The element A_0 of $\mathfrak{p}$ defined by $P = (0, \dots, 0, 1)$ generates a maximal $\mathbb{R}$ -diagonalizable subalgebra $\mathfrak{a}$ of $\mathfrak{su}(n, 1)$. Let f_0 be the linear functional on $\mathfrak{a}$ given by $f_0(A_0) = 1$. If $n > 1$, the set of roots of $(\mathfrak{su}(n, 1), \mathfrak{a})$ is $\Sigma = \{\pm f_0, \pm 2f_0\}$, the set $\Pi = \{f_0\}$ is a system of simple roots, and the corresponding positive root system is $\Sigma^+ = \{f_0, 2f_0\}$. If $n = 1$, $\Sigma = \{\pm 2f_0\}$, and $\Pi = \Sigma^+ = \{2f_0\}$.

Let E_{ij} be the matrix in $\mathfrak{gl}(n, \mathbb{C})$ such that the entry at the i -th row and the j -th column is 1 and the other entries are all zero. The root vector spaces are

$$\begin{aligned} \mathfrak{g}_{f_0} &= \langle Z_j, Z'_j : 1 \leq j \leq n-1 \rangle \text{ (if } n > 1), \quad \mathfrak{g}_{2f_0} = \langle U \rangle, \\ \mathfrak{g}_{-f_0} &= \langle W_j, W'_j : 1 \leq j \leq n-1 \rangle \text{ (if } n > 1), \quad \mathfrak{g}_{-2f_0} = \langle V \rangle, \end{aligned}$$

where

$$\begin{aligned} Z_j &= E_{jn} - E_{j,n+1} - E_{nj} - E_{n+1,j}, & Z'_j &= i(E_{jn} - E_{j,n+1} + E_{nj} + E_{n+1,j}), \\ W_j &= E_{jn} + E_{j,n+1} - E_{nj} + E_{n+1,j}, & W'_j &= i(E_{jn} + E_{j,n+1} + E_{nj} - E_{n+1,j}), \\ U &= i(E_{nn} - E_{n,n+1} + E_{n+1,n} - E_{n+1,n+1}), \\ V &= i(E_{nn} + E_{n,n+1} - E_{n+1,n} - E_{n+1,n+1}). \end{aligned}$$

If $n > 2$, the centralizer of $\mathfrak{a}$ in $\mathfrak{k}$ is $Z_{\mathfrak{k}}(\mathfrak{a}) = \langle C_r, F_{jk}, H_{jk} : r, j, k = 1, \dots, n-1, j < k \rangle \cong \mathfrak{u}(n-1)$, where

$$C_r = 2iE_{rr} - iE_{nn} - iE_{n+1,n+1}, \quad F_{jk} = E_{jk} - E_{kj}, \quad H_{jk} = i(E_{jk} + E_{kj})$$

and $\mathfrak{su}(n, 1) = (Z_{\mathfrak{k}}(\mathfrak{a}) + \mathfrak{a}) + \sum_{f \in \Sigma} \mathfrak{g}_f$ is the restricted-root space decomposition. We also have the Iwasawa decomposition $\mathfrak{su}(n, 1) = \mathfrak{k} + \mathfrak{a} + \mathfrak{n}$, where $\mathfrak{n} = \mathfrak{g}_{f_0} + \mathfrak{g}_{2f_0} = \langle U, Z_j, Z'_j : 1 \leq j \leq n-1 \rangle$.

If $n = 2$, we put $C = C_1 = \text{diag}(2i, -i, -i)$, $Z = Z_1$, $Z' = Z'_1$, and in this case C generates $Z_{\mathfrak{k}}(\mathfrak{a})$, and $\mathfrak{a} + \mathfrak{n} = \langle A_0, U, Z, Z' \rangle$. If $n = 1$, $Z_{\mathfrak{k}}(\mathfrak{a}) = 0$, we have the restricted-root space decomposition $\mathfrak{su}(1, 1) = \mathfrak{a} + (\mathfrak{g}_{2f_0} + \mathfrak{g}_{-2f_0}) = \langle A_0 \rangle + \langle U, V \rangle$, and the solvable part in the Iwasawa decomposition is $\mathfrak{a} + \mathfrak{n} = \langle A_0, U \rangle$.

By using the Cartan decomposition $\mathfrak{su}(n, 1) = \mathfrak{k} + \mathfrak{p}$, we express each element $X \in \mathfrak{su}(n, 1)$ as the sum $X = X_{\mathfrak{k}} + X_{\mathfrak{p}}$ ($X_{\mathfrak{k}} \in \mathfrak{k}$, $X_{\mathfrak{p}} \in \mathfrak{p}$). In particular, we have

$$\begin{aligned} U_{\mathfrak{k}} &= i(E_{nn} - E_{n+1,n+1}), & U_{\mathfrak{p}} &= i(E_{n+1,n} - E_{n,n+1}), \\ (Z_j)_{\mathfrak{k}} &= E_{jn} - E_{nj}, & (Z_j)_{\mathfrak{p}} &= -(E_{n+1,j} + E_{j,n+1}), \\ (Z'_j)_{\mathfrak{k}} &= i(E_{jn} + E_{nj}), & (Z'_j)_{\mathfrak{p}} &= i(E_{n+1,j} - E_{j,n+1}). \end{aligned}$$

From the basis $\{A_0, U, Z_j, Z'_j : 1 \leq j \leq n-1\}$ of $\mathfrak{a} + \mathfrak{n}$ and the generators of $Z_{\mathfrak{k}}(\mathfrak{a})$ above, we get the basis $\{C_r, F_{jk}, H_{jk}, U_{\mathfrak{k}}, (Z_r)_{\mathfrak{k}}, (Z'_r)_{\mathfrak{k}} : r, j, k = 1, \dots, n-1, j < k\}$ of $\mathfrak{k}$, and the basis $\{A_0, U_{\mathfrak{p}}, (Z_j)_{\mathfrak{p}}, (Z'_j)_{\mathfrak{p}} : 1 \leq j \leq n-1\}$ of $\mathfrak{p}$. Notice that, if $n = 1$, $\mathfrak{k} = \langle U_{\mathfrak{k}} \rangle$ and $\mathfrak{p} = \langle A_0, U_{\mathfrak{p}} \rangle$, and if $n = 2$ we have $\mathfrak{k} = \langle C, U_{\mathfrak{k}}, Z_{\mathfrak{k}}, Z'_{\mathfrak{k}} \rangle$, and $\mathfrak{p} = \langle A, U_{\mathfrak{p}}, Z_{\mathfrak{p}}, Z'_{\mathfrak{p}} \rangle$. We

also decompose $\mathfrak{k} = \mathfrak{k}' + \mathfrak{c}$, where $\mathfrak{k}' = [\mathfrak{k}, \mathfrak{k}] = \langle C_r - U_{\mathfrak{k}}, F_{jk}, H_{jk}, (Z_r)_{\mathfrak{k}}, (Z'_r)_{\mathfrak{k}} : r, j, k = 1, \dots, n-1, j < k \rangle \cong \mathfrak{su}(n)$, and $\mathfrak{c}$ is the center of $\mathfrak{k}$, which is generated by the element $E_J = \frac{1}{2(n+1)}(C_1 + \dots + C_{n-1} + (n+1)U_{\mathfrak{k}})$ such that $\text{ad}_{E_J} : \mathfrak{p} \rightarrow \mathfrak{p}$ defines the complex structure on $\mathbb{CH}(n)$. By the isomorphisms $\mathfrak{p} \cong \mathfrak{su}(n, 1)/\mathfrak{k} \cong \mathfrak{a} + \mathfrak{n}$, we obtain the complex structure J acting on $\mathfrak{a} + \mathfrak{n}$ as follows.

$$JA_0 = -U, \quad JU = A_0, \quad JZ_r = Z'_r, \quad JZ'_r = -Z_r. \quad (3.2)$$

We consider the scalar product $\langle \cdot, \cdot \rangle$ on $\mathfrak{a} + \mathfrak{n}$ defined by the isomorphism $\mathfrak{a} + \mathfrak{n} \cong \mathfrak{p}$ and $\frac{1}{4(n+1)}B|_{\mathfrak{p} \times \mathfrak{p}}$. Then $(\mathfrak{a} + \mathfrak{n}, \langle \cdot, \cdot \rangle, J)$ is a Hermitian vector space, and the basis $\{A_0, U, Z_r, Z'_r : 1 \leq r \leq n-1\}$ of $\mathfrak{a} + \mathfrak{n}$ is orthonormal. We consider the solvable factor AN (with Lie algebra $\mathfrak{a} + \mathfrak{n}$) of the Iwasawa decomposition of $SU(n, 1)$ with the invariant metric g and almost complex structure J defined by $\langle \cdot, \cdot \rangle$ and J , respectively.

The Lie brackets of the elements of the basis of $\mathfrak{a} + \mathfrak{n}$ are given by

$$\begin{aligned} [A_0, U] &= 2U, \quad [A_0, Z_j] = Z_j, \quad [A_0, Z'_j] = Z'_j, \quad [Z_j, Z'_r] = -\delta_{jr}2U, \\ [U, Z_j] &= [U, Z'_j] = [Z_j, Z_r] = [Z'_j, Z'_r] = 0. \end{aligned}$$

The Levi-Civita connection ∇ is given by $2g(\nabla_X Y, Z) = g([X, Y], Z) - g([Y, Z], X) + g([Z, X], Y)$ for all $X, Y, Z \in \mathfrak{a} + \mathfrak{n}$. So, the covariant derivatives between generators of $\mathfrak{a} + \mathfrak{n}$ are given by

$$\begin{aligned} \nabla_{A_0} A_0 &= \nabla_{A_0} U = \nabla_{A_0} Z_r = \nabla_{A_0} Z'_r = 0, \\ \nabla_U A_0 &= -2U, \quad \nabla_U U = 2A_0, \quad \nabla_U Z_r = Z'_r, \quad \nabla_U Z'_r = -Z_r, \\ \nabla_{Z_j} A_0 &= -Z_j, \quad \nabla_{Z_j} U = Z'_j, \quad \nabla_{Z_j} Z_r = \delta_{jr} A_0, \quad \nabla_{Z_j} Z'_r = -\delta_{jr} U, \\ \nabla_{Z'_j} A_0 &= -Z'_j, \quad \nabla_{Z'_j} U = -Z_j, \quad \nabla_{Z'_j} Z_r = \delta_{jr} U, \quad \nabla_{Z'_j} Z'_r = \delta_{jr} A_0. \end{aligned} \quad (3.3)$$

The components of the curvature tensor field R are given by

$$\begin{aligned} R_{A_0 U} A_0 &= -4U, \quad R_{A_0 U} U = 4A_0, \quad R_{A_0 U} Z_r = 2Z'_r, \quad R_{A_0 U} Z'_r = -2Z_r, \\ R_{A_0 Z_j} A_0 &= -Z_j, \quad R_{A_0 Z_j} U = Z'_j, \quad R_{A_0 Z_j} Z_r = \delta_{jr} A_0, \quad R_{A_0 Z_j} Z'_r = -\delta_{jr} U, \\ R_{A_0 Z'_j} A_0 &= -Z'_j, \quad R_{A_0 Z'_j} U = -Z_j, \quad R_{A_0 Z'_j} Z_r = \delta_{jr} U, \quad R_{A_0 Z'_j} Z'_r = \delta_{jr} A_0, \\ R_{U Z_j} A_0 &= -Z'_j, \quad R_{U Z_j} U = -Z_j, \quad R_{U Z_j} Z_r = \delta_{jr} U, \quad R_{U Z_j} Z'_r = \delta_{jr} A_0, \\ R_{U Z'_j} A_0 &= Z_j, \quad R_{U Z'_j} U = -Z'_j, \quad R_{U Z'_j} Z_r = -\delta_{jr} A_0, \quad R_{U Z'_j} Z'_r = \delta_{jr} U, \\ R_{Z_k Z_j} A_0 &= R_{Z_k Z_j} U = 0, \quad R_{Z_j Z'_r} A_0 = 2\delta_{jr} U, \quad R_{Z_j Z'_r} U = -2\delta_{jr} A_0, \\ R_{Z_k Z_j} Z_r &= \delta_{jr} Z_k - \delta_{kr} Z_j, \quad R_{Z_k Z_j} Z'_r = \delta_{jr} Z'_k - \delta_{kr} Z'_j, \quad R_{Z'_k Z'_j} = R_{Z_k Z_j}, \\ R_{Z_j Z'_j} Z_r &= -2(1 + \delta_{jr} Z'_r), \quad R_{Z_j Z'_j} Z'_r = 2(1 + \delta_{jr}) Z_r \\ R_{Z_k Z'_j} Z_r &= -\delta_{jr} Z'_k - \delta_{kr} Z'_j, \quad R_{Z_k Z'_j} Z'_r = \delta_{jr} Z_k - \delta_{kr} Z_j, \quad (k \neq j). \end{aligned}$$

In particular we see that the invariant metric on AN has constant holomorphic sectional curvature -4 .

3.2 Homogeneous Kähler structures on $\mathbb{CH}(n) \equiv AN$

We will determine the homogeneous Kähler structures on $\mathbb{CH}(n) \equiv AN$ in terms of the basis of left-invariant forms $\alpha, \beta, \gamma^j, \gamma'^j$, $1 \leq j \leq n-1$, dual to A_0, U, Z_j, Z'_j . If S is

a homogeneous Riemannian structure on AN and $\tilde{\nabla} = \nabla - S$, the condition $\tilde{\nabla}g = 0$ in (2.1) is equivalent to $S_{XYZ} + S_{XZY} = 0$ for all $X, Y, Z \in \mathfrak{a} + \mathfrak{n}$. Moreover, $\tilde{\nabla}R = 0$ is equivalent to the condition

$$(\nabla_X R)_{Y_1 Y_2 Y_3 Y_4} = -R_{S_X Y_1 Y_2 Y_3 Y_4} - R_{Y_1 S_X Y_2 Y_3 Y_4} - R_{Y_1 Y_2 S_X Y_3 Y_4} - R_{Y_1 Y_2 Y_3 S_X Y_4},$$

for all $Y_1, Y_2, Y_3, Y_4 \in \mathfrak{a} + \mathfrak{n}$. Replacing (Y_1, Y_2, Y_3, Y_4) by (A_0, U, A_0, Z_j) , (A_0, U, A_0, Z'_j) , (A_0, U, Z_k, Z_j) , and (A_0, U, Z_k, Z'_j) , one gets that $S_{XUZ_j} = S_{XA_0 Z'_j}$, $S_{XUZ'_j} = -S_{XA_0 Z_j}$, $S_{XZ_k Z'_j} = -S_{XZ'_k Z_j}$, and $S_{XZ_k Z_j} = S_{XZ'_k Z'_j}$, respectively. It is easy to see that the condition $\tilde{\nabla}R = 0$ holds if and only if the last four Eqs. are satisfied for all $X \in \mathfrak{a} + \mathfrak{n}$. These Eqs. also show (see (3.2)) that the condition $S \cdot J = 0$ of homogeneous Kähler structure (see Proposition 2.2) is fulfilled. We put

$$\omega(X) = S_{XA_0 U}, \quad \sigma^j(X) = S_{XA_0 Z_j} = -S_{XUZ'_j}, \quad \tau^j(X) = S_{XA_0 Z'_j} = S_{XUZ_j}, \quad (3.4)$$

$$\theta^{kj}(X) = S_{XZ_k Z'_j} = S_{XZ'_k Z_j}, \quad \psi^{kj}(X) = S_{XZ_k Z_j} = S_{XZ'_k Z'_j}. \quad (3.5)$$

We have $\theta^{kj} = \theta^{jk}$ and $\psi^{kj} = -\psi^{jk}$. Now, we must determine the conditions for the 1-forms ω , σ^j , τ^j , θ^{kj} and σ^{kj} under which the condition $\tilde{\nabla}S = 0$ in (2.1) is satisfied.

By (3.3), (3.4) and (3.5), the connection $\tilde{\nabla} = \nabla - S$ is given by

$$\begin{aligned} \tilde{\nabla}_X A_0 &= -(2\beta + \omega)(X)U - \sum_j (\gamma^j + \sigma^j)(X)Z_j - \sum_j (\gamma'^j + \tau^j)(X)Z'_j, \\ \tilde{\nabla}_X U &= (2\beta + \omega)(X)A_0 - \sum_j (\gamma'^j + \tau^j)(X)Z_j + \sum_j (\gamma^j + \sigma^j)(X)Z'_j, \\ \tilde{\nabla}_X Z_j &= (\gamma^j + \sigma^j)(X)A_0 + (\gamma'^j + \tau^j)(X)U + (\beta - \theta^j)(X)Z'_j \\ &\quad + \sum_{k \neq j} (\psi^{kj}(X)Z_k - \theta^{kj}(X)Z'_k), \\ \tilde{\nabla}_X Z'_j &= (\gamma'^j + \tau^j)(X)A_0 - (\gamma^j + \sigma^j)(X)U + (\theta^j - \beta)(X)Z_j \\ &\quad + \sum_{k \neq j} (\theta^{kj}(X)Z_k - \psi^{kj}(X)Z'_k). \end{aligned}$$

Now, replacing (V_1, V_2) in the Eq. $(\tilde{\nabla}_X S)(W, V_1, V_2) = 0$ by (A_0, U) , (A_0, Z_j) , (A_0, Z'_j) , (Z_k, Z_j) and (Z_k, Z'_j) , respectively, we obtain that the condition $\tilde{\nabla}S = 0$ is equivalent to the following conditions:

$$\begin{aligned} \tilde{\nabla}\omega &= 2\sum_j ((\gamma^j + \sigma^j) \otimes \tau^j - (\gamma'^j + \tau^j) \otimes \sigma^j), \\ \tilde{\nabla}\sigma^j &= -(\beta + \omega + \theta^j) \otimes \tau^j + (\gamma'^j + \tau^j) \otimes (\omega + \theta^j) \\ &\quad + \sum_{k \neq j} (\psi^{kj} \otimes \sigma^k - \theta^{kj} \otimes \tau^k + (\gamma'^k + \tau^k) \otimes \theta^{kj} - (\gamma^k + \sigma^k) \otimes \psi^{kj}), \\ \tilde{\nabla}\tau^j &= (\beta + \omega + \theta^j) \otimes \sigma^j - (\gamma^j + \sigma^j) \otimes (\omega + \theta^j) \\ &\quad + \sum_{k \neq j} (\theta^{kj} \otimes \sigma^k + \psi^{kj} \otimes \tau^k - (\gamma^k + \sigma^k) \otimes \theta^{kj} - (\gamma'^k + \tau^k) \otimes \psi^{kj}), \\ \tilde{\nabla}\theta^{kj} &= (\gamma^j + \sigma^j) \otimes \tau^k + (\gamma^k + \tau^k) \otimes \tau^j - (\gamma'^j + \tau^j) \otimes \sigma^k - (\gamma'^k + \tau^k) \otimes \sigma^j \\ &\quad + \sum_l \psi^{lk} \wedge \theta^{jl} + \sum_l \theta^{lk} \wedge \psi^{jl}, \\ \tilde{\nabla}\psi^{kj} &= (\gamma^k + \sigma^k) \otimes \sigma^j - (\gamma^j + \sigma^j) \otimes \sigma^k - (\gamma'^k + \tau^k) \otimes \tau^j - (\gamma'^j + \tau^j) \otimes \tau^k \\ &\quad + \sum_l \theta^{lk} \wedge \theta^{jl} - \sum_l \psi^{lk} \wedge \psi^{jl}, \end{aligned} \quad (3.6)$$

where $\theta^j = \theta^{jj}$. Thus, from (3.4) and (3.5), we have

Theorem 3.1. *All the homogeneous Kähler structures on $\mathbb{CH}(n) \equiv AN$ are given by*

$$S = \omega \otimes (\alpha \wedge \beta) + \sum_{j=1}^{n-1} (\sigma^j \otimes (\alpha \wedge \gamma^j - \beta \wedge \gamma'^j) + \tau^j \otimes (\alpha \wedge \gamma'^j + \beta \wedge \gamma^j) + \theta^{jj} \otimes (\gamma^j \wedge \gamma'^j)) \\ + \sum_{1 \leq k < j \leq n-1} (\psi^{kj} \otimes (\gamma^k \wedge \gamma^j + \gamma'^k \wedge \gamma'^j) + \theta^{kj} \otimes (\gamma^k \wedge \gamma'^j + \gamma^j \wedge \gamma'^k)),$$

where $\omega, \sigma^j, \tau^j, \theta^{kj}, \psi^{kj}$, ($1 \leq k, j \leq n-1$), are 1-forms on AN satisfying $\theta^{jk} = \theta^{kj}$, $\psi^{jk} = -\psi^{kj}$ and the Eqs. (3.6).

If $n = 2$, we put $\gamma = \gamma^1, \gamma' = \gamma'^1$, so that $\{\alpha, \beta, \gamma, \gamma'\}$ is the basis of left-invariant forms on $AN = \mathbb{CH}(2)$ dual to $\{A_0, U, Z, Z'\}$, and we have

Corollary 3.2. *All the homogeneous Kähler structures on the complex hyperbolic plane $\mathbb{CH}(2) \equiv AN$ are given by*

$$S = \omega \otimes (\alpha \wedge \beta) + \sigma \otimes (\alpha \wedge \gamma - \beta \wedge \gamma') + \tau \otimes (\alpha \wedge \gamma' + \beta \wedge \gamma) + \theta \otimes (\gamma \wedge \gamma'),$$

where ω, σ, τ and θ are 1-forms on AN satisfying

$$\begin{aligned} \tilde{\nabla} \omega &= 2(\gamma + \sigma) \otimes \tau - 2(\gamma' + \tau) \otimes \sigma = \tilde{\nabla} \theta, \\ \tilde{\nabla} \sigma &= -(\beta + \omega + \theta) \otimes \gamma + (\gamma' + \tau) \otimes (\omega + \theta), \\ \tilde{\nabla} \tau &= (\beta + \omega + \theta) \otimes \sigma - (\gamma + \sigma) \otimes (\omega + \theta). \end{aligned}$$

If $n = 1$, $\{\alpha, \beta\}$ is the basis of 1-invariant forms on the 2-dimensional solvable Lie group $AN = \mathbb{CH}(1)$ dual to the basis $\{A_0, U\}$ of $\mathfrak{a} + \mathfrak{n}$, and we have

Corollary 3.3. *All the homogeneous Kähler structures on the complex hyperbolic line (or real hyperbolic plane) $\mathbb{CH}(1) \equiv AN$ are given by $S = \omega \otimes (\alpha \wedge \beta)$, where ω is a 1-form on AN satisfying $\tilde{\nabla} \omega = 0$.*

Remark 3.4. If $S = \omega \otimes (\alpha \wedge \beta)$ is a homogeneous Kähler structure on $\mathbb{CH}(1)$, and $\omega = \lambda \alpha + \mu \beta$, where λ and μ are functions on $\mathbb{CH}(1)$, the condition $\tilde{\nabla} \omega = 0$ together with the structure Eq. $[A_0, U] = 2U$ gives $\lambda = \mu = 0$ or $\lambda^2 + \mu^2 = 4$, and we have that there are infinite homogeneous Kähler structures on $\mathbb{CH}(1)$. However (see [26], Th. 4.4), up to isomorphism, there are only two homogeneous structures on the real hyperbolic plane: one of them is $S = 0$ ($\lambda = \mu = 0$), and the other, which is given by $S_X Y = g(X, Y) \xi_0 - g(\xi_0, Y) X$, with $\xi_0 = 2A_0$ (for $X, Y \in \mathfrak{a} + \mathfrak{n} = \langle A_0, U \rangle$), corresponds to $S = \omega \otimes (\alpha \wedge \beta)$, with $\omega = -2\beta$ ($\lambda = 0, \mu = -2$).

Remark 3.5. For each $n > 0$, $S = 0$ is a homogeneous Kähler structure on $\mathbb{CH}(n) \equiv AN$, the corresponding canonical connection is $\tilde{\nabla} = \nabla$, its holonomy algebra is $\mathfrak{k} \cong \mathfrak{su}(n) \oplus \mathfrak{u}(1)$, the associated reductive decomposition is the Cartan decomposition $\mathfrak{su}(n, 1) = \mathfrak{k} + \mathfrak{p}$, and it gives the description of $\mathbb{CH}(n)$ as symmetric space $\mathbb{CH}(n) = SU(n, 1)/S(U(n) \times U(1))$.

Now, our purpose is to obtain nontrivial homogeneous Kähler structures on $\mathbb{CH}(n)$, $n \geq 2$, their associated reductive decompositions, and the corresponding descriptions as homogeneous Kähler spaces.

We will seek for solutions for which $\sigma^j = -\gamma^j$, $\tau^j = -\gamma'^j$. In this case, we have

$$\begin{aligned}\tilde{\nabla}\gamma^j &= (\beta - \theta^j) \otimes \gamma'^j + \sum_{k \neq j} (\psi^{kj} \otimes \gamma^k - \theta^{kj} \otimes \gamma'^k), \\ \tilde{\nabla}\gamma'^j &= (\theta^j - \beta) \otimes \gamma^j + \sum_{k \neq j} (\theta^{kj} \otimes \gamma^k + \psi^{kj} \otimes \gamma'^k).\end{aligned}$$

(Obviously, the last summands on the right hand-side in each one of the two Eqs. above do not appear if $n = 2$.) By the second and third Eqs. in (3.6), we must have $\omega = -2\beta$, which also satisfies the first Eqs. in (3.6), because $\tilde{\nabla}\beta = (2\beta + \omega) \otimes \alpha - \sum_j (\gamma'^j + \tau^j) \otimes \gamma^j + \sum_j (\gamma^j + \sigma^j) \otimes \gamma'^j = 0$. If $n = 2$, by Corollary 3.2, we only have to determine θ such that $\tilde{\nabla}\theta = 0$. If we put $\theta = a\alpha + b\beta + c\gamma + c'\gamma'$, by using also the structure Eqs. of $\mathfrak{a} + \mathfrak{n} = \langle A_0, U, Z, Z' \rangle$, we obtain that $c = c' = 0$ and a and b are constant. For $n > 2$ we put $\theta^j = \theta^{jj} = a_j\alpha + b_j\beta$, $\theta^{kj} = c_{kj}\alpha$, $\psi^{kj} = p_{kj}\alpha$, ($k \neq j$), with $a_j, b_j, c_{kj}, p_{kj} \in \mathbb{R}$. Then, if $\sigma^j = -\gamma^j$, $\tau^j = -\gamma'^j$ and $\omega = -2\beta$, Eqs. (3.6) are satisfied if and only if one has

$$p_{kj}(b_k - b_j) = c_{kj}(b_k - b_j) = 0.$$

Consequently, we get

Proposition 3.6. *For $n > 2$, the space $\mathbb{CH}(n)$ admits the multi-parametric family of homogeneous Kähler structures $S = S^{a_j, b_j, c_{kj}, p_{kj}}$ given in terms of the generators of $\mathfrak{a} + \mathfrak{n}$ by the following table.*

Table I	A_0	U	Z_j	Z'_j
S_{A_0}	0	0	$a_j Z'_j + \sum_{l \neq j} (p_{jl} Z_l + c_{jl} Z'_l)$	$-a_j Z_j + \sum_{l \neq j} (p_{jl} Z'_l - c_{jl} Z_l)$
S_U	$-2U$	$2A_0$	$b_j Z'_j$	$-b_j Z_j$
S_{Z_k}	$-Z_k$	Z'_k	$\delta_{kj} A_0$	$-\delta_{kj} U$
$S_{Z'_k}$	$-Z'_k$	$-Z_k$	$\delta_{kj} U$	$\delta_{kj} A_0$

The complex hyperbolic plane $\mathbb{CH}(2)$ admits the two-parametric family of homogeneous Kähler structures $S = S^{a, b}$ given in terms of the generators of $\mathfrak{a} + \mathfrak{n}$ by the following table.

Table II	A_0	U	Z	Z'
S_{A_0}	0	0	aZ'	$-aZ$
S_U	$-2U$	$2A_0$	bZ'	$-bZ$
S_Z	$-Z$	Z'	A_0	$-U$
$S_{Z'}$	$-Z'$	$-Z$	U	A_0

If $S = S^{a_j, b_j, c_{kj}, p_{kj}}$, with respect to the basis $\{A_0, U, Z_j, Z'_j\}$ of $\mathfrak{a} + \mathfrak{n}$, the connection $\tilde{\nabla} = \nabla - S$ is given by

$$\begin{aligned}\tilde{\nabla}_{A_0} Z_j &= -a_j Z'_j - \sum_{l \neq j} (p_{jl} Z_l + c_{jl} Z'_l), & \tilde{\nabla}_U Z_j &= (1 - b_j) Z'_j, \\ \tilde{\nabla}_{A_0} Z'_j &= a_j Z_j - \sum_{l \neq j} (p_{jl} Z'_l - c_{jl} Z_l), & \tilde{\nabla}_U Z'_j &= (b_j - 1) Z_j,\end{aligned}$$

with the rest vanishing. Hence, the components of the curvature tensor field are $\tilde{R}_{A_0 U} = -\tilde{R}_{Z_k Z'_k} = 2 \sum_j (1 - b_j)(Z'_j \otimes \gamma^j - Z_j \otimes \gamma'^j)$, and the rest zero.

If $b_j = 1$ for all $j = 1, \dots, n-1$, the holonomy algebra of $\tilde{\nabla}$ is trivial and the reductive decompositions associated to the homogeneous Kähler structures given in Proposition 3.6 are given by $\tilde{\mathfrak{g}}^{a_j, c_{kj}, p_{kj}} = \{0\} + (\mathfrak{a} + \mathfrak{n})$ with nonvanishing brackets, by (2.3), given by

$$\begin{aligned} [A_0, U] &= 2U, \quad [A_0, Z_j] = Z_j + a_j Z'_j + \sum_{l \neq j} (p_{jl} Z_l + c_{jl} Z'_l), \\ [A_0, Z'_j] &= -a_j Z_j + Z'_j + \sum_{l \neq j} (p_{jl} Z'_l + c_{jl} Z_l), \quad [Z_j, Z'_j] = -2U. \end{aligned} \quad (3.7)$$

On the other hand, the element $\hat{A}_0 = \lambda_1 C_1 + \dots + \lambda_{n-1} C_{n-1} + \sum_{j < l} (c_{jl} H_{jl} - p_{jl} F_{jl}) + A_0$ of $\mathfrak{su}(n, 1)$ generates a subspace $\mathfrak{e}^{\lambda_j, c_{kj}, p_{kj}}$ of $Z_{\mathfrak{k}}(\mathfrak{a}) + \mathfrak{a}$, and the structure Eqs. of the Lie subalgebra $\mathfrak{e}^{\lambda_j, c_{kj}, p_{kj}} + \mathfrak{n}$ of $\mathfrak{su}(n, 1)$ are

$$\begin{aligned} [\hat{A}_0, U] &= 2U, \quad [\hat{A}_0, Z_j] = Z_j + (3\lambda_j + \sum_{l \neq j} \lambda_l) Z'_j + \sum_{l \neq j} (p_{jl} Z_l + c_{jl} Z'_l), \\ [\hat{A}_0, Z'_j] &= -(3\lambda_j + \sum_{l \neq j} \lambda_l) Z_j + Z'_j + \sum_{l \neq j} (p_{jl} Z'_l + c_{jl} Z_l), \quad [Z_j, Z'_j] = -2U, \end{aligned} \quad (3.8)$$

with the rest vanishing. From (3.7) and (3.8), it follows that $\tilde{\mathfrak{g}}^{a_j, c_{kj}, p_{kj}}$ is isomorphic to $\mathfrak{e}^{\lambda_j, c_{kj}, p_{kj}} + \mathfrak{n}$.

Now, for the structure $S = S^{a_j, b_j, c_{kj}, p_{kj}}$ in Table I, suppose that $b_j \neq 1$ for some $j = 1, \dots, n-1$. Then, $\rho = \tilde{R}_{A_0 U} = -\tilde{R}_{Z_k Z'_k} = 2 \sum_j (1 - b_j)(Z'_j \otimes \gamma^j - Z_j \otimes \gamma'^j)$ generates the holonomy algebra $\tilde{\mathfrak{h}}^{a_j, b_j, c_{kj}, p_{kj}}$ of $\tilde{\nabla} = \nabla - S$, and the reductive decomposition associated to S is $\tilde{\mathfrak{g}}^{a_j, b_j, c_{kj}, p_{kj}} = \tilde{\mathfrak{h}}^{a_j, b_j, c_{kj}, p_{kj}} + (\mathfrak{a} + \mathfrak{n}) = \langle \rho, A_0, U, Z_j, Z'_j \rangle$ with structure Eqs., by (2.3), given by

$$\begin{aligned} [\rho, A_0] &= [\rho, U] = 0, \quad [\rho, Z_j] = 2(1 - b_j) Z'_j, \quad [\rho, Z'_j] = 2(b_j - 1) Z_j, \\ [A_0, U] &= \rho + 2U, \quad [A_0, Z_j] = Z_j + a_j Z'_j + \sum_{l \neq j} (p_{jl} Z_l + c_{jl} Z'_l), \\ [A_0, Z'_j] &= -a_j Z_j + Z'_j + \sum_{l \neq j} (p_{jl} Z'_l + c_{jl} Z_l), \\ [U, Z_j] &= (b_j - 1) Z'_j, \quad [U, Z'_j] = (1 - b_j) Z_j, \quad [Z_k, Z'_j] = -\delta_{kj}(\rho + 2U). \end{aligned} \quad (3.9)$$

If $\mathfrak{u} \cong \mathfrak{u}(1)$ is the subspace of $Z_{\mathfrak{k}}(\mathfrak{a})$ generated by $C = C_1 + \dots + C_{n-1}$, it is easy to see that the Lie algebra $\tilde{\mathfrak{g}}^{a_j, b_j, c_{kj}, p_{kj}}$ is isomorphic to the Lie subalgebra $\mathfrak{u} + \mathfrak{e}^{\lambda_j, c_{kj}, p_{kj}} + \mathfrak{n} = \langle C, \hat{A}_0, U, Z_j, Z'_j \rangle$ of $\mathfrak{su}(n, 1)$. We deduce

Theorem 3.7. *Let $S = S^{a_j, b_j, c_{kj}, p_{kj}}$ be the homogeneous Kähler structure on $\mathbb{CH}(n)$, $n > 2$, given by Table I, and let $\mathfrak{e}^{\lambda_j, c_{kj}, p_{kj}}$ be the subspace of $Z_{\mathfrak{k}}(\mathfrak{a}) + \mathfrak{a}$ generated by*

$$\hat{A}_0 = \sum_j \lambda_j C_j + \sum_{1 \leq j < l \leq n-1} (c_{jl} H_{jl} - p_{jl} F_{jl}) + A_0, \quad (\lambda_j = \frac{na_j - \sum_{l \neq j} a_l}{2n+2}),$$

and $\mathfrak{u} = \langle C_1 + \dots + C_{n-1} \rangle$. If $b_j = 1$ for all $j = 1, \dots, n-1$, the corresponding group of isometries is the connected subgroup $E^{\lambda_j, c_{kj}, p_{kj}} N$ of $SU(n, 1)$ whose Lie algebra is $\mathfrak{e}^{\lambda_j, c_{kj}, p_{kj}} + \mathfrak{n}$. If $b_j \neq 1$ for some $j = 1, \dots, n-1$, the corresponding group of isometries is the connected subgroup $U(1)E^{\lambda_j, c_{kj}, p_{kj}} N$ of $SU(n, 1)$ whose Lie algebra is $\mathfrak{u} + \mathfrak{e}^{\lambda_j, c_{kj}, p_{kj}} + \mathfrak{n}$.

If $S^{a,b}$ is the homogeneous Kähler structure on the complex hyperbolic plane $\mathbb{CH}(2)$ given by Table II, $\mathfrak{e}^\lambda = \langle \hat{A}_0 \rangle$, where $\hat{A}_0 = \lambda C + A_0$, ($\lambda = a/3$), and $\mathfrak{u} = \langle C \rangle$, then the corresponding group of isometries is (i) the subgroup $E^\lambda N$ of $SU(2,1)$ generated by the Lie subalgebra $\mathfrak{e}^\lambda + \mathfrak{n}$ of $\mathfrak{su}(2,1)$, if $b = 1$; (ii) the subgroup $U(1)E^\lambda N$ of $SU(2,1)$ generated by $\mathfrak{u} + \mathfrak{e}^\lambda + \mathfrak{n}$, if $b \neq 1$.

Remark 3.8. Each structure $S^{a_j, b_j, c_{kj}, p_{kj}}$, with $b_j = 1$ for all j , is also characterized by the fact that $\tilde{\nabla} = \nabla - S^{a_j, b_j, c_{kj}, p_{kj}}$ is the canonical connection for the Lie group $E^{\lambda_j, c_{kj}, p_{kj}} N$, which is the connection for which every left-invariant vector field on $E^{\lambda_j, c_{kj}, p_{kj}} N$ is parallel. Each one of these groups acts simply transitively on $\mathbb{CH}(n)$ and it provides a description of $\mathbb{CH}(n)$ as a homogeneous space. If all the parameters a_j, c_{kj}, p_{kj} are zero, then $\mathfrak{e}^{\lambda_j, c_{kj}, p_{kj}} = \mathfrak{a}$, and we get the usual description as a solvable Lie group $\mathbb{CH}(n) = AN$. In this case, the corresponding homogeneous structure is given by $S_X Y = \nabla_X Y$ for all $X, Y \in \mathfrak{a} + \mathfrak{n}$. If $b_j \neq 1$ for some $j = 1, \dots, n-1$, we get the descriptions as homogeneous space $\mathbb{CH}(n) = U(1)E^{\lambda_j, c_{kj}, p_{kj}} N/U(1)$.

3.3 Principal line bundle over $\mathbb{CH}(n)$

By (3.2), the fundamental 2-form of the Kähler structure (J, g) of $\mathbb{CH}(n) \equiv AN$ is given by $\Omega = \alpha \wedge \beta - \sum_{j=1}^{n-1} \gamma^j \wedge \gamma'^j = -\frac{1}{2} d\beta$, where $\{\alpha, \beta, \gamma^j, \gamma'^j : 1 \leq j \leq n-1\}$ is the basis of left-invariant 1-forms on AN dual to the basis $\{A_0, U, Z_j, Z'_j\}$ of $\mathfrak{a} + \mathfrak{n}$. We consider the principal line bundle $\pi: \bar{M} \rightarrow \mathbb{CH}(n)$, and identify the bundle space $\bar{M}$ with $AN \times \mathbb{R}$ and π with the projection on AN . The fundamental vector field ξ is identified with $\frac{d}{dt}$, and the 1-form $\eta = dt - \pi^* \beta$ is also regarded as a connection form on the bundle. If φ and $\bar{g}$ are given by (2.7), then $(\varphi, \xi, \eta, \bar{g})$ is a Sasakian structure on $\bar{M}$.

By (a) of Proposition 2.5, each homogeneous Kähler structure $S^{a_j, b_j, c_{kj}, p_{kj}}$ on $\mathbb{CH}(n)$ given in Theorem 3.7 defines a homogeneous Sasakian structure $\bar{S}^{a_j, b_j, c_{kj}, p_{kj}}$ on $\bar{M}$ which gives a description of $\bar{M}$ as either the connected subgroup $E^{\lambda_j, c_{kj}, p_{kj}} N \times \mathbb{R}$ of $SU(n, 1) \times \mathbb{R}$ (if $b_j = 1$ for all $j = 1, \dots, n-1$), or as the homogeneous space $(U(1)E^{\lambda_j, c_{kj}, p_{kj}} N \times \mathbb{R})/U(1)$.

On the other hand, from (b) of Proposition 2.5, it follows

Proposition 3.9. *The bundle space $\bar{M}$ of the line bundle $\pi: \bar{M} \rightarrow \mathbb{CH}(n)$ admits the family of homogeneous Sasakian structures $\{S^t : t \in \mathbb{R}\}$ given, in terms of the horizontal lifts of the generators of $\mathfrak{a} + \mathfrak{n}$ and the fundamental vector field ξ , by the following table.*

Table III	A_0^H	U^H	Z_j^H	$Z'_j{}^H$	ξ
$S_{A_0^H}^t$	0	$-\xi$	0	0	U^H
$S_{U^H}^t$	ξ	0	0	0	$-A^H$
$S_{Z_k^H}^t$	0	0	0	$\delta_{kj}\xi$	$-Z'_k{}^H$
$S_{Z'_k{}^H}^t$	0	0	$-\delta_{kj}\xi$	0	Z_k^H
S_ξ^t	tU^H	$-tA^H$	$-tZ'_j{}^H$	tZ_j^H	0

Remark 3.10. For each $p \in \bar{M}$, if $c_{12}(S^t)_p$ is the map from the tangent space $T_p(\bar{M})$ to its dual given by $c_{12}(S^t)_p(\tilde{X}) = \sum_{i=1}^{2n+1} S_{e_i e_i \tilde{X}}$, where $\{e_i\}$ is an orthonormal basis

of $T_p(\bar{M})$, then $c_{12}(S^t)_p$ vanishes for every $t \in \mathbb{R}$. According to Tricerri-Vanhecke's classification of homogeneous Riemannian structures in [26], each S^t is of type $\mathcal{T}_2 \oplus \mathcal{T}_3$. Moreover, if $t = -1$, we have $S_{\tilde{X}}\tilde{Y} + S_{\tilde{Y}}\tilde{X} = 0$, then S^{-1} is of type $\mathcal{T}_3$, which means that $\bar{M}$ is a naturally reductive Riemannian space. If $t = 2$, then each cyclic sum $\mathfrak{S}_{\tilde{X}\tilde{Y}\tilde{Z}}S_{\tilde{X}\tilde{Y}\tilde{Z}}$ vanishes, and hence $\bar{M}$ is of type $\mathcal{T}_2$, which may be also expressed by saying that $\bar{M}$ is a cotorsionless manifold (see [13]).

We will construct the reductive decomposition $\tilde{\mathfrak{g}}_t = \tilde{\mathfrak{h}}_t + \tilde{\mathfrak{m}}$ associated to each homogeneous Sasakian structure S^t , where $\tilde{\mathfrak{m}} = T_o(\bar{M})$, with $o \in \bar{M}$, is generated by $\tilde{A} = (A_0^H)_o$, $\tilde{U} = (U^H)_o$, $\tilde{Z}_j = (Z_j^H)_o$, $\tilde{Z}'_j = (Z'_j)^H_o$, $\tilde{\xi} = \xi_o$, ($1 \leq j \leq n-1$), and $\tilde{\mathfrak{h}}_t$ is the holonomy algebra of the connection $\tilde{D}^t = D - S^t$. Each connection $\tilde{D}^t$ is given by

Table IV	A_0^H	U^H	Z_j^H	$Z'_j{}^H$	ξ
$\tilde{D}_{A_0^H}^t$	0	0	0	0	0
$\tilde{D}_{U^H}^t$	$-2U^H$	$2A_0^H$	$Z_j'^H$	$-Z_j^H$	0
$\tilde{D}_{Z_k^H}^t$	$-Z_k^H$	$Z_k'^H$	$\delta_{kj}A_0^H$	$-\delta_{kj}U^H$	0
$\tilde{D}_{Z'_k{}^H}^t$	$-Z'_k{}^H$	$-Z_k^H$	$\delta_{kj}U^H$	$\delta_{kj}A_0^H$	0
$\tilde{D}_{\xi}^t$	$(1-t)U^H$	$(t-1)A^H$	$(t-1)Z_j'^H$	$(1-t)Z_j^H$	0

Let $\tilde{R}^t$ be the curvature of $\tilde{D}^t$, and let $\{\bar{\alpha}, \bar{\beta}, \bar{\gamma}^j, \bar{\gamma}'^j, \bar{\eta}\}$ be the basis dual to the basis $\{\tilde{A}, \tilde{U}, \tilde{Z}_j, \tilde{Z}'_j, \tilde{\xi}\}$ of $\tilde{\mathfrak{m}}$. The holonomy algebra $\tilde{\mathfrak{h}}_t$ of $\tilde{D}^t$ is generated by the curvature operators $\rho_0, \rho_r, \varphi_r, \psi_r, \sigma_{jk}, \tau_{jk}$ ($r, j, k = 1, \dots, n-1, j < k$), given by

$$\begin{aligned}
\rho_0 &= \tilde{R}_{\tilde{A}\tilde{U}}^t = 2(t-3)(\bar{\alpha} \otimes \tilde{U} - \bar{\beta} \otimes \tilde{A}) + 2(2-t) \sum_{j=1}^{n-1} (\bar{\gamma}^j \otimes \tilde{Z}'_j - \bar{\gamma}'^j \otimes \tilde{Z}_j), \\
\rho_r &= \tilde{R}_{\tilde{Z}_r \tilde{Z}'_r}^t = 2(2-t)(\bar{\alpha} \otimes \tilde{U} - \bar{\beta} \otimes \tilde{A}) + 2(t-3)(\bar{\gamma}^r \otimes \tilde{Z}'_r - \bar{\gamma}'^r \otimes \tilde{Z}_r) \\
&\quad + 2(t-2) \sum_{j \neq r} (\bar{\gamma}^j \otimes \tilde{Z}'_j - \bar{\gamma}'^j \otimes \tilde{Z}_j), \\
\varphi_r &= \tilde{R}_{\tilde{A}\tilde{Z}_r}^t = -\tilde{R}_{\tilde{U}\tilde{Z}'_r}^t = -\bar{\alpha} \otimes \tilde{Z}_r + \bar{\beta} \otimes \tilde{Z}'_r + \bar{\gamma}^r \otimes \tilde{A} - \bar{\gamma}'^r \otimes \tilde{U}, \\
\psi_r &= \tilde{R}_{\tilde{U}\tilde{Z}_r}^t = \tilde{R}_{\tilde{A}\tilde{Z}'_r}^t = -\bar{\alpha} \otimes \tilde{Z}'_r - \bar{\beta} \otimes \tilde{Z}_r + \bar{\gamma}^r \otimes \tilde{U} + \bar{\gamma}'^r \otimes \tilde{A}, \\
\sigma_{jk} &= \tilde{R}_{\tilde{Z}_j \tilde{Z}_k}^t = \tilde{R}_{\tilde{Z}'_j \tilde{Z}'_k}^t = -\bar{\gamma}^j \otimes \tilde{Z}_k - \bar{\gamma}'^j \otimes \tilde{Z}'_k + \bar{\gamma}^k \otimes \tilde{Z}_j + \bar{\gamma}'^k \otimes \tilde{Z}'_j, \\
\tau_{jk} &= \tilde{R}_{\tilde{Z}_j \tilde{Z}'_k}^t = \tilde{R}_{\tilde{Z}'_j \tilde{Z}_k}^t = -\bar{\gamma}^j \otimes \tilde{Z}'_k + \bar{\gamma}'^j \otimes \tilde{Z}_k - \bar{\gamma}^k \otimes \tilde{Z}'_j + \bar{\gamma}'^k \otimes \tilde{Z}_j.
\end{aligned}$$

(If $n = 2$, the operators σ_{jk} and τ_{jk} do not appear, that is $\tilde{\mathfrak{h}}_t = \langle \rho_0, \rho_1, \varphi_1, \psi_1 \rangle$, and if $n = 1$, $\tilde{\mathfrak{h}}_t$ is generated by $\rho_0 = \tilde{R}_{\tilde{A}\tilde{U}}^t = 2(t-3)(\bar{\alpha} \otimes \tilde{U} - \bar{\beta} \otimes \tilde{A})$.) The Lie structure of $\tilde{\mathfrak{g}}_t = \tilde{\mathfrak{h}}_t + \tilde{\mathfrak{m}}$ is defined by the Eqs. (2.3). If $t \neq (2n+1)/n$, the subalgebra $\tilde{\mathfrak{h}}_t$ is isomorphic to the Lie algebra $\mathfrak{k} = \mathfrak{s}(\mathfrak{u}(n) + \mathfrak{u}(1)) \cong \mathfrak{u}(n)$ in § 3.1, via the map $h: \tilde{\mathfrak{h}}_t \rightarrow \mathfrak{k}$ given by $h(\rho_0) = 2U_{\mathfrak{k}}$, $h(\rho_r) = -(C_r + U_{\mathfrak{k}})$, $h(\varphi_r) = (Z_r)_{\mathfrak{k}}$, $h(\psi_r) = (Z'_r)_{\mathfrak{k}}$, $h(\sigma_{jk}) = F_{jk}$, $h(\tau_{jk}) = -H_{jk}$. If we put $\hat{\rho}_0 = \frac{1}{2}(\rho_0 - 2\tilde{\xi})$, $\hat{\rho}_r = -\frac{1}{2}\rho_0 - \rho_r - \tilde{\xi}$, then $\hat{\mathfrak{su}}(n, 1) = \langle \hat{\rho}_0, \hat{\rho}_r, \varphi_r, \psi_r, \sigma_{jk}, \tau_{jk}, \tilde{A}, \tilde{U}, \tilde{Z}_r, \tilde{Z}'_r : r, j, k = 1, \dots, n-1, j < k \rangle$ is an ideal of $\tilde{\mathfrak{g}}_t$, and the map h extends to a Lie algebra isomorphism $\tilde{h}: \hat{\mathfrak{su}}(n, 1) \rightarrow \mathfrak{su}(n, 1) = \mathfrak{k} + \mathfrak{p}$, given by $\tilde{h}(\hat{\rho}_0) = U_{\mathfrak{k}}$, $\tilde{h}(\hat{\rho}_r) = C_r$, $\tilde{h}(\varphi_r) = (Z_r)_{\mathfrak{k}}$, $\tilde{h}(\psi_r) = (Z'_r)_{\mathfrak{k}}$, $\tilde{h}(\sigma_{jk}) = F_{jk}$, $\tilde{h}(\tau_{jk}) = -H_{jk}$,

$\tilde{h}(\tilde{A}) = A_0$, $\tilde{h}(\tilde{U}) = U_{\mathfrak{p}}$, $\tilde{h}(\tilde{Z}_r) = (Z_r)_{\mathfrak{p}}$, $\tilde{h}(\tilde{Z}'_r) = (Z'_r)_{\mathfrak{p}}$. Moreover, $\tilde{\mathfrak{g}}_t$ is the semidirect product of $\widehat{\mathfrak{su}}(n, 1)$ and the line generated by $\tilde{\xi}$ under the homomorphism $\delta_t: \langle \tilde{\xi} \rangle \rightarrow \text{Der}(\widehat{\mathfrak{su}}(n, 1))$, given by $\delta_t(\tilde{\xi})(\tilde{A}) = (t-1)\tilde{U}$, $\delta_t(\tilde{\xi})(\tilde{U}) = (1-t)\tilde{A}$, $\delta_t(\tilde{\xi})(\tilde{Z}_r) = (1-t)\tilde{Z}'_r$, $\delta_t(\tilde{\xi})(\tilde{Z}'_r) = (t-1)\tilde{Z}_r$, and $\delta_t(\tilde{\xi})(\langle \hat{\rho}_0, \hat{\rho}_r, \varphi_r, \psi_r, \sigma_{jk}, \tau_{jk} \rangle) = 0$. So, we have

Proposition 3.11. *The reductive decomposition associated to the homogeneous Sasakian structure S^t , $t \neq (2n+1)/n$, on the total space of the line bundle $\bar{M} \rightarrow \mathbb{CH}(n)$ is $\tilde{\mathfrak{g}}_t = \tilde{\mathfrak{h}}_t + \tilde{\mathfrak{m}}$, where $\tilde{\mathfrak{h}}_t \cong \mathfrak{s}(\mathfrak{u}(n) + \mathfrak{u}(1)) \cong \mathfrak{u}(n) \subset \mathfrak{su}(n, 1)$, and $\tilde{\mathfrak{m}} = \mathfrak{p} + \langle \tilde{\xi} \rangle = \langle A_0, U_{\mathfrak{p}}, (Z_r)_{\mathfrak{p}}, (Z'_r)_{\mathfrak{p}}, \tilde{\xi} : 1 \leq r \leq n-1 \rangle$. Moreover, $\tilde{\mathfrak{g}}_t$ is the semidirect product $\tilde{\mathfrak{g}}_t = \langle \tilde{\xi} \rangle \ltimes_{\delta_t} \mathfrak{su}(n, 1)$, where $\delta_t(\tilde{\xi})(A_0) = (t-1)U_{\mathfrak{p}}$, $\delta_t(\tilde{\xi})(U_{\mathfrak{p}}) = (1-t)A_0$, $\delta_t(\tilde{\xi})((Z_r)_{\mathfrak{p}}) = (1-t)(Z'_r)_{\mathfrak{p}}$, $\delta_t(\tilde{\xi})((Z'_r)_{\mathfrak{p}}) = (t-1)(Z_r)_{\mathfrak{p}}$, and $\delta_t(\tilde{\xi})(\tilde{\mathfrak{h}}_t) = 0$.*

If $n \geq 2$ and $t = (2n+1)/n$, then it is easy to see that $\rho_0 = \rho_1 + \dots + \rho_{n-1}$, and we put $\tilde{\rho}_r = \frac{1}{2}(\rho_0 + \rho_r)$, $1 \leq r \leq n-1$. In this case, $\tilde{\mathfrak{g}}_{\frac{2n+1}{n}} = \tilde{\mathfrak{h}}_{\frac{2n+1}{n}} + \tilde{\mathfrak{m}}$ coincides with the reductive decomposition $\mathfrak{su}(n, 1) = \mathfrak{k}' + \mathfrak{m}'$, where $\mathfrak{k}' = [\mathfrak{k}, \mathfrak{k}] \cong \mathfrak{su}(n)$, and $\mathfrak{m}' = \mathfrak{p} + \langle \mathfrak{c} \rangle$, being $\mathfrak{c}$ the center of $\mathfrak{k}$, which is generated by the element E_J such that $\text{ad}_{E_J}: \mathfrak{p} \rightarrow \mathfrak{p}$ defines the complex structure of $\mathbb{CH}(n)$. In fact, we have the isomorphism $f: \tilde{\mathfrak{g}}_{\frac{2n+1}{n}} \rightarrow \mathfrak{su}(n, 1)$ given by $f(\tilde{\rho}_r) = \frac{1}{2}(U_{\mathfrak{k}} - C_r)$, $f(\varphi_r) = (Z_r)_{\mathfrak{k}}$, $f(\psi_r) = (Z'_r)_{\mathfrak{k}}$, $f(\sigma_{jk}) = F_{jk}$, $f(\tau_{jk}) = -H_{jk}$, $f(\tilde{A}) = A_0$, $f(\tilde{U}) = U_{\mathfrak{p}}$, $f(\tilde{Z}_r) = (Z_r)_{\mathfrak{p}}$, $f(\tilde{Z}'_r) = (Z'_r)_{\mathfrak{p}}$, and $f(\tilde{\xi}) = -\frac{n+1}{n}E_J = -\frac{1}{2n}(C_1 + \dots + C_{n-1} + (n+1)U_{\mathfrak{k}})$, and, in particular, $f(\tilde{\mathfrak{h}}_{\frac{2n+1}{n}}) = \mathfrak{k}'$ and $f(\tilde{\mathfrak{m}}) = \mathfrak{m}'$. If $n = 1$ and $t = 3$, then $\rho_0 = 0$. In this case, $\tilde{\mathfrak{h}}_3 = 0$, $\mathfrak{k}' = [\mathfrak{k}, \mathfrak{k}] = 0$, $\mathfrak{c} = \langle E_J \rangle$, $E_J = \frac{1}{2}U_{\mathfrak{k}}$, $\tilde{\mathfrak{g}}_3 = \{0\} + \tilde{\mathfrak{m}}$ is the reductive decomposition $\mathfrak{su}(1, 1) = \{0\} + \mathfrak{m}'$, where $\tilde{\mathfrak{m}} = \langle \tilde{A}, \tilde{U}, \tilde{\xi} \rangle$, $\mathfrak{m}' = \langle A_0, U_{\mathfrak{p}}, U_{\mathfrak{k}} \rangle$, and $f: \tilde{\mathfrak{g}}_3 \rightarrow \mathfrak{su}(1, 1)$ such that $f(\tilde{A}) = A_0$, $f(\tilde{U}) = U_{\mathfrak{p}}$, $f(\tilde{\xi}) = -U_{\mathfrak{k}}$. Hence, we have obtained

Proposition 3.12. *The reductive decomposition associated to the homogeneous Sasakian structure S^t , with $t = (2n+1)/n$, on the total space of the line bundle $\bar{M} \rightarrow \mathbb{CH}(n)$ is $\mathfrak{su}(n, 1) = \mathfrak{k}' + \mathfrak{m}'$, where $\mathfrak{k}' = [\mathfrak{k}, \mathfrak{k}] \cong \mathfrak{su}(n)$, and $\mathfrak{m}' = \mathfrak{p} + \mathfrak{c}$, $\mathfrak{c} = \langle E_J \rangle$ being the center of $\mathfrak{k}$.*

Remark 3.13. The reductive decomposition $\mathfrak{su}(n, 1) = \mathfrak{k}' + \mathfrak{m}'$ associated to the homogeneous Sasakian structure S^t , with $t = \frac{2n+1}{n}$, provides the description of $\bar{M}$ as the homogeneous space $\widetilde{SU}(n, 1)/K'$, where $\widetilde{SU}(n, 1)$ is the universal covering of $SU(n, 1)$, and $K' \cong SU(n)$ is the connected subgroup of $\widetilde{SU}(n, 1)$ whose Lie algebra is $\mathfrak{k}' \cong \mathfrak{su}(n)$. (In particular, if $n = 1$, $\bar{M}$ is the universal covering space of $Sl(2, \mathbb{R})$.) These spaces appear in the classification by Jiménez and Kowalski [16] of complete simply connected φ -symmetric Sasakian manifolds, and they are also Sasakian space forms (they have constant φ -sectional curvature -7). Notice that for a Sasakian manifold, the condition of being a locally symmetric space is too strong, because in this case it is a space of constant curvature (Okumura [23]). For this reason, Takahashi [25] introduced φ -symmetric spaces in Sasakian geometry as generalizations of Sasakian space forms. They are also analogues of Hermitian symmetric spaces. A φ -symmetric space is a complete connected regular Sasakian manifold $\bar{M}$ that fibers over a Hermitian symmetric space M so that the geodesic involutions of M lift to involutive automorphisms of the Sasakian structure on $\bar{M}$. Moreover, each complete simply connected φ -symmetric space is a naturally reductive homogeneous space (Blair and Vanhecke [5]).

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References

- [1] E. ABBENA AND S. GARBIERO, Almost-Hermitian homogeneous structures, *Proc. Edinb. Math. Soc. (2)* **31** (1988) 375–395.
- [2] W. AMBROSE AND I. M. SINGER, On homogeneous Riemannian manifolds, *Duke Math. J.* **25** (1958) 647–669.
- [3] D. E. BLAIR, *Contact manifolds in Riemannian geometry* (Springer, Berlin, 1976).
- [4] D. E. BLAIR, *Riemannian geometry of contact and symplectic manifolds* (Birkhäuser, Boston, 2002).
- [5] D. E. BLAIR AND L. VANHECKE, New characterizations of φ -symmetric spaces, *Kōdai Math. J.* **10** (1987) 102–107.
- [6] W. M. BOOTHBY AND H. C. WANG, On contact manifolds, *Ann. of Math. (2)* **68** (1958) 721–734.
- [7] M. CASTRILLÓN LÓPEZ, P. M. GADEA AND A. F. SWANN, Homogeneous structures on real and complex hyperbolic spaces, *Illinois J. Math.*, to appear.
- [8] D. CHINEA AND C. GONZÁLEZ, An example of an almost cosymplectic homogeneous manifold, in *Differential geometry, Peñíscola, 1985, Lecture Notes in Mathematics 1209* (Springer, Berlin, 1986), pp. 133–142.
- [9] A. DÍAZ MIRANDA AND A. REVENTÓS, Homogeneous contact compact manifolds and homogeneous symplectic manifolds, *Bull. Sci. Math. (2)* **106** (1982), no. 4, 337–350.
- [10] J. DORFMEISTER AND K. NAKAJIMA, The fundamental conjecture for homogeneous Kähler manifolds, *Acta Math.* **161** (1988), no. 1–2, 23–70.
- [11] A. FINO, Almost contact homogeneous structures, *Boll. Un. Mat. Ital. (7)* **9-A** (1995) 299–311.
- [12] P. M. GADEA, A. MONTESINOS AMILIBIA AND J. MUÑOZ MASQUÉ, Characterizing the complex hyperbolic space by Kähler homogeneous structures, *Math. Proc. Cambridge Phil. Soc.* **27** (2000) 87–94.
- [13] P. M. GADEA AND J. A. OUBIÑA, Reductive homogeneous pseudo-Riemannian manifolds, *Monatsh. Math.* **124** (1997) 17–34.
- [14] C. GONZÁLEZ AND D. CHINEA, Quasi-Sasakian homogeneous structures on the generalized Heisenberg group $H(p, 1)$, *Proc. Amer. Math. Soc.* **105** (1989) 173–184.

- [15] S. HELGASON, *Differential geometry, Lie groups, and symmetric spaces* (Academic Press, New York, 1978).
- [16] J. JIMÉNEZ AND O. KOWALSKI, The classification of φ -symmetric Sasakian manifolds, *Monatsh. Math.* **115** (1993) 83–98.
- [17] V. F. KIRIČENKO, On homogeneous Riemannian spaces with an invariant structure tensor, *Soviet Math. Dokl.* **21** (1980) 734–737.
- [18] S. KOBAYASHI AND K. NOMIZU, *Foundations of differential geometry. Vol. I, II* (Wiley, New York, 1963, 1969).
- [19] T. KODA AND Y. WATANABE, Homogeneous almost contact Riemannian manifolds and infinitesimal models, *Boll. Un. Mat. Ital. B (7)* **11-B** (1997), Suppl., fasc. 2, 11–24.
- [20] K. NOMIZU, Invariant affine connections on homogeneous spaces, *Amer. J. Math.* **76** (1954) 33–65.
- [21] H. NISHINO, Alternative $N = (4, 0)$ superstring and σ -models, *Phys. Lett.* **B355** (1995) 117–126.
- [22] K. OGIUE, On fiberings of almost contact manifolds, *Kōdai Math. Sem. Rep.* **17** (1965) 53–62.
- [23] M. OKUMURA, Some remarks on spaces with a certain contact structure, *Tôhoku Math. J. (2)* **14** (1962) 135–145.
- [24] K. SEKIGAWA, Notes on homogeneous almost Hermitian manifolds, *Hokkaido Math. J.* **7** (1978) 206–213.
- [25] T. TAKAHASHI, Sasakian φ -symmetric spaces, *Tôhoku Math. J. (2)* **29** (1997) 91–113.
- [26] F. TRICERRI AND L. VANHECKE, *Homogeneous structures on Riemannian Manifolds* (Cambridge Univ. Press, Cambridge, 1983).
- [27] B. DE WIT AND P. VAN NIEUWENHUIZEN, Rigidly and locally supersymmetric two-dimensional nonlinear σ -models with torsion, *Nucl. Phys.* **B312** (1989) 58–94.