Scattering of light from an amplifying medium bounded by a randomly rough surface

Andriy V. Tutov* and Alexei A. Maradudin

Department of Physics and Astronomy and the Institute for Surface and Interface Science, University of California, Irvine, California 92697

Tamara A. Leskova

Institute of Spectroscopy, Russian Academy of Sciences, Troitsk 142092, Russia

Andreas P. Mayer

Institut für Theoretische Physik, Universität Regensburg, D-93040 Regensburg, Germany

José A. Sánchez-Gil

Instituto de Estructura de la Materia, CSIC, Serrano 121, 28006 Madrid, Spain

(Received 5 May 1999)

We study theoretically the scattering of *s*-polarized light from the one-dimensional, randomly rough surface of a homogeneous amplifying dielectric medium deposited as a film on the planar surface of a semi-infinite perfectly conducting substrate. The reflectivity of the rough film is found to be greater than that of the corresponding planar film if only true guided waves supported by the scattering structure exist at the frequency of the incident light; it can be smaller than that of the corresponding planar film if a leaky guided wave also exists at the frequency of the incident light. Although the reflectivity of an amplifying film with a planar surface is greater than unity for all angles of incidence, that of an amplifying film with a random surface can be smaller than unity in a certain range of angles of incidence as a consequence of the existence of a leaky guided wave. The contribution to the mean differential reflection coefficient from the incoherent component of the scattered light displays an enhanced backscattering peak and satellite peaks (the latter if the scattering structure supports two or more guided waves). The overall intensity of the light scattered incoherently from the surface of a rough amplifying film is always greater than that of the light scattered from the same film with the same magnitude of the imaginary part of its dielectric constant, but of opposite sign, irrespective of the presence or absence of a leaky wave at the frequency of the incident light. However, the height and width of the enhanced backscattering peak are nonmonotonic functions of the magnitude of the imaginary part of the dielectric constant of the film, when a leaky wave is present, but depend monotonically on it when no leaky wave is present. In the case of an absorbing film these functions depend monotonically on the imaginary part of the dielectric constant. [S0163-1829(99)02242-0]

I. INTRODUCTION

The transmission of light through, and its reflection from, volume disordered amplifying media has been the subject of several recent theoretical¹⁻¹⁰ and experimental¹¹⁻¹⁴ studies. A review of such investigations is given in Ref. 15. The interest in the interplay between phase-coherent multiple scattering and stimulated emission that these studies represent is due in large measure to the possibility of using such systems as random lasers,¹⁶ i.e. lasers in which the feedback is provided by multiple scattering due to disorder rather than by confinement by mirrors. In these investigations it has been predicted^{1,14} and verified experimentally¹⁴ that the enhanced backscattering cone narrows as a result of stimulated emission below the laser threshold; that in one-dimensional random systems amplification suppresses transmittance in just the same way as absorption does;⁹ and that all moments of the transmittance T of a one-dimensional random amplifying system $\langle T^m \rangle$ with $m \geq 1$ are infinite.¹⁰

However, volume disordered amplifying media are not the only systems in which the interplay between phase-coherent multiple scattering and stimulated emission can occur. In this paper we examine multiple-scattering effects oc-

curing in the scattering of light from a random surface that bounds an amplifying medium. In particular we study the coherent scattering (reflectivity) and the incoherent (diffuse) scattering of *s*-polarized light incident from vacuum on a dielectric film deposited on the planar surface of a perfect conductor, when the vacuum-dielectric interface is a one-dimensional random interface whose generators are perpendicular to the plane of incidence. The dielectric film is characterized by an isotropic, frequency-independent, complex dielectric constant $\epsilon = \epsilon_1 + i\epsilon_2$, where the real part ϵ_1 is assumed to be positive. The imaginary part ϵ_2 will be assumed to be positive or negative. In the former case the film is a passive (absorbing) medium; in the latter case the film is an active (amplifying) medium. Taking ϵ_2 to be negative is the simplest way to model stimulated emission in this system. A physical realization of such a medium could be¹⁴ Ti:sapphire optically pumped by a frequency-doubled Nd:YAG laser whose wavelength is 532 nm. However, we will be interested primarily in the qualitative nature of the effects arising from the interplay between the multiple scattering of the incident light caused by the randomness of the vacuum-dielectric interface and the gain in the medium represented by the negative value of ϵ_2 . For elucidating these effects the model assumed is adequate.

FIG. 1. The scattering system studied in this paper.

II. CHARACTERIZATION OF A RANDOM SURFACE

The physical system we study in this paper consists of vacuum in the region $x_3 > \zeta(x_1)$, a dielectric film in the region $-d < x_3 < \zeta(x_1)$, and a perfect conductor in the region $x_3 < -d$ (Fig. 1). The surface profile function $\zeta(x_1)$ is assumed to be a single-valued function of x_1 that is differentiable, and constitutes a zero-mean, stationary, Gaussian random process defined by the properties

$$\langle \zeta(x_1) \rangle = 0, \quad (1a)$$

$$\langle \zeta(x_1) \zeta(x'_1) \rangle = \delta^2 W(|x_1 - x'_1|). \quad (1b)$$

In Eqs. (1) the angle brackets denote an average over the ensemble of realizations of $\zeta(x_1)$, while $\delta = \langle \zeta^2(x_1) \rangle^{1/2}$ is the rms height of the surface. In numerical calculations we will use the Gaussian form

$$W(|x_1|) = \exp(-x_1^2/a^2) \quad (2)$$

for the surface height autocorrelation function. The characteristic length a appearing in this expression is called the transverse correlation length of the surface roughness.

It is convenient to introduce the Fourier integral representation of $\zeta(x_1)$,

$$\zeta(x_1) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} \hat{\zeta}(k) e^{ikx_1}. \quad (3)$$

The Fourier coefficient $\hat{\zeta}(k)$ is also a zero-mean Gaussian random process that possesses the following statistical properties:

$$\langle \hat{\zeta}(k) \rangle = 0 \quad (4a)$$

$$\langle \hat{\zeta}(k) \hat{\zeta}(k') \rangle = 2\pi \delta(k + k') \delta^2 g(|k|), \quad (4b)$$

where $g(|k|)$, the power spectrum of the surface roughness, is defined by

$$g(|k|) = \int_{-\infty}^{\infty} dx_1 W(|x_1|) e^{-ikx_1}. \quad (5)$$

The form of $g(|k|)$ that corresponds to the choice for $W(|x_1|)$ given by Eq. (2) is

FIG. 2. The dispersion curves of the s -polarized guided waves supported by a dielectric film of dielectric constant $\epsilon = 2.6896$ and thickness d deposited on a perfectly conducting substrate.

$$g(|k|) = \sqrt{\pi} a \exp(-a^2 k^2/4). \quad (6)$$

III. GUIDED WAVES

In the absence of the surface roughness [$\zeta(x_1) \equiv 0$] and when the dielectric constant of the film is real ($\epsilon_2 = 0$), the structure depicted in Fig. 1 supports s -polarized guided waves whose sagittal plane can be chosen with no loss of generality to be the $x_1 x_3$ -plane.¹⁷ The dispersion relation for these guided waves is

$$\beta_0(q) + \alpha(q) \cot \alpha(q) d = 0, \quad (7)$$

where $\beta_0(q) = [q^2 - (\omega^2/c^2)]^{1/2}$ and $\alpha(q) = [\epsilon(\omega/c)^2 - q^2]^{1/2}$. The number of such guided waves whose frequency is ω depends on the dielectric constant of the film and on the thickness of the film. In Fig. 2 we plot the dispersion curves of these guided waves for a film whose dielectric constant is $\epsilon = 2.6896$. They exist as true guided waves, for which q is real for real ω , only in the region of the (ω, q) plane defined by the inequalities $(\omega/c) < q < \sqrt{\epsilon}(\omega/c)$. The portions of the dispersion curves corresponding to this range of q values are depicted by solid lines in Fig. 2. The portions of the dispersion curves to the left of the vacuum light line $\omega = cq$ are the dispersion curves of leaky guided waves, for which q is complex, with positive real and imaginary parts, for real ω . Along a leaky wave branch of the dispersion curves, as the real part of q decreases, its imaginary part increases, and the corresponding leaky wave ceases to be a recognizable wave when $\text{Im } q$ equals $\text{Re } q$. At this value of $\text{Re } q$ the dispersion curve has been terminated. We note, moreover, that Eq. (7) has no solution for $\text{Re } q = 0, \text{Im } q \neq 0$. These leaky waves will play an important role in the results obtained in this paper. Finally, the four horizontal dashed lines in Fig. 2 correspond to the four thicknesses of the film that will be as-

sumed in the numerical calculations whose results are presented in this paper, namely $d=250, 442.96, 500$, and 750 nm.

IV. SCATTERING THEORY

The vacuum-dielectric interface $x_3=\zeta(x_1)$ is illuminated from the vacuum side by an s -polarized electromagnetic wave of frequency ω . The only nonzero component of the electric vector in the region $x_3>\zeta(x_1)_{max}$ is the sum of an incident wave and a scattered field:

$$E_2^>(x_1, x_3|\omega) = \exp[ikx_1 - i\alpha_0(k)x_3] + \int_{-\infty}^{\infty} \frac{dq}{2\pi} R(q|k) \exp[iqx_1 + i\alpha_0(q)x_3], \quad (8)$$

where

$$\alpha_0(q) = \begin{cases} [(\omega^2/c^2) - q^2]^{1/2} & |q| < \omega/c \\ i[q^2 - (\omega^2/c^2)]^{1/2} & |q| > \omega/c. \end{cases} \quad (9)$$

Within the film, $-d < x_3 < \zeta(x_1)$, the single nonzero component of the electric vector has the form

$$E_2^<(x_1, x_3|\omega) = \int_{-\infty}^{\infty} \frac{dq}{2\pi} T(q|k) \exp(iqx_1) \sin \alpha(q)(x_3 + d), \quad (10)$$

where

$$\alpha(q) = [\epsilon(\omega^2/c^2) - q^2]^{1/2}. \quad (11)$$

The real part of $\alpha(q)$ is always positive; the imaginary part is positive when $\epsilon_2 > 0$, but is negative when $\epsilon_2 < 0$.

The differential reflection coefficient $\partial R / \partial \theta_s$ is defined such that $(\partial R / \partial \theta_s) d\theta_s$ is the fraction of the total time-averaged flux incident on the surface that is scattered into the angular interval $d\theta_s$ about the scattering angle θ_s , in the limit as $d\theta_s \rightarrow 0$. The contribution to the mean differential reflection coefficient from the coherent (specular) component of the scattered field is given by

$$\left\langle \frac{\partial R}{\partial \theta_s} \right\rangle_{\text{coh}} = \frac{1}{L_1} \frac{\omega}{2\pi c} \frac{\cos^2 \theta_s}{\cos \theta_0} |\langle R(q|k) \rangle|^2; \quad (12)$$

the contribution to the mean differential reflection coefficient from the incoherent (diffuse) component of the scattered field is given by

$$\left\langle \frac{\partial R}{\partial \theta_s} \right\rangle_{\text{incoh}} = \frac{1}{L_1} \frac{\omega}{2\pi c} \frac{\cos^2 \theta_s}{\cos \theta_0} [\langle |R(q|k)|^2 \rangle - |\langle R(q|k) \rangle|^2]. \quad (13)$$

In these expressions L_1 is the length of the x_1 axis covered by the random surface, and the wave numbers k and q are to be expressed in terms of the angle of incidence θ_0 and the scattering angle θ_s , both measured from the normal to the mean surface (Fig. 1), according to

$$k = (\omega/c) \sin \theta_0, \quad q = (\omega/c) \sin \theta_s. \quad (14)$$

The scattering amplitude $R(q|k)$ entering Eqs. (12) and (13) is the solution of the reduced Rayleigh equation¹⁷

$$\int_{-\infty}^{\infty} \frac{dq}{2\pi} M(p|q) R(q|k) = N(p|k), \quad (15)$$

where

$$M(p|q) = \frac{e^{i\alpha(p)d}}{\alpha_0(q) + \alpha(p)} I[\alpha_0(q) + \alpha(p)|p - q] - \frac{e^{-i\alpha(p)d}}{\alpha_0(q) - \alpha(p)} I[\alpha_0(q) - \alpha(p)|p - q], \quad (16a)$$

$$N(p|k) = -\frac{e^{i\alpha(p)d}}{\alpha(p) - \alpha_0(k)} I[\alpha(p) - \alpha_0(k)|p - k] - \frac{e^{-i\alpha(p)d}}{\alpha(p) + \alpha_0(k)} I[-\alpha(p) - \alpha_0(k)|p - k], \quad (16b)$$

where

$$I(\gamma|Q) = \int_{-\infty}^{\infty} dx_1 e^{-iQx_1} e^{i\gamma\zeta(x_1)}. \quad (17)$$

We seek a solution of Eq. (15) in the form¹⁷

$$R(q|k) = 2\pi \delta(q - k) R_0(k) - 2i G_0(q) \tau(q|k) G_0(k) \alpha_0(k), \quad (18)$$

where $R_0(k)$ is the Fresnel coefficient for the scattering of an s -polarized electromagnetic wave from a dielectric film deposited on a perfectly conducting substrate, when the vacuum-dielectric and dielectric-substrate interfaces are planar and parallel. It is given by

$$R_0(k) = \frac{-i\alpha(k) \cos \alpha(k)d + \alpha_0(k) \sin \alpha(k)d}{i\alpha(k) \cos \alpha(k)d + \alpha_0(k) \sin \alpha(k)d}. \quad (19)$$

The function $G_0(k)$ is a Green's function for the same dielectric film on a perfectly conducting substrate,

$$G_0(k) = \frac{i \sin \alpha(k)d}{i\alpha(k) \cos \alpha(k)d + \alpha_0(k) \sin \alpha(k)d}. \quad (20)$$

The transition matrix $\tau(q|k)$ is postulated to satisfy the equations

$$\tau(q|k) = V(q|k) + \int_{-\infty}^{\infty} \frac{dp}{2\pi} V(q|p) G_0(p) \tau(p|k) \quad (21a)$$

$$= V(q|k) + \int_{-\infty}^{\infty} \frac{dp}{2\pi} \tau(q|p) G_0(p) V(p|k). \quad (21b)$$

Equations (15), (18), and (21b) define the scattering potential $V(q|k)$ as the solution of the integral equation¹⁷

$$\int_{-\infty}^{\infty} \frac{dq}{2\pi} [M(p|q) + N(p|q)] \frac{V(q|k)}{2i\alpha_0(q)} \\ = \frac{M(p|k)D_-(k) - N(p|k)D_+(k)}{2\alpha_0(k)\sin\alpha(k)d}, \quad (22)$$

where

$$D_{\pm}(k) = \alpha_0(k)\sin\alpha(k)d \pm i\alpha(k)\cos\alpha(k)d. \quad (23)$$

We also introduce the Green's function $G(q|k)$ associated with the random surface defined by the equation $x_3 = \zeta(x_1)$. It is the solution of the equation

$$G(q|k) = 2\pi\delta(q-k)G_0(k) + G_0(q) \int_{-\infty}^{\infty} \frac{dp}{2\pi} V(q|p)G(p|k) \\ = 2\pi\delta(q-k)G_0(k) + G_0(q)\tau(q|k)G_0(k). \quad (24)$$

If we combine Eqs. (18) and (25), and use Eqs. (19) and (20), we find that the scattering amplitude $R(q|k)$ is related to the Green's function $G(q|k)$ by

$$R(q|k) = -2\pi\delta(q-k) - 2iG(q|k)\alpha_0(k). \quad (26)$$

The calculation of the contributions to the mean differential reflection coefficient from the coherent and incoherent components of the scattered field, Eqs. (12) and (13), requires knowledge of the averaged Green's function $\langle G(q|k) \rangle$. Due to the stationarity of the surface profile function $\zeta(x_1)$, $\langle G(q|k) \rangle$ must be diagonal in q and k , and we write it in the form

$$\langle G(q|k) \rangle = 2\pi\delta(q-k)G(k). \quad (27)$$

An application of the smoothing method¹⁸ to Eq. (24) yields the result that $G(k)$ is given by

$$G(k) = \frac{1}{G_0^{-1}(k) - M(k)} \quad (28a) \\ = \frac{i\sin\alpha(k)d}{i\alpha(k)\cos\alpha(k)d + \alpha_0(k)\sin\alpha(k)d - i\sin\alpha(k)dM(k)}. \quad (28b)$$

The proper self-energy $M(k)$ appearing in these expressions is defined by

$$\langle M(q|k) \rangle = 2\pi\delta(q-k)M(k), \quad (29a)$$

where the unaveraged self-energy $M(q|k)$ is the solution of

$$M(q|k) = V(q|k) + \int_{-\infty}^{\infty} \frac{dp}{2\pi} M(q|p)G_0(p) \\ \times [V(p|k) - \langle V(p|k) \rangle]. \quad (29b)$$

V. COHERENT AND INCOHERENT SCATTERING

With the preceding results in hand we now turn to obtaining expressions for $\langle \partial R / \partial \theta_s \rangle_{\text{coh}}$ and $\langle \partial R / \partial \theta_s \rangle_{\text{incoh}}$ that will serve as the bases for their calculations. From Eqs. (26) and

(27) we find that the averaged scattering amplitude $\langle R(q|k) \rangle$ is given by

$$\langle R(q|k) \rangle \\ = -2\pi\delta(q-k)[1 + 2i\alpha_0(k)G(k)] \quad (30a)$$

$$= 2\pi\delta(q-k) \\ \times \frac{\alpha_0(k)\sin\alpha(k)d - i\alpha(k)\cos\alpha(k)d + iM(k)\sin\alpha(k)d}{\alpha_0(k)\sin\alpha(k)d + i\alpha(k)\cos\alpha(k)d - iM(k)\sin\alpha(k)d}. \quad (30b)$$

The contribution to the mean differential reflection coefficient from the coherent component of the scattered field, Eq. (12), therefore becomes

$$\left\langle \frac{\partial R}{\partial \theta_s} \right\rangle_{\text{coh}} = \delta(\theta_s - \theta_0)R(\theta_0), \quad (31)$$

where the reflectivity $R(\theta_0)$ is given by

$$R(\theta_0) = \left| \frac{\mathcal{D}_-(\theta_0)}{\mathcal{D}_+(\theta_0)} \right|^2 \quad (32a)$$

with

$$\mathcal{D}_{\pm}(\theta_0) = \cos\theta_0 \sin \left[\frac{\omega d}{c} (\epsilon - \sin^2\theta_0)^{1/2} \right] \\ \pm i \left\{ (\epsilon - \sin^2\theta_0)^{1/2} \cos \left[\frac{\omega d}{c} (\epsilon - \sin^2\theta_0)^{1/2} \right] \right. \\ \left. - (c/\omega)M[(\omega/c)\sin\theta_0] \sin \left[\frac{\omega d}{c} (\epsilon - \sin^2\theta_0)^{1/2} \right] \right\}. \quad (32b)$$

In obtaining Eq. (31) we have used the result that in one dimension

$$[2\pi\delta(q-k)]^2 = L_1 \frac{2\pi c}{\omega} \frac{\delta(\theta_s - \theta_0)}{\cos\theta_0}. \quad (33)$$

The real part of $(\epsilon - \sin^2\theta_0)^{1/2}$ in Eq. (32b) is positive; the imaginary part is positive if $\epsilon_2 > 0$, and is negative if $\epsilon_2 < 0$.

Turning now to the incoherent scattering, we find that when we combine Eqs. (13) and (26), we can write the contribution to the mean differential reflection coefficient from the incoherent component of the scattered light as

$$\left\langle \frac{\partial R}{\partial \theta_s} \right\rangle_{\text{incoh}} = \frac{1}{L_1} \frac{2}{\pi} \left(\frac{\omega}{c} \right)^3 \cos^2\theta_s \cos\theta_0 [\langle |G(q|k)|^2 \rangle \\ - |\langle G(q|k) \rangle|^2], \quad (34)$$

where $k = (\omega/c)\sin\theta_0$ and $q = (\omega/c)\sin\theta_s$. The two-particle Green's function $\langle |G(q|k)|^2 \rangle$ satisfies the Bethe-Salpeter equation

$$\begin{aligned} \langle |G(q|k)|^2 \rangle &= L_1 2\pi \delta(q-k) |G(q)|^2 \\ &+ |G(q)|^2 \int_{-\infty}^{\infty} \frac{ds}{2\pi} U(q|s) \langle |G(s|k)|^2 \rangle, \end{aligned} \quad (35)$$

where $U(q|s)$ is the irreducible vertex function. The solution of Eq. (35) can be written formally as

$$\begin{aligned} \langle |G(q|k)|^2 \rangle &= L_1 2\pi \delta(q-k) |G(q)|^2 \\ &+ L_1 |G(q)|^2 X(q|k) |G(k)|^2, \end{aligned} \quad (36)$$

where $X(q|k)$ is the reducible vertex function, and is the solution of

$$X(q|k) = U(q|k) + \int_{-\infty}^{\infty} \frac{dp}{2\pi} U(q|p) |G(p)|^2 X(p|k). \quad (37)$$

Since $\langle |G(q|k)|^2 \rangle = L_1 2\pi \delta(q-k) |G(q)|^2$, then on combining Eqs. (34) and (36) we obtain the result that

$$\left\langle \frac{\partial R}{\partial \theta_s} \right\rangle_{\text{incoh}} = \frac{2}{\pi} \left(\frac{\omega}{c} \right)^3 \cos^2 \theta_s \cos \theta_0 |G(q)|^2 X(q|k) |G(k)|^2, \quad (38)$$

where we recall that $k = (\omega/c) \sin \theta_0$ and $q = (\omega/c) \sin \theta_s$.

In order to evaluate the expressions given by Eqs. (32) and (38) we need to know the scattering potential $V(q|k)$. We can solve the equation it satisfies, Eq. (22), as an expansion in powers of the surface profile function $\zeta(x_1)$. Thus if we write

$$V(q|k) = \sum_{n=1}^{\infty} V^{(n)}(q|k), \quad (39)$$

where the superscript denotes the order of the corresponding term in $\zeta(x_1)$, we obtain for the first two terms in this expansion¹⁷

$$V^{(1)}(q|k) = (\epsilon - 1) \frac{\omega^2}{c^2} \hat{\zeta}(q-k), \quad (40a)$$

$$\begin{aligned} V^{(2)}(q|k) &= \frac{1}{2} (\epsilon - 1) \frac{\omega^2}{c^2} [\alpha(q) \cot \alpha(q) d \\ &+ \alpha(k) \cot \alpha(k) d] \int_{-\infty}^{\infty} \frac{dp}{2\pi} \hat{\zeta}(q-p) \hat{\zeta}(p-k). \end{aligned} \quad (40b)$$

In what follows we will make the small roughness approximation.¹⁹ This consists of approximating $V(q|k)$ by $V^{(1)}(q|k)$. The results we obtain are therefore limited to weakly rough surfaces.

In the small roughness approximation the self-energy $M(k)$ defined by Eqs. (29) is given to lowest nonzero order in $\zeta(x_1)$ by

$$M(k) = W^2 \int_{-\infty}^{\infty} \frac{dp}{2\pi} g(|k-p|) G_0(p), \quad (41)$$

FIG. 3. The contribution to $U(q|k)$ from the sum of all maximally crossed diagrams.

where

$$W = \delta(\epsilon - 1) \frac{\omega^2}{c^2}. \quad (42)$$

This expression will be used in numerical calculations of the reflectivity and of the Green's function $G(k)$ entering the calculation of $\langle \partial R / \partial \theta_s \rangle_{\text{incoh}}$.

For the irreducible vertex function $U(q|k)$ we will use the sum of the contributions from all maximally crossed diagrams (Fig. 3). It is these contributions that describe the coherent interference between a given multiple-scattering sequence and its reciprocal partner that gives rise to such effects as enhanced backscattering and satellite peaks. In the diagrams presented in Fig. 3 a horizontal line labeled by a wave number Q and directed to the left denotes the Green's function $G(Q)$; a horizontal line labeled by a wave number Q and directed to the right denotes the Green's function $G^*(Q)$; a dashed line labeled by a wave number Q and directed downward denotes $|W|^2 g(|Q|)$. The sum of the wave numbers labeling the lines entering a vertex equals the sum of the wave numbers labeling the lines leaving it. The incoming and outgoing lines in each diagram labeled k and q , however, do not have Green's functions associated with them, and serve only to define the wave-number conservation conditions at the vertices they enter and leave. Finally, all wave numbers other than q and k are integrated over, with weight $(2\pi)^{-1}$. With the use of these rules we find that $U(q|k)$ is given by

$$\begin{aligned} U(q|k) &= |W|^2 g(|q-k|) + |W|^4 \int_{-\infty}^{\infty} \frac{dp_1}{2\pi} g(|k-p_1|) \\ &\times g(|p_1-q|) G(p_1) G^*(q+k-p_1) \\ &+ |W|^6 \int_{-\infty}^{\infty} \frac{dp_1}{2\pi} \int_{-\infty}^{\infty} \frac{dp_2}{2\pi} g(|k-p_1|) \\ &\times g(|p_1-p_2|) g(|p_2-q|) G(p_1) \\ &\times G^*(q+k-p_1) G(p_2) G^*(q+k-p_2) + \dots \end{aligned} \quad (43)$$

To evaluate this sum we proceed as follows. We write the power spectrum of the surface roughness $g(|q-k|)$ in the separable form

FIG. 4. The contribution to $X(q|k)$ from the sum of all ladder diagrams, starting with the two-rung diagram.

$$g(|q-k|) = \sqrt{\pi a} \exp[-a^2(q-k)^2/4] = \sum_{l=0}^{\infty} \phi_l(q) \phi_l(k), \quad (44a)$$

where

$$\phi_l(q) = \left(\frac{\sqrt{\pi a}^{2l+1}}{2^l l!} \right)^{1/2} q^l e^{-(a^2/4)q^2}. \quad (44b)$$

In numerical calculations the upper limit on the sum in Eq. (44a) is replaced by an integer N , which is increased until a convergent result for $U(q|k)$ is obtained. With the use of this representation we find that $U(q|k)$ can be represented as

$$U(q|k) = |W|^2 g(|q-k|) + |W|^2 \sum_{l=0}^N \sum_{l'=0}^N \phi_l(q) \times \{[\mathbf{I} - \mathbf{K}(q+k)]^{-1} \mathbf{K}(q+k)\}_{ll'} \phi_{l'}(k) \quad (45a)$$

$$\equiv |W|^2 g(|q-k|) + \Lambda_C(q|k). \quad (45b)$$

The elements of the matrix $\mathbf{K}(Q)$ appearing in Eq. (45a) are given by

$$K_{ll'}(Q) = |W|^2 \int_{-\infty}^{\infty} \frac{dp}{2\pi} \phi_l(p) G(p) G^*(Q-p) \phi_{l'}(p). \quad (46)$$

The result for $U(q|k)$ given by Eq. (45b) must now be substituted into Eq. (37), which is then solved by iteration:

$$X(q|k) = |W|^2 g(|q-k|) + \Lambda_C(q|k) + \int_{-\infty}^{\infty} \frac{dp_1}{2\pi} [|W|^2 g(|q-p_1|) + \Lambda_C(q|p_1)] |G(p_1)|^2 [|W|^2 g(|p_1-k|) + \Lambda_C(p_1|k)] + \dots \quad (47)$$

In summing this series we keep in the integrals only the terms containing products of $|W|^2 g$, and neglect all terms containing factors of Λ_C . The result is just the sum of the contributions from the ladder diagrams starting with the two-rung ladder diagram (Fig. 4), and we obtain for $X(q|k)$ the expression

$$X(q|k) = |W|^2 g(|q-k|) + \Lambda_C(q|k) + \Lambda_L(q|k), \quad (48)$$

where

$$\Lambda_L(q|k) = |W|^2 \sum_{l=0}^N \sum_{l'=0}^N \phi_l(q) \{[\mathbf{I} - \mathbf{K}(0)]^{-1} \mathbf{K}(0)\}_{ll'} \phi_{l'}(k). \quad (49)$$

In obtaining this result we have used the fact that $G(k)$ is an even function of k . We note that $\Lambda_C(q|k)$ and $\Lambda_L(q|k)$ are equal when $q = -k$, i.e., for scattering into the retroreflection direction. This is why we have included the contribution $\Lambda_L(q|k)$ in $X(q|k)$. Our result for $\langle \partial R / \partial \theta_s \rangle_{\text{incoh}}$ is finally

$$\left\langle \frac{\partial R}{\partial \theta_s} \right\rangle_{\text{incoh}} = \frac{2}{\pi} \left(\frac{\omega}{c} \right)^3 \cos^2 \theta_s \cos \theta_0 |G(q)|^2 [|W|^2 g(|q-k|) + \Lambda_C(q|k) + \Lambda_L(q|k)] |G(k)|^2, \quad (50)$$

where $k = (\omega/c) \sin \theta_0$ and $q = (\omega/c) \sin \theta_s$.

We note here that, although it was not necessary to do so, in calculating $M(k)$ from Eq. (41) to keep our approximations consistent we used the representation (44a), with the result that

$$M(k) = W^2 \sum_{l=0}^N \phi_l(k) \int_{-\infty}^{\infty} \frac{dp}{2\pi} \phi_l(p) G_0(p). \quad (51)$$

Since $G_0(p)$ is an even function of p , only even values of l give nonzero contributions to the sum on the right-hand side of this equation. In the numerical calculations of $M(k)$ values of $N=40-50$ were used.

When calculating numerically the self-energy and the elements of the matrix $\mathbf{K}$ the poles of the Green's functions $G_0(q)$ and $G(q)$ should be treated with utmost care, since they are situated in the close vicinity of, or even on, the real axis. To do this we chose to integrate numerically in the complex plane of the complex variable $q = q_1 + iq_2$, rather than along the real q axis. We deformed the integration path so that it left the real axis at the branch point $q = \omega/c$ of $\alpha_0(q)$ and entered the upper/lower half plane of the variable q depending on whether the poles of the Green's functions are in the lower/upper half plane, respectively, and returned to the real axis at a point well beyond the poles. Of course, the integrands should be analytically continued into the complex plane, and the deformation of the integration path should be carried out in accordance with the theory of complex variables. When calculating the elements of the matrix $\mathbf{K}$ one can encounter the case where the poles of the integrand $G(q)G^*(x-q)$ are in both the upper and lower halves of the complex plane. In this case, in accordance with the theory of complex variables, the residues at any poles that were crossed in the deformation of the contour should be calculated analytically. In our calculations we determined the poles of the Green's function $G_0(q)$ by finding the complex roots of the equation $G_0^{-1}(q) = 0$. The poles of the renormalized Green's function $G(q) = [G_0^{-1}(q) - M(q)]^{-1}$ were determined numerically using the Newton-Raphson procedure. Although $\mathbf{K}$ matrices as large as 9×9 were used in the numerical calculations of $\Lambda_C(q|k)$ and $\Lambda_L(q|k)$, accurate results were already obtained by the use of 4×4 matrices.

VI. NUMERICAL RESULTS AND DISCUSSION

The numerical calculations based on the preceding results have been carried out on the assumption that the wavelength of the incident light is $\lambda = 632.8$ nm, and that the mean thickness of the film is $d = 250, 442.96, 500$, and 750 nm. The rms height of each random surface has been chosen to

FIG. 5. The reflectivity of an amplifying film with a random surface (—) and a planar surface (---). $\lambda = 632.8$ nm, $d = 250$ nm, $\delta = 15$ nm, $a = 100$ nm, and $\epsilon_1 = 2.6896$. (a) $\epsilon_2 = -2.5 \times 10^{-5}$; (b) $\epsilon_2 = -1 \times 10^{-5}$; (c) $\epsilon_2 = -0.5 \times 10^{-5}$.

be $\delta = 15$ nm, while the transverse correlation length of the surface roughness is $a = 100$ nm. The real part of the dielectric constant of the film has been chosen to be $\epsilon_1 = 2.6896$, while its imaginary part varies from $\epsilon_2 = -2.5 \times 10^{-5}$ to $\epsilon_2 = 2.5 \times 10^{-5}$. The gain factor for an amplifying medium, which is given by

$$g = \frac{2\pi}{\lambda} \frac{|\epsilon_2|}{\sqrt{\epsilon_1}},$$

for the values of ϵ_2 chosen for study reaches its maximum value $g = 1.51 \text{ cm}^{-1}$ for $\epsilon_2 = -2.5 \times 10^{-5}$. Such values of g are achievable experimentally, for example, in Ti:sapphire optically pumped by a frequency-doubled Nd:YAG laser whose wavelength is 532 nm.¹⁴

We consider first the deviation from unity, $\Delta R(\theta_0) = 1 - R(\theta_0)$, of the reflectivity of an amplifying film with a random surface. In Figs. 5(a)–(c) we present $\Delta R(\theta_0)$ as a function of the angle of incidence for a film of mean thickness $d = 250$ nm, for $\epsilon_2 = -2.5 \times 10^{-5}$, -1×10^{-5} , and -0.5×10^{-5} , respectively, and compare these results with those for the same film but in the absence of roughness. From Fig. 2 we see that for this film thickness the film supports one guided wave and no leaky waves. For each value of ϵ_2 the reflectivity [$R(\theta_0) = 1 - \Delta R(\theta_0)$] of the film with a random vacuum-dielectric interface is greater than that of the film with a planar vacuum-dielectric interface. The reflectiv-

FIG. 6. The same as Fig. 5, but for $d = 500$ nm.

ity of the planar film is a monotonically decreasing function of the angle of incidence θ_0 , while that of the film with a random vacuum-dielectric interface is a nonmonotonic function of θ_0 . However, both are greater than unity for all angles of incidence.

When the mean thickness of the film is increased to 500 nm, it supports two guided waves and one leaky wave. The reflectivities for $\epsilon_2 = -2.5 \times 10^{-5}$, -1×10^{-5} , and -0.5×10^{-5} are plotted in Figs. 6(a)–(c). In this case we see that the reflectivities of the films with a planar vacuum-dielectric interface are larger than unity while those of the films with the random vacuum-dielectric interfaces are smaller than unity, although greater than the reflectivities for the same films with $\epsilon_2 = 2.5 \times 10^{-5}$, 1×10^{-5} , and 0.5×10^{-5} for each angle of incidence (not shown). In addition, both reflectivities are nonmonotonic functions of the angle of incidence. The same qualitative behavior is displayed by films whose mean thickness is 750 nm, and which, according to Fig. 2, support three guided waves and a leaky wave [Figs. 7(a)–(c)].

What accounts for the difference between the results plotted in Fig. 5, where the films with the random vacuum-dielectric interface have a larger reflectivity than the corresponding films with a planar vacuum-dielectric interface, and the results plotted in Figs. 6 and 7 where the opposite is the case? It appears to be that in the first case the film supports only guided waves but no leaky wave, while in the other two cases it supports a leaky wave in addition to guided waves.

This conjecture is supported by the results presented in Fig. 8 where the reflectivities for films with a mean thickness

FIG. 7. The same as Fig. 5, but for $d = 750$ nm.

$d = 442.96$ nm are plotted. According to Fig. 2 a film of this thickness supports two guided waves and no leaky wave. For each value of ϵ_2 the film with a random vacuum-dielectric interface has the larger reflectivity than the film with a planar vacuum-dielectric interface.

We can explain these results qualitatively in the following way. The absorptivity [$\Delta R(\theta_0) = 1 - R(\theta_0)$] of a planar film on a perfectly conducting substrate is proportional to the imaginary part of the dielectric function of the film. When a leaky mode is excited, and if ϵ_2 is positive, $\Delta R(\theta_0)$ displays a peak in the absorptivity at the angle of incidence corresponding to the wave number of the leaky wave. The reflectivity, consequently, will display a dip at the corresponding wave number. If ϵ_2 is negative, i.e., if the film is amplifying rather than absorbing, the absorptivity becomes negative, and there will be a dip at the angle of excitation of the leaky wave. The corresponding reflectivity will display a peak at this angle. In the presence of the surface roughness, the absorptivity $\Delta R(\theta_0)$ becomes proportional to the sum of the imaginary parts of the dielectric function of the film and of the self-energy, with a positive coefficient. Therefore the absorptivity of the film is now determined by the competition between a contribution proportional to the imaginary part of the dielectric constant of the film ϵ_2 , and the imaginary part of the self-energy. If no leaky wave is present, only guided waves, the imaginary part of the self-energy in the radiative region is proportional to ϵ_2 with a positive coefficient. Thus the absorptivity of an amplifying film is negative both in the case of a planar surface and in the case of a random surface, but it is more negative in the latter case due to the contribu-

FIG. 8. The same as Fig. 5, but for $d = 442.9$ nm.

tion from the self-energy. Therefore the reflectivity is larger when the surface is rough than when the surface is planar. The situation is different when a leaky wave is present. In contrast to guided waves, the decay rate of the leaky wave is positive, independent of the sign of ϵ_2 , even for an unrealistically high gain in an amplifying medium. As a result, the self-energy is positive throughout the radiative region. The positive self-energy outweighs the negative contribution associated with a negative ϵ_2 , for the small values of ϵ_2 assumed here, so that the absorptivity of the film is positive and will display a peak associated with the excitation of the leaky wave. A dip therefore occurs in the reflectivity at the corresponding angle of incidence even in the case of an amplifying film.

These qualitative arguments can be justified and made more quantitative, if we note that from Eqs. (12), (30a), and (33) the reflectivity can be written equivalently as

$$R(\theta_0) = |1 + 2i\alpha_0(k)G(k)|^2 \quad (52a)$$

$$= 1 - 4\alpha_0(k)\text{Im} G(k) + 4\alpha_0^2(k)|G(k)|^2, \quad (52b)$$

where $k = (\omega/c)\sin\theta_0$. The function $\Delta R(\theta_0) = 1 - R(\theta_0)$, which describes losses in the film, can be written in the form

$$\Delta R(\theta_0) = -4\alpha_0(k)|G(k)|^2 \text{Im}[G_0^{-1}(k) - M(k) + i\alpha_0(k)]. \quad (53)$$

In the present case, where $|\epsilon_2| \ll 1$, we find that

$$\begin{aligned} \Delta R(\theta_0) &\simeq 4\alpha_0 \\ &\times \frac{M_2 \sin^2 \alpha_1 d + \alpha_2(\alpha_1 d - \sin \alpha_1 d \cos \alpha_1 d)}{(\alpha_1 \cos \alpha_1 d - M_1 \sin \alpha_1 d)^2 + (\alpha_0 + M_2)^2 \sin^2 \alpha_1 d}, \end{aligned} \quad (54)$$

where

$$\alpha_1(k) = \text{Re } \alpha(k) \simeq \left(\epsilon_1 \frac{\omega^2}{c^2} - k^2 \right)^{1/2} > 0, \quad (55a)$$

$$\alpha_2(k) = \text{Im } \alpha(k) \simeq \epsilon_2 \frac{\omega^2/c^2}{2\alpha_1(k)}, \quad (55b)$$

$$M_1(k) = \text{Re } M(k), \quad M_2(k) = \text{Im } M(k). \quad (55c)$$

To simplify the notation we have suppressed the argument k in writing Eq. (54).

We first consider the case where the vacuum-dielectric interface is planar [$M(k) \equiv 0$]. The function $\Delta R(\theta_0)$ then takes the form

$$\Delta R(\theta_0)_{pl} = 4\alpha_0 \alpha_2 \frac{(\alpha_1 d - \sin \alpha_1 d \cos \alpha_1 d)}{\alpha_1^2 \cos^2 \alpha_1 d + \alpha_0^2 \sin^2 \alpha_1 d}. \quad (56)$$

Then, since $\alpha_1 d - \sin \alpha_1 d \cos \alpha_1 d$ is non-negative, we see from Eq. (56) that $\Delta R(\theta_0)_{pl}$ is positive when $\epsilon_2 > 0$, and is

negative when $\epsilon_2 < 0$. This means that $R(\theta_0)_{pl}$ is smaller than unity when $\epsilon_2 > 0$ and is larger than unity when $\epsilon_2 < 0$. In addition, if the scattering structure supports a leaky wave, the real part of its wave number at the frequency ω of the incident light is the value of k at which the denominator $\alpha_1^2 \cos^2 \alpha_1 d + \alpha_0^2 \sin^2 \alpha_1 d$ in Eq. (56) is a minimum. As a result, $|\Delta R(\theta_0)_{pl}|$ has a broad maximum associated with the excitation of the leaky wave, so that $R(\theta_0)_{pl}$ has a minimum when $\epsilon_2 > 0$ and a maximum when $\epsilon_2 < 0$. In the absence of a leaky wave $|\Delta R(\theta_0)_{pl}|$ is a monotonically decreasing function of $k = (\omega/c) \sin \theta_0$, that vanishes at $\theta_0 = 90^\circ$ due to the factor α_0 , so that $R(\theta_0)_{pl}$ decreases from its maximum value at $\theta_0 = 0^\circ$, to a value of unity at $\theta_0 = 90^\circ$ when $\epsilon_2 < 0$, and increases from its minimum value, at $\theta_0 = 0^\circ$, to a value of unity at $\theta_0 = 90^\circ$ when $\epsilon_2 > 0$. These conclusions are in agreement with the results for the reflectivity of a planar amplifying surface presented in Figs. 5–8.

Turning now to the case where the vacuum-dielectric interface is randomly rough, $M(k) \neq 0$, we write the self-energy in the form

$$\begin{aligned} M(k) = & W^2 \int_{-\infty}^{\infty} \frac{dp}{2\pi} g(|k-p|) \text{Re } G_0(p) \\ & + i W^2 \int_{-\infty}^{\infty} \frac{dp}{2\pi} g(|k-p|) \text{Im } G_0(p). \end{aligned} \quad (57)$$

For $|\epsilon_2| \ll 1$, the imaginary part of $G_0(p)$ takes the following forms in the indicated regions of p :

$$\text{Im } G_0(p) = \frac{\alpha_0 \sin^2 \alpha_1 d + \alpha_2(\alpha_1 d - \sin \alpha_1 d \cos \alpha_1 d)}{\alpha_0^2 \sin^2 \alpha_1 d + \alpha_1^2 \cos^2 \alpha_1 d} \quad |p| < \frac{\omega}{c} \quad (58a)$$

$$= \frac{\alpha_2(\alpha_1 d - \sin \alpha_1 d \cos \alpha_1 d)}{(\beta_0 \sin \alpha_1 d + \alpha_1 \cos \alpha_1 d)^2 + \alpha_2^2(\beta_0 d \cos \alpha_1 d + \cos \alpha_1 d - \alpha_1 d \sin \alpha_1 d)^2} \quad \frac{\omega}{c} < |p| < \sqrt{\epsilon_1} \frac{\omega}{c} \quad (58b)$$

$$= \frac{\beta_2(\sinh \beta_1 d \cosh \beta_1 d - \beta_1 d)}{(\beta_0 \sinh \beta_1 d + \beta_1 \cosh \beta_1 d)^2 + \beta_2^2(\beta_0 d \cosh \beta_1 d + \cosh \beta_1 d + \beta_1 d \sinh \beta_1 d)^2} \quad |p| > \sqrt{\epsilon_1} \frac{\omega}{c}, \quad (58c)$$

where

$$\beta_1(p) = \left(p^2 - \epsilon_1 \frac{\omega^2}{c^2} \right)^{1/2}, \quad \beta_2(p) = \epsilon_2 \frac{(\omega/c)^2}{2\beta_1(p)}. \quad (59)$$

It follows therefore that when $\epsilon_2 > 0$, $\text{Im } M(k) = M_2(k)$ is positive, because $\alpha_1 d - \sin \alpha_1 d \cos \alpha_1 d > 0$ and $\sinh \beta_1 d \cosh \beta_1 d - \beta_1 d > 0$. Therefore since $|G(k)|^2$ differs very little from $|G_0(k)|^2$ for the roughness parameters and the values of ϵ_2 assumed here, it can be seen from Eq. (53), that $\Delta R(\theta_0)_{pl} < \Delta R(\theta_0)_{rough}$ [$R(\theta_0)_{pl} > R(\theta_0)_{rough}$] for all angles of incidence except those defined by the condition $\sin \alpha_1(k)d = 0$, i.e., at which

$$\sin \theta_0 = \left[\epsilon_1 - \left(n \frac{\pi c}{d\omega} \right)^2 \right]^{1/2} \quad n = 1, 2, \dots \quad (60)$$

At these angles the Green's function $G(k)$ vanishes when $\epsilon_2 = 0$, and the reflectivity of the rough surface then equals that of the planar surface. When $\epsilon_2 > 0$, as in the case we are considering, $G(k)$ no longer vanishes at the angles of incidence given by Eq. (60), and the reflectivity of the rough surface remains smaller than that of the planar surface.

When $\epsilon_2 < 0$, $\text{Im } M(k)$ can be positive or negative. This case can be analyzed with the aid of the pole approximation for $G_0(p)$ in the region $|p| > (\omega/c)$ given by

$$G_0(p) = \sum_j' \frac{C_j(\omega)}{p - q_j(\omega) - i\Delta_j^{(\epsilon)}(\omega)}, \quad (61)$$

where $q_j(\omega)$ is the real part of the wave number of the j th waveguide mode at the frequency ω of the incident light, and

$\Delta_j^{(\epsilon)}(\omega)$ is the decay (amplification) rate of the j th waveguide mode when $\epsilon_2 > 0$ ($\epsilon_2 < 0$). The sum on j runs from $-n$ to n , where n is the number of waveguide modes supported by the scattering structure, and the prime on the sum means that the term with $n=0$ is omitted. The functions $C_j(\omega)$, $q_j(\omega)$, and $\Delta_j^{(\epsilon)}(\omega)$ entering Eq. (61) satisfy the relations $C_{-j}(\omega) = -C_j(\omega)$, $q_{-j}(\omega) = -q_j(\omega)$, and $\Delta_{-j}^{(\epsilon)}(\omega) = -\Delta_j^{(\epsilon)}(\omega)$. It follows from Eqs. (57) and (61) that $\text{Im} M(k)$ can be written in the form

$$\text{Im} M(k) \approx \text{Im} M_r(k) + \frac{1}{2} \sum_j C_j W^2 g(|k - q_j|) \text{sgn} \Delta_j^{(\epsilon)}, \quad (62)$$

where $M_r(k)$ is the contribution to $M(k)$ from the radiative region [$|p| < (\omega/c)$ in Eq. (57)]. It contains a contribution from leaky waves, if any exist at the frequency ω of the incident light. $\text{Im} M_r(k)$ is positive for both $\epsilon_2 > 0$ and $\epsilon_2 < 0$. In contrast, the contribution from the poles of $G_0(p)$, which occur in the nonradiative region $|p| > (\omega/c)$, changes its sign when ϵ_2 changes its sign, since $\Delta_j^{(\epsilon)} > 0$ when $\epsilon_2 > 0$, and $\Delta_j^{(\epsilon)} < 0$ when $\epsilon_2 < 0$. In the absence of leaky waves $\text{Im} M_r(k)$ is small, so that $\text{Im} M(k) > 0$ when $\epsilon_2 > 0$, and $\text{Im} M(k) < 0$ when $\epsilon_2 < 0$. In this case $|\Delta R(\theta_0)_{\text{rough}}| > |\Delta R(\theta_0)_{\text{pl}}|$, so that $R(\theta_0)_{\text{rough}} > R(\theta_0)_{\text{pl}}$ for all angles of incidence, except in the immediate vicinity of those defined by the condition $\sin \alpha_1(k)d = 0$, where the inequality can be reversed. However, in the latter case the difference $R(\theta_0)_{\text{pl}} - R(\theta_0)_{\text{rough}} > 0$ is too small to show up on the scale of Fig. 8.

When a leaky wave is present, its contribution to the integral for $M(k)$ becomes important. $\text{Im} M_r(k)$ is quite large in this case and, as a result, the pole contributions cannot change the sign of $\text{Im} M(k)$ when the sign of ϵ_2 is changed. Thus $\text{Im} M(k) > 0$ both when $\epsilon_2 > 0$ and when $\epsilon_2 < 0$. Therefore we have that $|\Delta R(\theta_0)_{\text{rough}}| < |\Delta R(\theta_0)_{\text{pl}}|$ when $\epsilon_2 < 0$, so that $R(\theta_0)_{\text{rough}} < R(\theta_0)_{\text{pl}}$, and $R(\theta_0)_{\text{rough}}$ is smaller than unity for the values of ϵ_2 used in our calculations. For larger values of $|\epsilon_2|$ the reflectivity of the film with the rough vacuum-film interface becomes larger than unity, and eventually reaches and becomes larger than that of the planar film. In addition, the roughness of the vacuum-film interface influences strongly the damping of the leaky mode. For example, in the planar film of thickness $d = 250$ nm, the leaky wave is overdamped, and in Fig. 2 we have terminated its dispersion curve at the point where $\text{Im} q = \text{Re} q$. There is therefore no intersection of this portion of its dispersion curve with the horizontal line corresponding to the value of $d = 250$ nm. Consequently, the reflectivity of this film does not display a maximum associated with the excitation of a leaky mode. In the presence of surface roughness, however, the value of $\text{Im} q$ corresponding to a given value of $\text{Re} q$ is reduced from its value for a planar film. The dispersion curve for the leaky wave can therefore be continued to smaller values of $\text{Re} q$ before $\text{Im} q$ equals $\text{Re} q$, and as a result intersects the horizontal line corresponding to $d = 250$ nm. As a result, the reflectivity of a rough film with this mean thickness displays a maximum associated with excitation of the leaky mode.

Turning now to incoherent scattering, in Fig. 9 we have

FIG. 9. The contribution to the mean differential reflection coefficient from the incoherent component of the light scattered from a dielectric film with a random surface. $\lambda = 632.8$ nm, $\theta_0 = 0^\circ$, $d = 250$ nm, $\delta = 15$ nm, $a = 100$ nm, and $\epsilon_1 = 2.6896$, for six positive and negative values of ϵ_2 .

plotted the contribution to the mean differential reflection coefficient (DRC) for s -polarized light incident normally on a film of mean thickness $d = 250$ nm as ϵ_2 takes the values 2.5×10^{-5} , 1×10^{-5} , 0.5×10^{-5} , -0.5×10^{-5} , -1×10^{-5} , and -2.5×10^{-5} . The remaining experimental and roughness parameters have the values given above. From the results plotted we see that for each value of $|\epsilon_2|$, the DRC corresponding to the negative value of ϵ_2 is larger in magnitude than the DRC corresponding to the positive value of ϵ_2 ; as $|\epsilon_2|$ increases the DRC of the absorbing film decreases in magnitude as does the DRC of the amplifying film; the height of the enhanced backscattering peak decreases with increasing $|\epsilon_2|$ both for the absorbing and amplifying films, while its width increases with increasing $|\epsilon_2|$ both for the absorbing film and for the amplifying film. However, for each value of $|\epsilon_2|$, the enhanced backscattering peak corresponding to the negative value of ϵ_2 is narrower than the peak corresponding to the positive value of ϵ_2 . The last results are more readily seen from the results plotted in Fig. 10, where the enhanced backscattering peaks in Fig. 9 have been shifted vertically so that their maxima coincide.

When the mean thickness of the dielectric film is increased to $d = 442.96$ nm, the resulting DRC displays the same qualitative properties present in the results plotted in Fig. 9 (Fig. 11). However, we also see that for each value of $|\epsilon_2|$ a pair of very weakly defined satellite peaks is present at

FIG. 10. The enhanced backscattering peaks present in the results plotted in Fig. 9 shifted vertically so that their maxima coincide.

$\theta_s \cong \pm 22^\circ$, and a minimum is present in the corresponding DRC at scattering angles $\theta_s \cong \pm 53^\circ$.

The results presented in Figs. 9 and 11 correspond to scattering systems that support one and two guided waves, respectively, but no leaky wave. When the mean thickness of

FIG. 12. The same as Fig. 9, but for $d = 500$ nm.

FIG. 11. The same as Fig. 9, but for $d = 442.96$ nm.

the film is increased to $d = 500$ nm (Fig. 12) the scattering system supports a leaky wave in addition to two guided waves. The DRC of the amplifying film is larger than that of the corresponding absorbing film for each value of $|\epsilon_2|$; as $|\epsilon_2|$ is increased the magnitude of the DRC decreases for the absorbing film and increases for the amplifying film; the height of the enhanced backscattering peak decreases/increases with increasing $|\epsilon_2|$; and its width increases/decreases, for the absorbing/amplifying film. The two satellite peaks at $\theta_s \cong \pm 17.7^\circ$ are more pronounced in Fig. 12 than they are in Fig. 11. In addition, the background to the enhanced backscattering and satellite peaks now has a minimum at $\theta_s = 0^\circ$ instead of a maximum as in Figs. 9 and 11.

The preceding results can be understood if we recall that the height of the enhanced backscattering peak is proportional to $|\Delta_{gw}|^{-1}$, and its width is proportional to $|\Delta_{gw}|$, where $|\Delta_{gw}|$ is the smallest decay/amplification rate of the guided waves supported by the film in the presence of the surface roughness.¹⁷ With the use of the pole approximation Δ_{gw} can be written in the form

$$\Delta_{gw} = \Delta^{(\epsilon)} + C \operatorname{Im} M(k_{gw})$$

$$= \Delta^{(\epsilon)} + C \operatorname{Im} M_r(k_{gw}) + \frac{1}{2} C^2 W^2 g(0) \operatorname{sgn} \Delta^{(\epsilon)}.$$

(63)

FIG. 13. The enhanced backscattering peaks present in the DRC for an amplifying film of thickness $d = 258$ nm, calculated for seven values of ϵ_2 . These peaks have been shifted vertically so that their maxima coincide.

We have seen earlier that the function $\Delta^{(\epsilon)}$ is positive when ϵ_2 is positive and is negative when ϵ_2 is negative. We have also seen that in the absence of a leaky wave $\text{Im } M(k) < 0$ when $\epsilon_2 < 0$, so that $\Delta_{gw} < 0$, $|\Delta_{gw}|$ increases with increasing ϵ_2 , the height of the enhanced backscattering peak decreases, and its width increases. When a leaky wave is present $\text{Im } M_r(k)$ is positive and becomes important. When $\epsilon_2 < 0$ and $|\epsilon_2|$ is very small, $|\Delta^{(\epsilon)}|$ is small compared to $C \text{Im } M(k_{gw})$, and Δ_{gw} is therefore positive; it decreases with increasing $|\epsilon_2|$, passes through zero and, finally, $|\Delta_{gw}|$ begins to increase. The height of the enhanced backscattering peak in this case first increases with increasing $|\epsilon_2|$ until $|\epsilon_2|$ reaches the value at which Δ_{gw} vanishes, at which point the height of the enhanced backscattering peak reaches its maximum and then decreases with a further increase of $|\epsilon_2|$. This behavior is not seen in the results plotted in Fig. 12, which shows a monotonic increase of the height of the enhanced backscattering peak with increasing $|\epsilon_2|$ when $\epsilon_2 < 0$. This is because the values of $|\epsilon_2|$ used in the numerical calculations are too small to overcome the effect of the presence of the leaky wave (the value of ϵ_2 at which the height of the peak reaches its maximum is $\epsilon_2 = -0.00262$ and is too large to be considered as experimentally achievable). We illustrate the nonmonotonic dependence of the width of the enhanced backscattering peak as a function of ϵ_2 in the results plotted in Fig. 13, where the enhanced backscattering peaks obtained from DRC's calculated for a film of the thickness $d = 258$ nm for $\epsilon_2 = -2.5 \times 10^{-5}$, -5×10^{-5} , -1×10^{-4} , -1.5×10^{-4} , -2×10^{-4} , -2.5×10^{-4} , and -3×10^{-4} , have been shifted vertically so that their maxima coincide. In this case the value of ϵ_2 at which the height of the enhanced backscattering peak reaches its maximum is about $\epsilon_2 = -1.2 \times 10^{-4}$, and the gain is $g = 7.23 \text{ cm}^{-1}$. It is seen that the width narrows significantly as ϵ_2 varies from -2.5×10^{-5} to -1×10^{-4} , and then broadens as ϵ_2 varies

from -1.5×10^{-4} to -3×10^{-4} .

Finally, the changes in the overall shape of the background of the DRC as the mean thickness of the dielectric film is increased, from a bell-shaped curve in Fig. 9 to a bell-shaped curve with subsidiary minima at $\theta_s \cong \pm 53^\circ$ in Fig. 11, and then to a curve with a minimum at $\theta_s = 0^\circ$ and shoulders at $\theta_s \cong \pm 25^\circ$ in Fig. 12, are caused by the changes in the dependence of the factor $|G(q)|^2 = |G((\omega/c) \sin \theta_s)|^2$ in Eq. (50) on the scattering angle θ_s as the mean thickness of the film is increased. The subsidiary minima in Fig. 11 are due to the minima of the Green's function given by Eq. (60), while the shoulders in Fig. 12 are due to the excitation of the leaky wave existing in the film of 500-nm mean thickness.

VII. CONCLUSIONS

In this paper we have studied the reflectivity of an amplifying dielectric film, deposited on the planar surface of a perfect conductor, and illuminated by s -polarized light, in the cases that the vacuum-dielectric interface is planar or a one-dimensional, randomly rough interface. The reflectivity of an amplifying film with a planar vacuum-dielectric interface, is greater than unity for all angles of incidence, irrespective of whether or not the film supports a leaky wave at the frequency of the incident light, in addition to the guided waves at that frequency that it supports. In contrast, for the experimental and roughness parameters assumed in this work, it has been found that the reflectivity of an amplifying film with a randomly rough vacuum-dielectric interface depends strongly on whether or not the film supports a leaky wave in addition to guided waves. If it does, the reflectivity of the rough film is smaller than that of the corresponding film with a planar vacuum-dielectric interface, and can be smaller than unity for some range of values of the angle of incidence. If it does not, the reflectivity of the rough film is larger than that of the corresponding film with a planar vacuum-dielectric interface, and is therefore larger than unity for all angles of incidence.

We have also shown that the contribution to the mean differential reflection coefficient (DRC) from the incoherent component of the scattered light can depend significantly on whether the scattering system does or does not support a leaky wave at the frequency of the incident light. If no leaky wave is present in addition to guided waves, the overall scattered intensity and the height of the enhanced backscattering peak decrease monotonically with increasing $|\epsilon_2|$, while the width of the enhanced backscattering peak increases monotonically with increasing $|\epsilon_2|$ for both absorbing and amplifying media. When a leaky wave is present the overall scattered intensity of an absorbing film and the height of the enhanced backscattering peak decreases monotonically with increasing ϵ_2 , while the width of the enhanced backscattering peak increases monotonically with increasing ϵ_2 . In contrast, for an amplifying film the overall scattered intensity and the height of the enhanced backscattering peak initially increase with increasing $|\epsilon_2|$, and then decrease, while the width of the enhanced backscattering peak initially decreases with increasing $|\epsilon_2|$, and then increases. However, whether a leaky wave is present or not, the DRC for an amplifying film with a given value of $|\epsilon_2|$ is larger than the DRC for an

absorbing film with the same magnitude of ϵ_2 but of opposite sign.

The qualitative results obtained in this work for the reflectivity of an amplifying film with a random surface and for the angular dependence of the intensity of the light scattered incoherently from such a film, should be observable experimentally.

ACKNOWLEDGMENTS

This research was supported in part by Army Research Office Grant No. DAAG 55-98-C-0034. The research of A.V.T. was supported in part by a COBASE grant. The research of J.A.S.-G. was supported in part by DGEIC Grant No. PB97-1221.

*Permanent address: B. I. Verkin Institute for Low Temperature Physics and Engineering, National Academy of Sciences of Ukraine, 47 Lenin Avenue, 310164 Kharkov, Ukraine.

¹A.Yu. Zyuzin, *Europhys. Lett.* **26**, 517 (1994).

²P. Pradhan and N. Kumar, *Phys. Rev. B* **50**, 9644 (1994).

³A.Yu. Zyuzin, *Pis'ma Zh. Éksp. Teor. Fiz.* **61**, 961 (1995) [*JETP Lett.* **61**, 990 (1995)].

⁴A.Yu. Zyuzin, *Phys. Rev. E* **51**, 5274 (1994).

⁵Z.Q. Zhang, *Phys. Rev. B* **52**, 7960 (1995).

⁶S. Feng and Z.Q. Zhang, *Phys. Rev. B* **54**, 3240 (1996).

⁷S. John and G. Pang, *Phys. Rev. A* **54**, 3642 (1996).

⁸C.W.J. Beenakker, J.C.J. Paasschens, and P.W. Brouwer, *Phys. Rev. Lett.* **76**, 1368 (1996).

⁹J.C.J. Paasschens, T.Sh. Misirpashaev, and C.W.J. Beenakker, *Phys. Rev. B* **54**, 11 887 (1996).

¹⁰V. Freilikher, M. Pustilnik, and I. Yurkevich, *Phys. Rev. B* **56**, 5974 (1997).

¹¹A.Z. Genack and J.M. Drake, *Nature (London)* **368**, 400 (1994).

¹²N.M. Lawandy, R.M. Balachandran, A.S.L. Gomes, and E. Sauvain, *Nature (London)* **368**, 436 (1994).

¹³W.L. Sha, C.-H. Liu, and R.R. Alfano, *Opt. Lett.* **19**, 1922 (1994).

¹⁴D.S. Wiersma, M.P. van Albada, and A. Lagendijk, *Phys. Rev. Lett.* **75**, 1739 (1995).

¹⁵D.S. Wiersma and A. Lagendijk, *Phys. World* **10**, 33 (1997).

¹⁶M. Kempe, G. A. Berger, and A. Z. Genack, in *Handbook of Optical Properties*, edited by R. E. Hummel and P. Wissman (CRC Press, Boca Raton, FL, 1997), Vol. II, pp. 301–330.

¹⁷J.A. Sánchez-Gil, A.A. Maradudin, Jun Q. Lu, V.D. Freilikher, M. Pustilnik, and I. Yurkevich, *Phys. Rev. B* **50**, 15 353 (1994).

¹⁸G. Brown, V. Celli, M. Haller, A.A. Maradudin, and A. Marvin, *Phys. Rev. B* **31**, 4993 (1985), Appendix A.

¹⁹F. Toigo, A. Marvin, V. Celli, and N.R. Hill, *Phys. Rev. B* **15**, 5618 (1977).