

Canonical Realizations of Doubly Special Relativity

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Doubly Special Relativity is usually formulated in momentum space, providing the explicit nonlinear action of the Lorentz transformations that incorporates the deformation of boosts. Various proposals have appeared in the literature for the associated realization in position space. While some are based on noncommutative geometries, others respect the compatibility of the spacetime coordinates. Among the latter, there exist several proposals that invoke in different ways the completion of the Lorentz transformations into canonical ones in phase space. In this paper, the relationship between all these canonical proposals is clarified, showing that in fact they are equivalent. The generalized uncertainty principles emerging from these canonical realizations are also discussed in detail, studying the possibility of reaching regimes where the behavior of suitable position and momentum variables is classical, and explaining how one can reconstruct a canonical realization of doubly special relativity starting just from a basic set of commutators. In addition, the extension to general relativity is considered, investigating the kind of gravity's rainbow that arises from this canonical realization and comparing it with the gravity's rainbow formalism put forward by Magueijo and Smolin, which was obtained from a commutative but noncanonical realization in position space.

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1. INTRODUCTION

Most approaches to quantum gravity support the existence of a fundamental scale (expected to be of Planck order) that would act as a threshold for the onset of quantum effects, so that beyond it the classical picture of spacetime should be replaced by a genuine quantum description [1, 2]. For instance, in loop quantum gravity the classical smooth geometry of the spacetime is replaced by a discrete quantum geometry at small scales, with a minimal nonzero eigenvalue for the area operator that is indeed of the Planck order [3, 4, 5]. Similarly, different analyses of scattering processes in string theory suggest the existence of a minimal length [6, 7, 8, 9]. The possible emergence of this fundamental scale seems to pose a problem if Lorentz symmetry has to be approximately valid (at least in some asymptotic regions) in the effective theory that describes the flat spacetime limit of quantum gravity. Since, in general, lengths are not invariant under Lorentz transformations, different inertial observers would disagree on the value of the scale at which the quantum discreteness of spacetime begins to manifest itself. In this sense, the existence of a fundamental scale would contradict the relativity principle of inertial frames.

This apparent inconsistency may nonetheless be overcome by modifying special relativity so as to preserve an additional scale, apart from the one provided by the speed of light [10, 11, 12, 13, 14, 15, 16]. Since the action of the Lorentz transformations must now leave invariant two dimensionful quantities, the family of theories that incorporate this suggestion is generically called Doubly Special Relativity (DSR) [10]. DSR theories are usually formulated in momentum space because they lead to modified dispersion relations [17], which imply interesting observational consequences [10, 18, 19, 20, 21, 22, 23]. The new scale is an energy and/or momentum scale, and its invariance is possible only because the action of the Lorentz group becomes nonlinear in momentum space.

In order to determine the corresponding action of the Lorentz group in position space, various proposals have been discussed in the literature [24, 25, 26, 27, 28]. A class of proposals is based on the use of noncommutative geometries. Namely, they introduce spacetime coordinates that are not mutually compatible in the quantum theory, as it happens e.g. in the case of κ -deformed Minkowski spacetime [16, 24]. Nevertheless, it is perfectly feasible to implement DSR theories in position space while keeping the

commutativity of the spacetime geometry [17, 25, 26, 27, 28, 29, 30, 31]. For example, Kimberly, Magueijo and Medeiros [25] proposed a realization that preserves the Lorentz invariance of the contraction between energy-momentum and position. This requirement ensures that free field theories based upon DSR admit planewave solutions [25]. Such a position space realization was later reformulated and generalized to the framework of general relativity by Magueijo and Smolin, including the effect of curvature [29]. The resulting formalism describes geometries that depend explicitly on the energy-momentum of the particle that probes them (in the language of the renormalization group, geometry “runs” [29]). For this reason, this proposal is called gravity’s rainbow.

There exists another class of proposals that, while maintaining the use of a commutative geometry for the description of the spacetime, are based (in a way or another) on a canonical implementation of the Lorentz transformations rather than on the planewaves requirement of Ref. [25]. Probably, the first of these proposals was made by Mignemi [26], who demanded covariance on phase space when studying the transformation law for commuting coordinates in DSR. Nonetheless, instead of pursuing this alternative, he finally focused his study on noncommutative geometries. Another proposal for a canonical implementation appeared in a work by the authors [27]. This proposal employed (the existence of) a nonlinear map that transforms the physical energy-momentum into an auxiliary one on which the Lorentz action becomes linear [32]. The realization of DSR in position space was deduced by completing this map into a canonical transformation, so that the symplectic form on phase space remains invariant [27]. Similarly, Hinterleitner [28] proposed a prescription for DSR in position space such that the complete Lorentz transformation on phase space turned out to be canonical. On the other hand, Cortés and Gamboa [30] analyzed the modification of quantum commutators in DSR under the assumption that one may adopt commuting spacetime coordinates corresponding to the generators of translations in the auxiliary energy-momentum, so that they form altogether a canonical set.

Although all these proposals were put forward independently, it is clear that they are interrelated. For instance, Hossenfelder pointed out recently [17] that it is possible to use a generalized uncertainty principle to find the nonlinear map between the physical and the auxiliary energy-momentum. This map, together with the results of Ref. [32], suffices to fix the DSR theory. The main aim of the present work is to discuss in detail

the connection between the different canonical realizations suggested for DSR. In spite of the confusion that seems to exist in the literature, we will prove that they are actually equivalent. Moreover, extending Hossenfelder's analysis, we will study in detail the generalized uncertainty principles associated with this kind of realizations. In particular, we will provide an explicit construction of the bijective map between DSR theories and sets of modified commutators between physical (or auxiliary) energy-momentum and auxiliary (or respectively physical) position variables. We will also investigate the possibility that there exist limiting regimes in which these mixed (auxiliary and physical) elementary phase space variables reach a classical behavior. Finally, following exactly the line of reasoning defended by Magueijo and Smolin [29] to extend the DSR realization in position space to general relativity, we will discuss the gravity's rainbow formalism that arises from a canonical realization. We will not consider here other more tentative suggestions for the extension of DSR to general relativity, like e.g. that explored in Ref. [28].

The organization of the rest of the paper is as follows. In Sec. II we briefly review some basic aspects of DSR theories formulated in momentum space. We prove in Sec. III that the different canonical realizations of DSR in position space that have appeared in the literature are in fact equivalent. Section IV deals with the modified quantum commutators that correspond to canonical realizations of DSR. In Sec. V we summarize the proposal for a gravity's rainbow and compare the formalism put forward by Magueijo and Smolin with the corresponding counterpart obtained from a canonical realization of DSR. As an important example, we consider in Sec. VI the Schwarzschild solution to the modified Einstein equations obtained in these two types of gravity's rainbow. Finally, Sec. VII contains the conclusions.

2. DSR IN MOMENTUM SPACE

In DSR, the nonlinear action of the Lorentz transformations in momentum space can be easily captured in a nonlinear invertible map U between the physical energy-momentum $P_a := (-E, p_i)$ ^[a] and an auxiliary energy-momentum that transforms like the usual momentum variables in standard special relativity, $\Pi_a := (-\epsilon, \Pi_i)$ [32]. Representing the

[a] Lowercase Latin letters from the beginning and the middle of the alphabet denote Lorentz and flat spatial indices, respectively.

conventional linear action of the Lorentz group by \mathcal{L} , the nonlinear Lorentz transformations are given by [15, 32]

$$L_a(P_b) = [U^{-1} \circ \mathcal{L} \circ U]_a(P_b), \quad (2.1)$$

where the symbol \circ denotes composition. Demanding that the standard action of rotations is not modified, the map U is totally determined by two scalar functions g and f [15, 24]. Using a notation similar to that of Refs. [27, 31], U can be expressed ^[b]

$$P_a = U_a^{-1}(\Pi_b) \Rightarrow \begin{cases} E = g(\epsilon, \Pi), \\ p_i = f(\epsilon, \Pi) \frac{\Pi_i}{\Pi}. \end{cases} \quad (2.2)$$

Here, Π is the (Euclidean) norm of the auxiliary momentum Π_i . Each admissible choice of functions f and g leads to a different DSR theory. We remember that the map U^{-1} must be invertible in its range. In addition, in order to recover standard special relativity, U^{-1} must reduce to the identity (i.e., $g \approx \epsilon$, $f \approx \Pi$) in the region of negligible energies and momenta compared to the scale of the DSR theory. This scale corresponds to the limit of the functions f and/or g when the auxiliary energy-momentum reaches infinity (on the mass shell $\epsilon^2 - \Pi^2 = \mu^2$, where μ is the Casimir invariant of the auxiliary momentum space). So, at least one of the two considered functions must have a finite limit.

3. CANONICAL IMPLEMENTATION OF DSR IN POSITION SPACE

For the realization of DSR in position space we will focus on a family of proposals that, in the context of commutative geometry, are based on a canonical implementation of DSR [17, 26, 27, 28, 30, 31].

Let us start by considering the proposal introduced by Mignemi in Ref. [26]. Mignemi investigated the Hamiltonian formalism for a free particle with physical energy-momentum P_a and associated commuting spacetime coordinates x^a . From Eq. (2.1), the nonlinear action of the Lorentz transformations in momentum space is

$$P_a \rightarrow P'_a = L_a(P_b). \quad (3.1)$$

[b] Our definition of the functions f and g coincides with that introduced in Refs. [27, 31], but differs in general from other conventions found in the literature. Although our convention simplifies the calculations, when comparing our results with those of other works it is important to take into account this possible discrepancy.

Here, we have conveniently adapted all expressions to our notation ^[c]. Mignemi identifies then the position space as the cotangent one to the momentum space. He also introduces implicitly the demand that the origin of the spacetime coordinates remain invariant under all Lorentz transformations. Appealing to covariance, he deduces the following transformation of coordinates [26]:

$$x^a \rightarrow x'^a = \left(\frac{\partial P'_a}{\partial P_b} \right)^{-1} x^b := \bar{L}^a(x^b, P_c). \quad (3.2)$$

Note that, from Eq. (3.1), $\partial P'_a / \partial P_b = \partial L_a / \partial P_b$. The above transformation in position space is linear in the coordinates x^a , but depends nontrivially on the physical energy-momentum.

Soon after Mignemi's proposal, Hinterleitner [28] suggested independently that the realization of DSR in position space can be determined by completing the Lorentz transformations into canonical ones on phase space. Although he restricted his analysis to two-dimensional spacetimes for simplicity, we will reproduce his arguments in four dimensions. He considered canonical transformations between phase space coordinates in two different inertial frames, (x^a, P_a) and (x'^a, P'_a) , imposing that their restriction to momentum space coincided with the corresponding Lorentz transformations in DSR. The canonical transformation is then totally fixed if one requires that its action on the spacetime coordinates be homogeneous (i.e. the origin be left invariant). Calling $P_a = P_a(P'_b)$, it is straightforward to conclude that $x'^a = (\partial P_b / \partial P'_a) x^b$, so that (employing the inverse function theorem) one arrives precisely at transformation (3.2). Thereby, we see that Mignemi's and Hinterleitner's proposals are in fact the same. Let us note that, in spite of the fact that the two proposals have appeared separately in the literature, their equivalence is actually obvious once the condition of covariance is understood as the invariance of the symplectic form $\mathbf{d}x^a \wedge \mathbf{d}P_a$, which in turn implies that the considered transformations must be canonical.

On the other hand, in Ref. [27] we proposed to find the realization of DSR in position space by demanding that the extension of the nonlinear map (2.2) to phase space

[c] Mignemi denotes the action of the deformed boosts by W , rather than using the symbol L for all Lorentz transformations in DSR. In addition, he calls p_a the physical energy-momentum and q^a the spacetime coordinates conjugate to it, whereas we denote these physical variables by P_a and x^a , respectively [27, 31]. We reserve the notation q^a for the auxiliary spacetime coordinates conjugate to the auxiliary energy-momentum Π_a .

preserve the form $\mathbf{d}q^a \wedge \mathbf{d}\Pi_a$. Remember that q^a are the auxiliary spacetime coordinates, canonically conjugate to Π_a [27, 31]. The above requirement assigns to the system new, modified spacetime coordinates x^a that are conjugate to the physical energy-momentum P_a , so that the relation between (x^a, P_a) and (q^a, Π_a) is a canonical transformation. Demanding that the origins of the two different sets of spacetime coordinates coincide, one then gets

$$q^a = \frac{\partial P_b}{\partial \Pi_a} x^b. \quad (3.3)$$

With this relationship between auxiliary and physical coordinates one can fully determine the nonlinear action of the Lorentz group both in position and momentum spaces. It is given by

$$\tilde{L}(x^a, P_a) = [\tilde{U}^{-1} \circ \tilde{\mathcal{L}} \circ \tilde{U}](x^a, P_a). \quad (3.4)$$

Here, $\tilde{\mathcal{L}}$ represents the standard linear action of the Lorentz group on phase space, while \tilde{U} is the canonical map between physical and auxiliary variables determined by U and Eq. (3.3).

Since \tilde{U} is a canonical transformation, so is its inverse, \tilde{U}^{-1} . Moreover, the standard Lorentz action $\tilde{\mathcal{L}}$ corresponds as well to a canonical transformation, which maps auxiliary variables from an inertial frame to another one. Hence the composition $\tilde{L} = \tilde{U}^{-1} \circ \tilde{\mathcal{L}} \circ \tilde{U}$ provides a canonical transformation. In addition, its restriction to momentum space reproduces Eq. (2.1). Thus, \tilde{L} can be regarded as the canonical extension to phase space of the DSR transformation of the energy-momentum. Besides, \tilde{L} leaves the origin of the spacetime coordinates invariant (because so do \tilde{U} and $\tilde{\mathcal{L}}$). Therefore, our proposal leads exactly to the same Lorentz transformation on physical phase space that was put forward by Hinterleitner. In this way, our canonical realization of DSR turns out to be equivalent to those of Mignemi and Hinterleitner.

The main distinction between our proposal and the ones suggested by Mignemi and Hinterleitner is that the latter introduce directly the canonical transformation between physical variables on phase space as measured by two different observers, related by means of a nonlinear Lorentz transformation, whereas our realization of DSR relies on the canonical map between auxiliary and physical variables. In other words, Mignemi and Hinterleitner provide the nonlinear map \tilde{L} appearing in Eq. (3.4), while our proposal gives the map \tilde{U} .

Still in the context of commutative geometry, Cortés and Gamboa [30] analyzed the modification of the quantum commutation relations in the framework of DSR. They considered spacetime coordinates interpretable as the generators of translations in the auxiliary energy and momentum variables $(-\epsilon, \Pi_i)$. These spacetime coordinates are therefore canonically conjugate to the auxiliary energy-momentum and, according to our notation (which differs from that of Ref. [30]), they can be identified with q^a . Similarly, one can consider spacetime coordinates that generate the translations in the physical energy-momentum $(-E, p_i)$. These are what we have called the physical spacetime coordinates x^a . The relation between (q^a, Π_a) and (x^a, P_a) is obviously a canonical transformation. In particular, one can then straightforwardly assign commutators to the complete set of mixed (i.e. half auxiliary and half physical) phase space variables (q^a, P_a) by multiplying their Poisson brackets just by an imaginary factor i (with the Planck constant \hbar set equal to the unity). In this way, one recovers the result of Cortés and Gamboa, i.e.

$$[\hat{q}^a, \hat{P}_b] = i \frac{\widehat{\partial P_b}}{\partial \Pi_a}, \quad (3.5)$$

the rest of commutators vanishing. Finally, remember that $\partial P_b / \partial \Pi_a = \partial U_b^{-1} / \partial \Pi_a$ from Eq. (2.2).

In Ref. [17] Hossenfelder adopts a very similar viewpoint. The wave vector k_a introduced in that work is what we have generically referred to as physical energy-momentum, P_a , and is canonically conjugate to the physical coordinates x^a . This wave vector is related to the auxiliary energy-momentum via the nonlinear map U . The commutators considered by Hossenfelder are those corresponding to the mixed phase space variables (x^a, Π_a) . Regarded as fundamental commutators, obtained directly from the Poisson brackets, they are then given by

$$[\hat{x}^a, \hat{\Pi}_b] = i \frac{\widehat{\partial \Pi_b}}{\partial P_a}, \quad (3.6)$$

with $\partial \Pi_b / \partial P_a = \partial U_b / \partial P_a$.

In conclusion, although the literature contains several proposals for the canonical realization of DSR in phase space, we have seen that they are indeed equivalent, leading to the same transformations of the spacetime coordinates. We expect that our discussion helps to clarify this issue definitively.

4. MODIFIED COMMUTATION RELATIONS

At the end of the previous section we have shown that a canonical realization of DSR yields modified commutation relations if one chooses as elementary phase space variables any of the two mixed sets (q^a, P_a) or (x^a, Π_a) (formed by both auxiliary and physical variables). We want to discuss now the different behavior of the generalized uncertainty principles obtained with these two sets. We will also provide a specific recipe to obtain the map U that determines the DSR theory starting from the modified basic commutators.

A. Fundamental commutators

We first regard as elementary variables the set formed by the auxiliary spacetime coordinates and the physical energy-momentum, (q^a, P_a) , i.e. we adopt the choice made in Ref. [30]. From Eqs. (2.2) and (3.5), we obtain the following nonvanishing commutators:

$$\begin{aligned} [\hat{q}^0, \hat{E}] &= -i \frac{\widehat{\partial g}}{\partial \epsilon}, & [\hat{q}^0, \hat{p}_i] &= -i \frac{\widehat{\partial f}}{\partial \epsilon} \frac{\widehat{\Pi}_i}{\Pi}, \\ [\hat{q}^i, \hat{E}] &= i \frac{\widehat{\partial g}}{\partial \Pi} \frac{\widehat{\Pi}^i}{\Pi}, & [\hat{q}^i, \hat{p}_j] &= i \frac{\widehat{\partial f}}{\partial \Pi} \frac{\widehat{\Pi}^i}{\Pi} \frac{\widehat{\Pi}_j}{\Pi} + i \frac{\widehat{f}}{\Pi} \left(\delta_j^i - \frac{\widehat{\Pi}^i \widehat{\Pi}_j}{\Pi} \right). \end{aligned} \quad (4.1)$$

We can view these relations as a generalized uncertainty principle, noticing that the usual Heisenberg relations are recovered for energies and momenta well below the DSR (Planck) scale, because $g \approx \epsilon$ and $f \approx \Pi$ in this limit. It is also worth pointing out that all the operators on the right-hand side of these expressions commute, because they are determined only by functions of the energy-momentum, with no contribution from the position variables q^a . In terms of these fundamental commutators, we can also define the commutator of other phase space functions.

In order to simplify our expressions and make more clear the difference between this choice of fundamental commutators and the one considered by Hossenfelder, we will restrict our attention in this subsection to a subfamily of DSR theories in which the physical energy is just a function of the auxiliary one, with no dependence on the auxiliary momentum, namely $E = g(\epsilon)$. These theories have received special attention in the literature [17, 29, 30, 33]. As an immediate consequence of the fact that $(\partial g / \partial \Pi) = 0$, the commutator $[\hat{q}^i, \hat{E}]$ becomes equal to zero. Moreover, let us consider DSR theories of the so-called DSR2 and DSR3 types, whose physical energy is bounded from above. We re-

member that DSR3 theories have a bounded physical energy but an unbounded physical momentum, whereas DSR2 theories possess an invariant scale both in the physical energy and momentum. In these types of theories, one has that $(\partial g/\partial\epsilon) \rightarrow 0$ when ϵ approaches infinity and hence the commutator $[\hat{q}^0, \hat{E}]$ vanishes for infinitely large auxiliary energy.

On the other hand, the physical momentum is bounded from above not only in theories of the DSR2 type, but also in the so-called DSR1 theories, whose physical energy is nonetheless unbounded. In a theory of the DSR1 or DSR2 type, the partial derivatives $\partial f/\partial\epsilon$ and $\partial f/\partial\Pi$ may both vanish as ϵ and Π tend to infinity (on the mass shell). We note, however, that while this vanishing is allowed, it is not a necessary consequence of the boundedness of the physical momentum (on the mass shell), because the function f depends on both ϵ and Π . If $(\partial f/\partial\epsilon) \rightarrow 0$ when the auxiliary energy-momentum approaches infinity, the commutator $[\hat{q}^0, \hat{p}_i]$ becomes zero in that limit. Similarly, if $(\partial f/\partial\Pi) \rightarrow 0$, the commutators $[\hat{q}^i, \hat{p}_j]$ are negligible when the auxiliary energy-momentum gets large, because for theories of the DSR1 or DSR2 type one also has that $(f/\Pi) \rightarrow 0$ at infinity. Thus, the operators representing the auxiliary spatial coordinates would commute with those representing the physical momentum.

We therefore conclude that all the modified phase space commutators may vanish for DSR2 theories when the auxiliary energy-momentum tends to infinity (on the mass shell). It is then possible to find DSR theories in which the behavior of the phase space variables (q^a, P_a) becomes classical when one approaches the region of invariant energy and momentum scales.

Let us now analyze the choice of fundamental commutators made by Hossenfelder [17], which correspond to a set of elementary phase space variables formed by the physical spacetime coordinates and the auxiliary energy-momentum. With the restriction to the subfamily of DSR theories with $E = g(\epsilon)$, we obtain

$$\begin{aligned} [\hat{x}^0, \hat{\epsilon}] &= -i\frac{\widehat{1}}{\partial_{\epsilon}g}, & [\hat{x}^0, \hat{\Pi}_i] &= i\frac{\widehat{1}}{\partial_{\epsilon}g}\frac{\widehat{1}}{\partial_{\Pi}f}\frac{\widehat{\partial f}}{\partial\epsilon}\frac{\widehat{\Pi}_i}{\Pi}, \\ [\hat{x}^i, \hat{\epsilon}] &= 0, & [\hat{x}^i, \hat{\Pi}_j] &= i\frac{\widehat{1}}{\partial_{\Pi}f}\frac{\widehat{\Pi}^i}{\Pi}\frac{\widehat{\Pi}_j}{\Pi} + i\frac{\widehat{\Pi}}{f}\left(\delta_j^i - \frac{\widehat{\Pi}^i}{\Pi}\frac{\widehat{\Pi}_j}{\Pi}\right). \end{aligned} \quad (4.2)$$

In order to make more compact our expressions, we have employed a notation of the type $\partial_{\epsilon}g := \partial g/\partial\epsilon$ for partial derivatives that appear in denominators.

As we have commented above, $(\partial g/\partial\epsilon) \rightarrow 0$ when the auxiliary energy approaches

infinity in DSR2 and DSR3 theories. Thus, $[\hat{x}^0, \hat{\epsilon}]$ explodes in this limit and there is a total lack of resolution for simultaneous measurements of x^0 and ϵ . On the other hand, for a generic DSR theory, $[\hat{x}^0, \hat{\Pi}_i]$ may either vanish or tend to infinity depending just on the functional form of the partial derivatives of f and g , with no a priori restrictions. Finally, for theories of the DSR1 and DSR2 type, the ratio Π/f (and possibly $1/\partial_\Pi f$ as well) tends to infinity for large auxiliary energy-momenta (on the mass shell). So, for these theories, $[\hat{x}^i, \hat{\Pi}_j]$ diverges. We hence see that all the nontrivial fundamental commutators may diverge for certain DSR2 theories, with a complete loss of resolution when one reaches the regime of the invariant (physical) energy and momentum scales. This result strongly contrasts with the conclusions obtained for the alternative choice of modified commutators (4.1) that we have analyzed previously.

One can also study the behavior of the commutators when the energy of the system is restricted to be a certain function of the momentum. This interdependence makes the expressions of the commutators change “on shell”. After the reduction to energy surfaces, and for example with the choice of modified commutators (4.1) made by Cortés and Gamboa, one obtains the generalized uncertainty principle:

$$[\hat{q}^i, \hat{p}_j] = i \frac{\widehat{df}}{d\Pi} \frac{\widehat{\Pi}^i}{\Pi} \frac{\widehat{\Pi}_j}{\Pi} + i \frac{\widehat{f}}{\Pi} \left(\delta_j^i - \frac{\widehat{\Pi}^i \widehat{\Pi}_j}{\Pi \Pi} \right). \quad (4.3)$$

This expression is similar to the last one in Eq. (4.1), but the partial derivative of f with respect to Π has been replaced by a total derivative. This replacement simplifies the analysis of the emergence of a classical behavior. For DSR1 and DSR2 theories, it is straightforward to see that the ratio f/Π and the total derivative $df/d\Pi$ tend to zero when the auxiliary momentum approaches infinity. Therefore, the commutators of the operators that represent the auxiliary (spatial) position variables and the physical momentum vanish in this limit, which corresponds to the regime in which the physical momentum reaches the value of the invariant scale.

Although it may seem initially counterintuitive, the emergence of a regime with this kind of classical behavior poses in fact no inconsistency. If one measures the auxiliary position with infinite resolution then, according to the standard Heisenberg principle, one must have a complete uncertainty in its conjugate momentum, which is the auxiliary one. However, since DSR1 and DSR2 theories map infinite auxiliary momenta into a finite physical momentum scale, the uncertainty in the latter of these momenta can be kept

under control, yielding a vanishing product with the uncertainty in the auxiliary position.

B. Derivation of a DSR theory from a generalized uncertainty principle

To conclude this section, we will derive explicit formulas for the nonlinear map U that characterizes the DSR theory using a set of modified commutators. These formulas, together with Eq. (4.1) or Eq. (4.2), prove that the relevant information present in a canonical realization of a DSR theory and in a generalized uncertainty principle is equivalent. We will study explicitly the case of the fundamental commutators $[\hat{q}^a, \hat{P}_b]$. The analysis can be straightforwardly extended to the alternative case of the commutators $[\hat{x}^a, \hat{\Pi}_b]$.

Note that the functions f and g that determine the DSR theory, and that in a generic situation depend on both ϵ and Π , appear in the expressions (4.1) as partial derivatives. Therefore, given a generalized uncertainty principle of the form (4.1), we will obtain the map U (or equivalently U^{-1}) by mere integration. The reconstruction of an arbitrary function $z(\epsilon, \Pi)$ from its partial derivatives $u := \partial z / \partial \epsilon$ and $v := \partial z / \partial \Pi$ is immediate (assuming the integrability condition $\partial u / \partial \Pi = \partial v / \partial \epsilon$):

$$z(\epsilon, \Pi) = \int_0^\Pi v(\epsilon, \tilde{\Pi}) d\tilde{\Pi} + \int_0^\epsilon u(\tilde{\epsilon}, 0) d\tilde{\epsilon}. \quad (4.4)$$

The function z represents either f or g , and we have employed that $f(0, 0) = g(0, 0) = 0$ to fix an additive integration constant. It then finally suffices to realize that

$$\begin{aligned} \frac{\partial g}{\partial \epsilon} &= -\{q^0, E\}, & \frac{\partial g}{\partial \Pi} &= \{q^i, E\} \frac{\Pi^i}{\Pi}, \\ \frac{\partial f}{\partial \epsilon} &= -\{q^0, P_i\} \frac{\Pi^i}{\Pi}, & \frac{\partial f}{\partial \Pi} &= \{q^i, P_j\} \frac{\Pi_i \Pi^j}{\Pi \Pi}, \end{aligned} \quad (4.5)$$

where the above Poisson brackets $\{\cdot, \cdot\}$ are the straight classical counterpart of the fundamental commutators (4.1) (namely, these brackets are obtained by replacing energy-momentum operators –that commute among themselves– by classical variables, and removing an imaginary factor i).

5. GRAVITY'S RAINBOW

As we commented in the Introduction, Magueijo and Smolin put forward a generalization of DSR that attempts to incorporate the effect of curvature and, therefore, provide

an extension to general relativity [29]. This generalization is called gravity's rainbow and rests on a realization of DSR in position space that is compatible with the commutativity of the geometry. The realization selected by Magueijo and Smolin is not a canonical one; instead, it is constructed from a specialization of the requirement (introduced by Kimberly, Magueijo and Medeiros [25]) that free field theories in DSR continue to admit planewave solutions. This specialization can be formulated as the invariance under Lorentz transformations of the contraction between the energy-momentum and an infinitesimal spacetime displacement [29]. In our notation, this condition can be expressed in the form $\Pi_a \mathbf{d}q^a = P_a \mathbf{d}\tilde{x}^a$. We have denoted the physical coordinates by \tilde{x}^a to emphasize that their expressions in terms of auxiliary phase space variables will now differ from those obtained with a canonical realization of DSR.

From the above requirement, Magueijo and Smolin derived an energy dependent set of orthonormal frame fields. The expressions of the one-forms that they obtained depend nontrivially on the auxiliary energy-momentum,

$$d\tilde{x}^0 = \frac{\epsilon}{g} dq^0, \quad d\tilde{x}^i = \frac{\Pi}{f} dq^i. \quad (5.1)$$

Owing to this dependence, a rainbow of metrics emerge in the formalism, each particle being associated with a different metric according to its energy-momentum [29].

Actually, from Eq. (3.3) we see that a canonical realization of DSR leads also to spacetime displacements that depend explicitly on the energy and momentum (which can be understood as corresponding to the probe used by the observer to test the geometry). In this sense, canonical DSR theories can be extended as well to a type of gravity's rainbow formalism [27, 31, 33]. In the following, we restrict again our attention to the subfamily of DSR theories with $E = g(\epsilon)$. It is not difficult to see then that the requirement of invariance of the symplectic form $\mathbf{d}q^a \wedge \mathbf{d}\Pi_a = \mathbf{d}x^a \wedge \mathbf{d}P_a$ leads to the following one-forms [27], rather than to those in Eq. (5.1):

$$\begin{aligned} dx^0 &= \det U \left(\frac{\partial f}{\partial \Pi} dq^0 + \frac{\partial f}{\partial \epsilon} \frac{\Pi_i}{\Pi} dq^i \right), \\ dx^i &= \det U \left(\frac{\partial g}{\partial \epsilon} \frac{\Pi^i \Pi_j}{\Pi^2} dq^j \right) + \frac{\Pi}{f} \left(dq^i - \frac{\Pi^i \Pi_j}{\Pi^2} dq^j \right). \end{aligned} \quad (5.2)$$

Here, $\det U = 1/[(\partial g/\partial \epsilon)(\partial f/\partial \Pi)]$. The analysis carried out by Magueijo and Smolin in Ref. [29] can then be generalized by substituting relations (5.1) with Eq. (5.2), and will not be repeated here. Instead, in the next section we consider a particularly interesting

example of the solutions to the modified Einstein equations that arise in this type of gravity's rainbow formalism, namely, the generalization of the Schwarzschild solution.

6. MODIFIED SCHWARZSCHILD SOLUTION

Since we want to study spherically symmetric solutions of the modified Einstein equations, we will impose a natural symmetry reduction on the physical phase space (of the test particle that probes the geometry), restricting the physical momentum to be parallel or antiparallel to the spatial position. Therefore, for each given physical momentum, $\varepsilon_{jk}^i p_i dx^j = 0$ [33]. Here, ε_{ijk} is the Levi-Civita symbol. Since $p_i/p = \Pi_i/\Pi$, this condition can be rewritten as $dx^i = dx^j \Pi_j \Pi^i / \Pi^2$. With this reduction, we find from Eq. (5.2)

$$d\tilde{x}_{\pm}^0 = \frac{1}{\partial_{\epsilon} g} dq^0, \quad dx^i = \frac{1}{\partial_{\Pi} f} dq^i, \quad (6.1)$$

where we have defined

$$\tilde{x}_{\pm}^0 := x^0 \pm \frac{1}{\partial_{\epsilon} g} \frac{\partial f}{\partial \epsilon} x, \quad x := \sqrt{x^i x^i}. \quad (6.2)$$

The sign in the definition of \tilde{x}_{\pm}^0 depends on whether p^i and dx^i are parallel or antiparallel. Note that the new coordinate \tilde{x}_{\pm}^0 , although not canonically conjugate to the energy, differs from the canonical time x^0 only in the inclusion of a radial shift vector that is constant in spacetime.

Apart from this constant shift, we notice from Eqs. (5.1) and (6.1) that, in the two realizations of DSR that we are considering (the canonical one and that of Ref. [25]), the geometry is affected just by two independent scalings that are constant in spacetime, but depend on the auxiliary energy-momentum. They consist in a time dilation and a conformal transformation of the spatial components. The associated (spherically symmetric) gravity's rainbow formalisms can be explored simultaneously if one adopts the generic notation $dq^0 = G(\epsilon) d\tilde{x}_{\pm}^0$, $dq^i = F(\epsilon, \Pi) dx^i$, with \tilde{x}_{\pm}^0 designating x^0 for the case of the gravity's rainbow elaborated by Magueijo and Smolin. Explicitly

$$G(\epsilon) := \frac{g(\epsilon)}{\epsilon}, \quad F(\epsilon, \Pi) := \frac{f(\epsilon, \Pi)}{\Pi}, \quad (6.3)$$

for Magueijo and Smolin, whereas for a canonical realization

$$G(\epsilon) := \frac{\partial g}{\partial \epsilon}(\epsilon), \quad F(\epsilon, \Pi) := \frac{\partial f}{\partial \Pi}(\epsilon, \Pi). \quad (6.4)$$

One can now easily derive the modified Schwarzschild solution following the same line of reasoning that was explained in Ref. [29]. In terms of the physical coordinates (called energy independent by Magueijo and Smolin), we can generically express a spherically symmetric metric as [29]

$$ds^2 = -\tilde{A}(x)[G(\epsilon)]^2(d\tilde{x}_\pm^0)^2 + [F(\epsilon, \Pi)]^2 \left(\tilde{B}(x)dx^2 + x^2d\Omega^2 \right). \quad (6.5)$$

Translated into auxiliary spacetime coordinates (energy dependent in Magueijo and Smolin's terminology), we obtain

$$ds^2 = -A(q|\epsilon, \Pi)(dq^0)^2 + B(q|\epsilon, \Pi)dq^2 + q^2d\Omega^2, \quad (6.6)$$

where

$$\begin{aligned} q &:= \sqrt{q^i q^i}, & q &= F(\epsilon, \Pi)x, \\ A(q|\epsilon, \Pi) &:= \tilde{A}[x(q, \epsilon, \Pi)] & B(q|\epsilon, \Pi) &:= \tilde{B}[x(q, \epsilon, \Pi)], \end{aligned} \quad (6.7)$$

and the round metric on S^2 is not changed. According to Birkoff's theorem, the functions A and B must satisfy $A(q|\epsilon, \Pi) = B(q|\epsilon, \Pi)^{-1} = 1 - C(\epsilon, \Pi)/q$ for every fixed auxiliary energy-momentum.

Taking into account the relationship between q and x , we see that, in order to recover the independence of the function $\tilde{A}(x)$ on the energy-momentum, the ratio $C(\epsilon, \Pi)/F(\epsilon, \Pi)$ must be a genuine constant, not only with respect to its spacetime dependence, but also with respect to ϵ and Π [29]. On the other hand, in the regime of low energies and momenta $F(\epsilon, \Pi)$ approaches the unity and $C(\epsilon, \Pi)$ coincides with $2M$, where M is the Schwarzschild mass of general relativity and we are setting G , the effective Newton constant in the low energy limit, equal to one. Therefore, we must have $C(\epsilon, \Pi) = 2MF(\epsilon, \Pi)$.

This completely determines the modified Schwarzschild solution. In physical coordinates, the explicit expressions in the gravity's rainbow formalisms proposed by Magueijo and Smolin, on the one hand, and arising from a canonical realization, on the other hand, are respectively

$$\begin{aligned} ds^2 &= -\left(1 - \frac{2M}{x}\right) \left(\frac{g}{\epsilon}\right)^2 (dx^0)^2 + \left(1 - \frac{2M}{x}\right)^{-1} \left(\frac{f}{\Pi}\right)^2 dx^2 + \left(\frac{f}{\Pi}\right)^2 x^2 d\Omega^2; \\ ds^2 &= -\left(1 - \frac{2M}{x}\right) \left(\frac{\partial g}{\partial \epsilon}\right)^2 (d\tilde{x}_\pm^0)^2 + \left(1 - \frac{2M}{x}\right)^{-1} \left(\frac{\partial f}{\partial \Pi}\right)^2 dx^2 + \left(\frac{\partial f}{\partial \Pi}\right)^2 x^2 d\Omega^2. \end{aligned} \quad (6.8)$$

In particular, notice that the horizon is placed at the same location as in general relativity, $x = 2M$. The corresponding notions of null and spatial infinities and of asymptotic flatness in the context of DSR can be introduced as explained in Ref. [34].

7. SUMMARY AND CONCLUSIONS

We have studied several proposals for the realization of DSR in position space that are based on a canonical implementation in phase space. We have shown that, although these proposals have been discussed separately in the literature, they are actually equivalent. Some of these proposals have been formulated as the requirement that the action of the Lorentz transformations on the physical spacetime coordinates be such that the total transformation in the resulting phase space be canonical [26, 28]. In another case, the demand of a canonical transformation has been introduced instead on the nonlinear map that relates the physical phase space with an auxiliary one where the Lorentz transformations have the standard linear action [27, 31]. Finally, in other cases the realization in position space has been encoded in a generalized uncertainty principle, with modified commutators that correspond either to the physical energy-momentum and the spacetime coordinates that generate translations in the auxiliary energy-momentum [30] or, alternatively, to the opposite choice of auxiliary versus physical variables [17].

We have also analyzed the behavior of those two different kinds of modified commutation relations. For the first kind, namely, when one considers auxiliary spacetime coordinates and physical energy-momentum, we have shown that all phase space commutators may vanish in the limit of large auxiliary energy and momentum (on the mass shell) for certain DSR theories of the so-called DSR2 type. Therefore, it is in principle possible to approach a classical regime for the mentioned elementary phase space variables when the physical energy-momentum tends to the invariant scale of the DSR theory. On the contrary, for the same type of DSR theories, the alternative choice of physical spacetime coordinates and auxiliary energy-momentum as elementary variables leads to modified commutators that generally explode when the system approaches the invariant energy-momentum scale. In terms of these elementary variables, there is a complete loss of resolution in the considered limit. In addition, we have shown how to construct the nonlinear map U that determines the DSR theory from the sole knowledge of a given set

of modified phase space commutators. In this construction, it has been essential that both the fundamental commutators and certain projections of them in the momentum direction are not affected by factor ordering ambiguities, because they are defined exclusively in terms of functions of the energy-momentum.

Finally, we have also discussed the possibility of generalizing the gravity's rainbow formalism put forward by Magueijo and Smolin in order to find an extension to general relativity of the canonical realization of DSR theories. We have shown that the generalization is possible and we have compared the gravity's rainbow that results from the canonical realization with that elaborated in Ref. [29] starting from a different proposal for the realization of DSR in position space [25]. In both cases, one actually arrives to a rainbow of metrics that depend nontrivially on the energy-momentum of the test particles. We have studied in detail the particular example of the generalized Schwarzschild solution to the modified Einstein equations. The main modification with respect to the standard solution in general relativity consists in a time dilation and a spatial conformal transformation, both of them dependent on the energy-momentum. We have deduced the explicit form of the dilation and conformal factors in the case of the canonical realization, showing that they generally differ from those obtained by Magueijo and Smolin. This difference may have important consequences for black hole thermodynamics [33].

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