Optimal electron propagation on a quantum chain by a topological phase

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We study the quantum diffusion of an electron in a quantum chain starting from an initial state localized around a given site. As the wavepacket diffuses, the probability of reconstructing the initial state on another site diminishes drastically with the distance. In order to optimize the state transmission we find that a topological quantum phase can be introduced. The effect of this phase is the reduction of wavepacket spreading together with almost coherent group propagation. In this regime, the electron has a quasi-linear dispersion and high fidelity can be achieved also over large distances in terms of lattice spacing.

I. INTRODUCTION

The spatial transmission of a quantum state is an important and nontrivial task in quantum communication. State encoding usually occurs on a local device and an efficient channel is required to transmit it elsewhere. Quantum communication can be achieved both transmitting the particle on which the state has been encoded (flying qubit) [1], by teleportation [2] or by a quantum state transfer in which only information is transmitted without transfer of matter or light. The first case can be, for example, a laser beam that coherently propagates carrying the information codified on the polarization. Photons can travel with low loss, in optical fibres or even in free space, and can be readily measured by a receiving party. This is because of the small interaction with the environment an the linear dispersion, allowing for a propagation without spreading of the wavepacket. This is why optical flying qubits represent a very efficient channel over long distances.

Nevertheless, different solutions can be more suitable over smaller (micro and nanometric) distances typical of solid state devices. To this aim, ion trap based device has been proposed[3, 4]. Other schemes have been also described for short distance communication by a spin chain used as channel (see [5, 6, 7, 8, 9, 10] and references therein). Here, the transfer is based on the diffusion through the chain of an initially localized spin state by means of typical collective modes induced by the particular phase in which we prepare it. The use of local excitations requires an optimization of the interference for the state reconstruction[11, 12, 13, 14, 15, 16], because there is, together with the usual problem of decoherence, also a diffusion phenomenon due to typically non-lineardispersion laws. Other different physical realizations of quantum channels have been also suggested with different strategies to optimize the state transfer: Josephson arrays [17], nanoelectromechanical oscillators [18], quantum chains as quantum bus [19, 20, 21, 22], spin-1 chains[23, 24], dot chain [25, 26, 27, 28, 29].

In this paper, we propose a scheme of quantum state transfer over a quantum chain in which an electron, and the spin state encoded on it, plays the role of flying qubit. We study the diffusion of an electron in a tight-binding chain, starting from an initial localized state, under the action of a topological phase, induced, for example, by a magnetic Aharonov-Bohm flux [30, 31]. The electron wavepacket propagation in chain of quantum dots was studied in [32], while a general treatment of the Aharonov-Bohm scattering for a free particle was given by Stelitano [33]. In Ref. [34], it has been shown that the Aharonov-Bohm effect enables the generation of entanglement in mesoscopic rings.

We derive the dynamics of the electron in a discrete one-dimensional lattice, showing how a suitable topological phase can induce an almost linear dispersion. As a result, in the limit of a large number of sites, the electron moves coherently with a reduced wavepacket spreading. Such a scheme allows for communication over an intermediate range, contrary to what happens with other dispersive channels. A different use of the topological phase was used in [35] for

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communication on spin rings.

We show that an initial amount of delocalization is necessary for the scheme to work properly, otherwise an initial completely localized state does not feel the effect of the topological phase, spreading rapidly. The transmission efficiency can be characterized by the fidelity [36, 37]. Even if the definition of this quantity is more general, for a pure state it reduces to the squared modulus of the projection of the evolved state on the transmitted one. It quantifies how much the initial state, centered around the 0−site, can be reproduced around another given site of the chain after a suitable time. The other quantity we are going to study is the evolution of the probability distribution, to better emphasize the two aspects of the evolution: the propagation, due to the presence of a group velocity over different chain modes, and the wavepacket spreading due to different phase velocities.

In the next section we describe the system and calculate exactly both the fidelity and the probability distribution in the general case. In the section III, we show how the localized preparation is not useful for communication. In section IV we study the evolution of an initial square packet, comparing the result with a simple analytical approximation, and show how a good communication can be achieved over a large number of sites. Finally we give our conclusions.

II. MODEL

Let us consider a single electron moving in a one-dimensional ring-shaped lattice, with N sites and lattice constant a. In the reciprocal lattice space, and in the tight-binding approximation, the diagonal Hamiltonian is

$$H_R = \sum_{q,\sigma} \epsilon_q c^\dagger_{q,\sigma} c_{q,\sigma}, \tag{1}$$

where $\epsilon_q = -2w\cos(aq)$ are the energies forming the lattice band, w is the half band width, $q = \frac{2\pi}{N}n$ are the vectors of the reciprocal lattice and $c_{q,\sigma}$ the electron fermionic operators. It is simple to see that the operators time dependence is given by $c^\dagger_{q,\sigma}(t) = e^{-i\epsilon_q t} c^\dagger_{q,\sigma}$. Hereafter, we shall impose $\hbar = 1$.

In a perspective of quantum communication, we can encode the qubit on the spin state since this latter does not change during the evolution. From now on, we shall omit the σ index, denoting with $c^\dagger = \alpha_1 c^\dagger_\uparrow + \alpha_2 c^\dagger_\downarrow$ the electron operator with generic spin state.

We start preparing the electron in a state localized around the site $j = 0$

$$|\psi_0(0)\rangle = \sum_j g_j c^\dagger_j |0\rangle = \frac{1}{\sqrt{N}} \sum_q \tilde{g}_q c^\dagger_q |0\rangle, \tag{2}$$

where g_l is an amplitude distribution centered around the site $j = 0$ and

$$\tilde{g}_q = \sum_j g_j e^{iqaj}, \tag{3}$$

is its Fourier transform. The probability distribution of the site occupancy is

$$P_j(0) = |\langle 0|c_j|\psi_0(0)\rangle|^2 = |g_j|^2 . \tag{4}$$

As soon as the electron evolves, a diffusion process starts. The state evolution is easily calculated

$$|\psi_0(t)\rangle = \frac{1}{\sqrt{N}} \sum_q \tilde{g}_q e^{-i\epsilon_q t} c^\dagger_q |0\rangle, \tag{5}$$

so as the time-dependent probability distribution

$$P_j(t) = |\langle 0|c_j|\psi_0(t)\rangle|^2 = \frac{1}{N^2} \sum_{qq'} \tilde{g}_q \tilde{g}^*_{q'} e^{-i\left[(\epsilon_q - \epsilon_{q'})t - (q-q')aj\right]}. \tag{6}$$

As expected, since $\epsilon_q = \epsilon_{-q}$, one can see from Eq. (6) that $P_j(t)$ remains centered around $j = 0$, and a pure diffusion occurs. In order to obtain a real transport of the wave packet, it is necessary to introduce something breaking the translational invariance of the Hamiltonian. This result can be achieved introducing a topological phase which changes the hopping terms as follows: $c^\dagger_j c_{j+1} \to e^{i\theta} c^\dagger_j c_{j+1}$. As a consequence, the energies become

$$\epsilon_q(\theta) = -2w\cos(aq - \theta) . \tag{7}$$

As we shall show, tuning the θ phase makes possible to reduce the wave packet spread and optimize the particle transmission. Such a phase can be obtained, for instance, introducing an suitable magnetic orthogonal field B trough the ring. By means of the Aharonov-Bhom effect the phase shift is $\theta = (2\pi\Phi)/(N\Phi_0)$ where Φ is the magnetic flux through the ring and $\Phi_0 = hc/e$ is the quantum unit of magnetic flux. Topological phases can be also created in different ways, for example by means of the electric Aharonov-Bohm effect [38, 39, 40]. We will not insist on this point, referring to a generic phase independently on how to generate it.

In a quantum information context, where the conducting ring can be used as a communication channel (quantum bus), it is useful to consider another quantity, the so-called fidelity. It represents how the initial state, centered on the $j = 0$ site, can be efficiently reproduced around another site $j = d$ (the *receiver* site). Starting from the transfer amplitude

$$f_d(t) = \langle \psi_d | \psi_0(t) \rangle = \frac{1}{N} \sum_q |\tilde{g}_q|^2 e^{i(qad - \epsilon_q t)}, \tag{8}$$

the fidelity is defined as

$$F_d(t) = |f_d(t)|^2 . \tag{9}$$

In the following sections we shall examine two cases: in the first one, the electron is initially localized on the site $j = 0$; in the other case, the electron is prepared with a squared distribution around the same site.

III. ATOMIC PREPARATION

To start with, we study the case in which the electron is prepared on one site, say $j = 0$, putting $g_j = \delta_{j,0}$ and $\tilde{g}_q = 1$. We shall refer to this preparation as the *atomic* one. The transfer amplitude is simply

$$f_d(t) = \frac{1}{N} \sum_q e^{i(qad - \epsilon_q t)}. \tag{10}$$

In this case, the probability distribution is equal to the fidelity $P_j(t) = F_j(t)$. In the limit of large number of sites, f_d is proportional to a Bessel Function [41] J_d

$$f_d(t) = e^{i\frac{\pi}{2}d} J_d(2wt). \tag{11}$$

For fixed d, the fidelity has a maximum for $t \simeq d/2w$, which is also an absolute maximum in the $N \to \infty$ limit. This can be seen in Fig. 1 where the time dependence of the fidelity is plotted for different final sites. For finite value of N, other maxima appear, higher than the first, because of the interference between the two different propagating wave packet in which the initial wave function is splitted. This effect occurs also in the context of quantum communication on spin chains[6]. Here, the interference is due to the counterpropagating spin waves and the constructive peaks are used to optimize the short range transport (few sites). This mechanism is not very interesting on larger distances because of the difficulty of estimating the optimal times of the better constructive interference and their strong dependence on the site number. For this reason we work in the limit of large number of sites, fixing the first maximum time as the useful one for the transport process.

The atomic preparation appears to be highly inefficient, as it can be seen looking at the very small fidelities in Fig. 1. Moreover is easy to demonstrate that the introduction of a topological phase does not change anything. As we shall discuss in the next section, fidelity enhances if the electron wavefuncion is initially delocalized over few sites.

IV. SQUARE PACKET PREPARATION

The second configuration we consider consists in an electron prepared to be equally delocalized around the site $j = 0$. The initial state is given by a square wave packet centered on the 0 site and extended over $2M + 1$ sites. The amplitude distribution is

$$g_l = \begin{cases} \frac{1}{\sqrt{2M+1}} & \text{If } -M \leq l \leq M \\ 0 & \text{Elsewhere,} \end{cases} \tag{12}$$

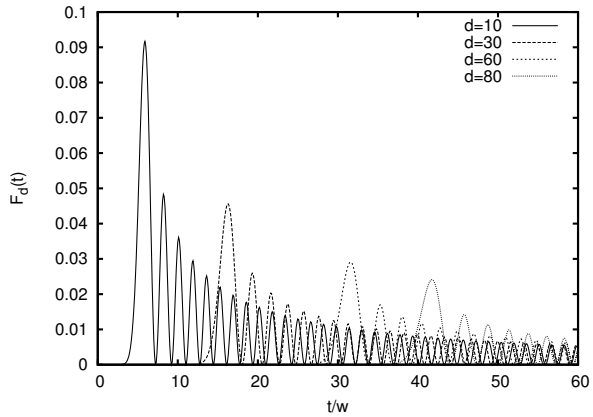

FIG. 1: Time evolution of the fidelity in the atomic limit with fixed final site d and $N = 500$. Fidelity is plotted for $d = 10, 30, 60, 80$.

corresponding to the form factor

$$\tilde{g}_q = \frac{1}{\sqrt{\lambda}} \frac{\sin \lambda \frac{qa}{2}}{\sin \frac{qa}{2}}, \tag{13}$$

with $\lambda = (2M + 1)$.

The fidelity is plotted in Fig. 2 for a 500-site chain and without any phase θ. One can see that a completely different behavior occurs, even if the efficiency still remains low. The reason is to be ascribed to the fact that the wave packet actually does not moves but only diffuses, as it is show in Fig 3, where the probability distribution is reported for different times.

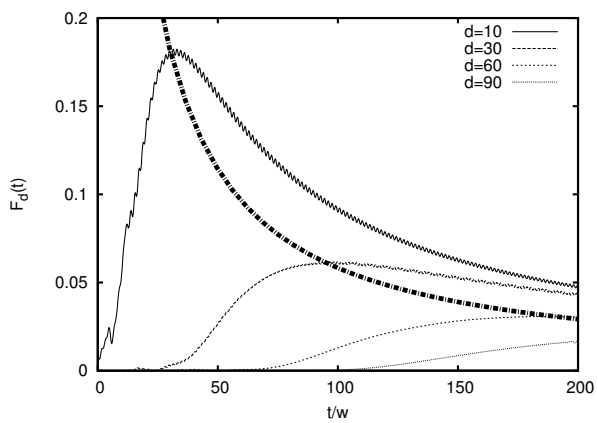

FIG. 2: Time evolution of the fidelity in the square packet preparation ($M = 5$), without phase θ, with fixed final site d and $N = 500$. Fidelity is plotted for $d = 10, 30, 60, 90$. Thick line indicate the maxima of fidelity reached by each site at different times.

The introduction of a suitable phase θ induces a wavepacket propagation so to increase the transmission efficiency. In order to estimate the optimal value of the phase, we derive here an approximate analytical expression for $F_d(t)$ and $P_d(t)$. Considering a very large N we can substitute the sum (8) by an integral introducing the continuous variable $x = qa/2$. The expression for the transfer amplitude, with a generic θ becomes

$$f_d(t) \simeq \frac{1}{\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} dx G(x) e^{i(2xd + 2wt(\cos(2x-\theta)-1))}. \tag{14}$$

The form factor $G(x) = \lambda^{-1} (\sin \lambda x / \sin x)^2$ in the integral is a periodic function whose peaked principal maxima are spaced by an amount of π from each other. The integration interval contains only the principal maximum centered

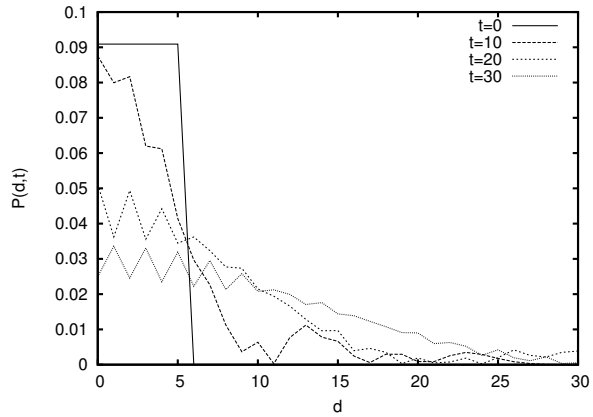

FIG. 3: Probability distribution for $N = 500$, in the square packet preparation ($M = 5$) and without phase θ. The function is symmetric and here is plotted only for d positive. The wavepacket spread is shown for different times. $t = 0, 10, 20, 30$.

on the origin and so we can expand $G(x)$ around $x = 0$ approximating it by a Gaussian function

$$G(x) \simeq \lambda \left(1 - \frac{\lambda^2 - 1}{3} x^2\right) \simeq \lambda e^{-\frac{\lambda^2-1}{3}x^2}. \tag{15}$$

Now we expand the argument of the phase factor into the integral. We note that the expansion is to be performed, at less, until the second order because the first term vanishes as soon as the phase is turned off. So, if we want to take into account of the effect of phase-free propagation we have to consider the expansion

$$\cos(2x - \theta) \simeq \cos\theta + 2\sin\theta x - 2\cos\theta x^2. \tag{16}$$

Isolating the inessential phase factors we obtain

$$f(t) \simeq e^{i\phi} \frac{\lambda}{\pi} \int_{-\infty}^{\infty} dx e^{-\left[\frac{\lambda^2-1}{3} - i4wt\cos\theta\right]x^2 + i2x(d+2wt\sin\theta)}, \tag{17}$$

that can be integrated as a common Gaussian integral. The resulting fidelity is

$$F_d(t) = A(t) e^{-\frac{[d+2wt\sin\theta]^2}{2\sigma_F^2(t)}}, \tag{18}$$

with

$$A(t) = \frac{3\lambda^2}{\pi\sqrt{(\lambda^2-1)^2 + 144w^2t^2\cos^2\theta}}, \tag{19}$$

and

$$\sigma_F^2(t) = \frac{(\lambda^2-1)^2 + 144w^2t^2\cos^2\theta}{12(\lambda^2-1)}. \tag{20}$$

The same calculation can be done for the probability distribution. Expanding the form factor (13) into

$$\tilde{g}(x) \simeq \sqrt{\lambda} e^{-\frac{\lambda^2-1}{6}x^2}, \tag{21}$$

the wave function (6) becomes

$$\psi_d(t) =\simeq e^{i\phi} \frac{\sqrt{\lambda}}{\pi} \int_{-\infty}^{\infty} dx e^{-\left[\frac{\lambda^2-1}{6} - i4wt\cos\theta\right]x^2 + i2x(d+2wt\sin\theta)}. \tag{22}$$

Integrating and squaring, we obtain

$$P_l(t) = B(t) e^{-\frac{[d+2wt\sin\theta]^2}{2\sigma_P^2(t)}}, \tag{23}$$

with

$$B(t) = \frac{6\lambda}{\pi\sqrt{(\lambda^2-1)^2 + 576w^2t^2\cos^2\theta}} \tag{24}$$

$$\sigma_P^2(t) = \frac{(\lambda^2-1)^2 + 576w^2t^2\cos^2\theta}{24(\lambda^2-1)}. \tag{25}$$

By this calculation, both $F_d(t)$ and $P_d(t)$ are approximated with Gaussian functions of the variable d, which propagate and diffuses in time. The propagation velocity is given by $v = -2w\sin\theta$, while the diffusion is quantified by the variance $\sigma^2_{F,P}$. We shall study $F_d(t)$ as a function of time with fixed d (the *receiver* site), and $P_d(x)$ as a spatial distribution at fixed times. Two different behaviors appear, corresponding to $\theta = 0$ and $\theta = -\pi/2$.

In the first case, with $\theta = 0$, the quadratic term in the expansion (16) prevails. There is no propagation of the probability distribution, $v = 0$. The wave packet diffuses only, as one can see in Fig. 3. The fidelity has a maximum in correspondence of the time

$$t^* = \frac{\sqrt{(\lambda^2-1)(12d^2-\lambda^2+1)}}{12w}. \tag{26}$$

In Fig. 2 is depicted the fidelity at different sites with the curve of the maxima. The result is better than the atomic case but the efficiency rapidly decreases with the distance.

In the second case, with $\theta = -\pi/2$, only the linear term in (16) remain. The dispersion becomes approximately linear and the wavepacket (23)) does not diffuse (Fig. 5). Moreover, it propagates with velocity $v = 2w$ causing an enhancement in the fidelity. In particular, $F_d(t)$ assumes its maximum value at the approximate time

$$t^* = \frac{d}{2w}. \tag{27}$$

In Fig. 4 time dependent fidelity is reported for different receiving sites, together with the time evolution of the maxima. As one can see, in this case a fidelity of the order of about 0.8 is achieved even for distances of the order of 100 sites. Notice that the maximum value of the fidelity one can achieve is almost independent of the chain length. In Fig. 6 the maximum of fidelity is reported as a function of distance for the two extremal values of θ

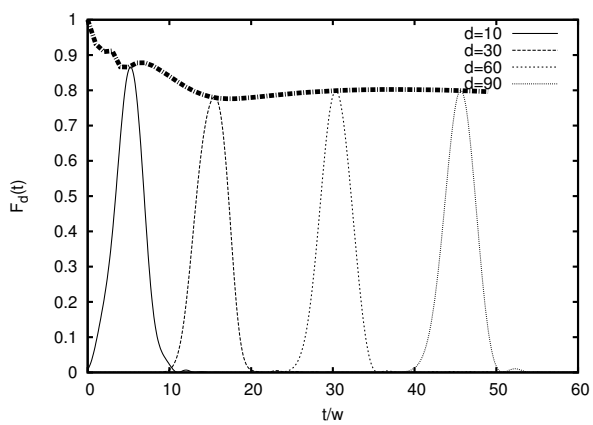

FIG. 4: Time evolution of the fidelity in the square packet preparation ($M = 5$), with phase $\theta = -\pi/2$, with fixed final site d and $N = 500$. Fidelity is plotted for $d = 10, 30, 60, 90$. The thick line indicate the maxima of fidelity reached by each site at different times.

V. CONCLUSIONS

We studied the quantum diffusion of an electron in a periodic lattice, in the tight-binding regime. We shown that,even in the presence of a nonlinear dispersion, it is possible to approach a linear regime where the electron wave packet spreading is reduced. This is possible introducing in the system a suitable topological quantum phase. The new advances in the scalable quantum information and communication on mesoscopic solid state devices gives rise to a need for new communication channel beyond the usual photon flying qubit. In this perspective, the possibility of a flying qubit carried by electrons appears a promising resource for state transfer along mesoscopic scales. .

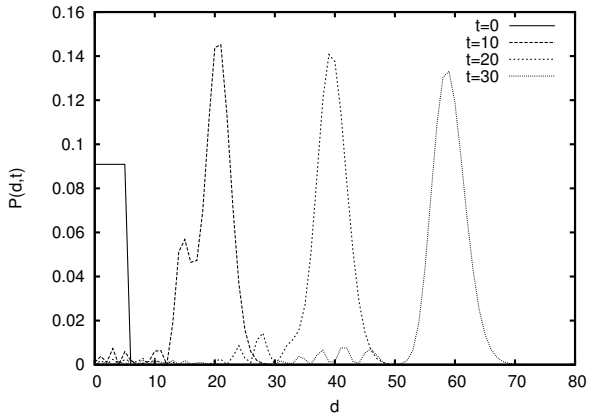

FIG. 5: Probability distribution for $N = 500$, in the square packet preparation ($M = 5$) and with phase $\theta = -\pi/2$. Here a propagation of the wavepacket can be observed. The wavepacket spread is shown for different times. $t = 0, 10, 20, 30$.

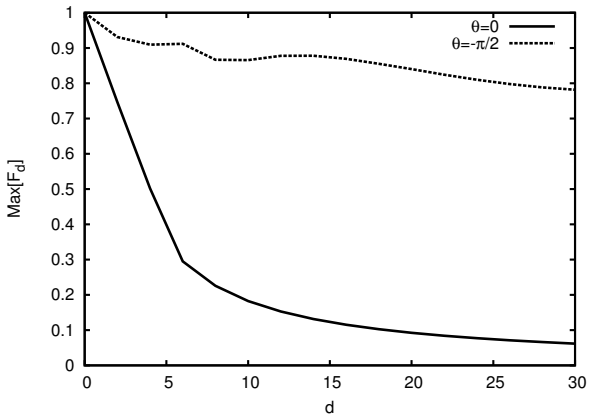

FIG. 6: Maximum of fidelity with distance.$N = 500$ and $\theta = 0, -\pi/2$. The topological phase allows an efficient long range state transfer.

VI. ACKNOWLEDGMENT

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