[7] Skura, T., *On decision procedures for sentential logics*, ***Studia Logica***, vol.50 (1991), no.2, pp. 173–179.

[8] Smiley, Timothy, *Rejection*, ***Analysis (Oxford)***, vol.56 (1996), no.1, pp. 1–9.

▸ PILAR DELLUNDE, ÀNGEL GARCíA-CERDAÑA, AND CARLES NOGUERA, *Advances on elementary equivalence in model theory of fuzzy logics.*
Dep. of Philosophy, Universitat Autònoma de Barcelona, Campus UAB, 08193 Bellaterra, Catalonia, Spain.
E-mail: pilar.dellunde@uab.cat.
Artificial Intelligence Research Institute (IIIA), Spanish National Research Council (CSIC), Campus UAB, 08193 Bellaterra, Catalonia, Spain
Dep. of Information and Communication Technologies, University Pompeu Fabra, Tànger 122-140, 08018 Barcelona, Catalonia, Spain.
E-mail: angel@iiia.csic.es, angel.garcia@upf.edu.
Institute of Information Theory and Automation, Academy of Sciences of the Czech Republic, Pod vodárenskou věží 4, 182 00 Prague, Czech Republic.
E-mail: noguera@utia.cas.cz.

Elementary equivalence is a central notion in classical model theory that allows to classify first-order structures. It was defined by Tarski in [10] who, together with Vaught, also proved fundamental results on elementary extensions and elementary chains in [11]. Later it has received several useful characterizations, among others, in terms of systems of back-and-forth, and has yielded many important results like the general forms of Löwenheim-Skolem theorems. (For general surveys on the subject and historical overviews we refer the reader to [1, 8].)

In the context of fuzzy predicate logics, the notion of elementarily equivalent structures was defined in [7]. There the authors presented a characterization of conservative extension theories using the elementary equivalence relation (see Theorems 6 and 11 of [7]). A related approach is the one presented in [9] where models can be elementary equivalent *in a degree d*. Following the definitions of [7], a few recent papers have contributed to the development of model theory of predicate fuzzy logics (see e.g. [4, 3]). However, the understanding of the central notion of elementary equivalence is still far from its counterpart in classical model theory. The present contribution intends to provide some advances towards this goal. After some preliminaries on first-order fuzzy logics in the first section, we list some of our new results in Section 2.

§1. The framework. In the following let L be a fixed core semilinear logic in a propositional language $\mathcal{L}$ (i.e. an expansion of the logic SL of [2], possibly with additional connectives with a congruence property, that is complete with respect to a semantics of linearly ordered algebras). The language of a first-order extension of L is defined in the same way as in classical first-order logic. A *predicate language* $\mathcal{P}$ is a triple $\langle Pred_\mathcal{P}, Func_\mathcal{P}, Ar_\mathcal{P}\rangle$, where $Pred_\mathcal{P}$ is a non-empty set of *predicate symbols*, $Func_\mathcal{P}$ is a set (disjoint with $Pred_\mathcal{P}$) of *function symbols,* and $Ar_\mathcal{P}$ is the *arity function,* assigning to each predicate or function symbol a natural number called the *arity* of the symbol. The function symbols f with $Ar_\mathcal{P}(f) = 0$ are called *object* or *individual constants.* The predicates symbols P for which $Ar_\mathcal{P}(P) = 0$ are called *truth constants.*

$\mathcal{P}$-terms and (atomic) $\mathcal{P}$-formulae of a given predicate language are defined as in classical logic (note that the notion of formula also depends on propositional connectives in $\mathcal{L}$). A $\mathcal{P}$*-theory* is a set of $\mathcal{P}$-formulae. The notions of free occurrence of a variable, substitutability, open formula, and closed formula (or, synonymously, *sentence*) are defined in the same way as in classical logic. Unlike in classical logic, in fuzzy logics without involutive negation the quantifiers $\forall$ and $\exists$ are not mutually definable and have to be both primitive symbols.

There are several variants of the first-order extension of a propositional fuzzy logic L that can be defined. Following Hájek's approach in [5, 6] and the general presentation of [2], we restrict to logics of models over linearly ordered algebras and introduce the first-order logics L$\forall$ and L$\forall^{\mathrm{w}}$ (respectively, complete w.r.t. all models or w.r.t. witnessed models). The logic L$\forall$ in language $\mathcal{P}$ has the following axioms:

(P)	The axioms of L
($\forall$1)	$(\forall x)\phi(x) \to \phi(t)$, where the $\mathcal{P}$-term t is substitutable for x in ϕ
($\exists$1)	$\phi(t) \to (\exists x)\phi(x)$, where the $\mathcal{P}$-term t is substitutable for x in ϕ
($\forall$2)	$(\forall x)(\chi \to \phi) \to (\chi \to (\forall x)\phi)$, where x is not free in χ
($\exists$2)	$(\forall x)(\phi \to \chi) \to ((\exists x)\phi \to \chi)$, where x is not free in χ
($\forall$3)	$(\forall x)(\chi \vee \phi) \to \chi \vee (\forall x)\phi$, where x is not free in χ.

The deduction rules of L$\forall$ are those of L plus the rule of *generalization:*

(Gen) $\langle \phi, (\forall x)\phi \rangle$.

The logic L$\forall^{\mathrm{w}}$ is the extension of L$\forall$ by the axioms:

(C$\forall$) $(\exists x)(\phi(x) \to (\forall y)\phi(y))$

(C$\exists$) $(\exists x)((\exists y)\phi(y) \to \phi(x))$.

A $\mathcal{P}$*-structure* is $\langle A, \mathbf{M} \rangle$ where A is an L-algebra and $\mathbf{M} = \langle S, \langle P_{\mathbf{M}} \rangle_{P \in \mathbf{P}}, \langle f_{\mathbf{M}} \rangle_{f \in \mathbf{F}} \rangle$, where M is a non-empty domain; $P_{\mathbf{M}}$ is an n-ary fuzzy relation, i.e. a function $S^n \to A$, for each n-ary predicate symbol $P \in \mathbf{P}$ with $n \geq 1$ and an element of A if P is a truth constant; $f_{\mathbf{M}}$ is a function $M^n \to M$ for each n-ary $f \in \mathbf{F}$ with $n \geq 1$ and an element of M if f is an object constant.

Let $\langle A, \mathbf{M} \rangle$ be a $\mathcal{P}$-structure. An $\mathbf{M}$-*evaluation* of the object variables is a mapping v which assigns to each variable an element from S. Let v be an $\mathbf{M}$-evaluation, x a variable, and $a \in M$. Then $\mathrm{v}[x{\to}a]$ is an $\mathbf{M}$-evaluation such that $\mathrm{v}[x{\to}a](x) = a$ and $\mathrm{v}[x{\to}a](y) = \mathrm{v}(y)$ for each $y \neq x$.

Let $\langle A, \mathbf{M} \rangle$ be a $\mathcal{P}$-structure and v an $\mathbf{M}$-evaluation. We define *values* of $\mathcal{P}$-terms and *truth values* of $\mathcal{P}$-formulae in $\mathbf{M}$ for an evaluation v as:

$$\begin{array}{rcll}
||x||^{\mathbf{M}}_{\mathrm{v}} & = & \mathrm{v}(x), & \\
||f(t_1, \ldots, t_n)||^{\mathbf{M}}_{\mathrm{v}} & = & f_{\mathbf{M}}(||t_1||^{\mathbf{S}}_{\mathrm{v}}, \ldots, ||t_n||^{\mathbf{M}}_{\mathrm{v}}), & \text{for } f \in \mathbf{F} \\
||P(t_1, \ldots, t_n)||^{\mathbf{M}}_{\mathrm{v}} & = & P_{\mathbf{M}}(||t_1||^{\mathbf{M}}_{\mathrm{v}}, \ldots, ||t_n||^{\mathbf{M}}_{\mathrm{v}}), & \text{for } P \in \mathbf{P} \\
||c(\phi_1, \ldots, \phi_n)||^{\mathbf{M}}_{\mathrm{v}} & = & c^A(||\phi_1||^{\mathbf{M}}_{\mathrm{v}}, \ldots, ||\phi_n||^{\mathbf{M}}_{\mathrm{v}}), & \text{for } c \in \mathcal{L} \\
||(\forall x)\phi||^{\mathbf{M}}_{\mathrm{v}} & = & \inf_{\leq^A}\{||\phi||^{\mathbf{M}}_{\mathrm{v}[x\to a]} \mid a \in M\}, & \\
||(\exists x)\phi||^{\mathbf{M}}_{\mathrm{v}} & = & \sup_{\leq^A}\{||\phi||^{\mathbf{M}}_{\mathrm{v}[x\to a]} \mid a \in M\}. &
\end{array}$$

If the infimum or supremum does not exist, we take its value as undefined. We say that $\langle A, \mathbf{M} \rangle$ is *safe* iff $||\phi||^{\mathbf{M}}_{\mathrm{v}}$ is defined for each $\mathcal{P}$-formula ϕ and each $\mathbf{M}$-evaluation v. $\langle A, \mathbf{M} \rangle$ is a *model* of a set of formulae Γ if it is safe and for every $\phi \in \Gamma$, $||\phi||^{\mathbf{M}}_{\mathrm{v}} \in F^A$ (where F^A is the filter of designated elements of the algebra A). If $\phi(x_1, \ldots, x_n)$ has $x_1, \ldots, x_n$ as free variables and $d_1, \ldots, d_n \in M$, by $||\phi(d_1, \ldots, d_n)||^{A}_{\mathbf{M}}$ we denote the truth value for any evaluation v such that $v(x_i) = d_i$ for each i. Finally, we call $\langle A, \mathbf{M} \rangle$ a *witnessed model* if all interpretations of quantifiers are actually maxima or minima reached by elements of the domain.

The semantical notion of consequence is defined in the usual way (every model of the premises is also a model of the conclusion) and corresponding completeness theorems are proved (see [7]).

§1. Results on elementary equivalence and elementary substructures. In this section we give a compressed sample list of the kind of results we can achieve regarding elementary substructures and elementary equivalence for predicate fuzzy logics. See [3, 4, 7] for any unexplained notion.

Definition 1 ([4]). Let $\langle A, \mathbf{N}\rangle$ be a $\mathcal{P}$-structure, $K \subseteq N$, $e_1, \dots, e_n \in K$, and $\phi(x, y_1, \dots, y_n)$ a $\mathcal{P}$-formula. We denote by $X^{\langle A,\mathbf{N}\rangle}_{\phi,e_1,\dots,e_n,K}$ the following subset of A: $\{||\phi(d, e_1, \dots, e_n)||^A_{\mathbf{N}} \mid d \in K\}$. It is said that a subset Y of A is *definable with parameters in* $\langle A, \mathbf{N}\rangle$ if there are $K \subseteq N$, $e_1, \dots, e_n \in K$, and a $\mathcal{P}$-formula $\phi(x, y_1, \dots, y_n)$ such that $Y = X^{\langle A,\mathbf{N}\rangle}_{\phi,e_1,\dots,e_n,K}$.

Definition 2. The *cardinality* of $\langle B, \mathbf{M}\rangle$ is the cardinality of the domain M, denoted by $|M|$.

Definition 3. We denote by $p(B)$ the minimum cardinal γ satisfying that, for every $X \subseteq B$ definable with parameters in $\langle B, \mathbf{M}\rangle$ such that its infimum and supremum exist, there is a $Y \subseteq X$ of cardinality $\leq \gamma$, which also has infimum and supremum and such that $\inf X = \inf Y$ and $\sup X = \sup Y$.

Theorem 4 (Downward Löwenheim-Skolem Theorem). *Let $\langle B, \mathbf{M}\rangle$ be an infinite $\mathcal{P}$-structure. Assume that every subset of B definable with parameters in $\langle B, \mathbf{M}\rangle$ has infimum and supremum. Then, for every cardinal κ with $\max\{p(B), |\mathcal{P}|, \omega\} \leq \kappa \leq |M|$ and every $Z \subseteq M$ with $|Z| \leq \kappa$, there is $\langle B, \mathbf{O}\rangle$ which is an elementary substructure of $\langle B, \mathbf{M}\rangle$ of cardinality $\leq \kappa$ and $Z \subseteq O$.*

Theorem 5 (Upward Löwenheim-Skolem Theorem). *For every $\langle B, \mathbf{M}\rangle$ and every $\kappa \geq \max\{|M|, |\mathcal{P}|\}$, there is a structure $\langle B, \mathbf{O}\rangle$ of cardinality κ such that $\langle B, \mathbf{M}\rangle$ is elementary mapped in $\langle B, \mathbf{O}\rangle$.*

Theorem 6 (Upward Löwenheim-Skolem Theorem for relational languages). *Assume that $\mathcal{P}$ is a purely relational predicate language. For every $\mathcal{P}$-structure and every $\kappa \geq \max\{|M|, |\mathcal{P}|\}$, there is $\langle B, \mathbf{O}\rangle$ of cardinality κ such that $\langle B, \mathbf{M}\rangle$ is an elementary substructure of $\langle B, \mathbf{O}\rangle$.*

Definition 7. We say that two $\mathcal{P}$-structures $\langle B_1, \mathbf{M}_1\rangle$ and $\langle B_2, \mathbf{M}_2\rangle$ are *elementary equivalent* (in symbols: $\langle B_1, \mathbf{M}_1\rangle \equiv \langle B_2, \mathbf{M}_2\rangle$) if for every $\mathcal{P}$-sentence σ, $||\sigma||^{B_1}_{\mathbf{M_1}} \in F^{B_1}$ iff $||\sigma||^{B_2}_{\mathbf{M_2}} \in F^{B_2}$.

Definition 8. Two $\mathcal{P}$-structures over the same chain $\langle B, \mathbf{M}_1\rangle$ and $\langle B, \mathbf{M}_2\rangle$ are *filter-strongly elementary equivalent* (in symbols: $\langle B, \mathbf{M}_1\rangle \equiv^{fs} \langle B, \mathbf{M}_2\rangle$) if for each $\mathcal{P}$-sentence σ, $||\sigma||^B_{\mathbf{M_1}} \in F^B$ iff $||\sigma||^B_{\mathbf{M_2}} \in F^B$ and, in this case, $||\sigma||^B_{\mathbf{M_1}} = ||\sigma||^B_{\mathbf{M_2}}$.

Definition 9. We say that two $\mathcal{P}$-structures over the same chain $\langle B, \mathbf{M}_1\rangle$ and $\langle B, \mathbf{M}_2\rangle$ are *strongly elementary equivalent* (in symbols: $\langle B, \mathbf{M}_1\rangle \equiv^s \langle B, \mathbf{M}_2\rangle$) if for every $\mathcal{P}$-sentence σ, $||\sigma||^B_{\mathbf{M_1}} = ||\sigma||^B_{\mathbf{M_2}}$.

Example 10. The notions of elementary equivalent and strongly elementary equivalent structures are different. Consider Gödel–Dummett logic G, a predicate language with only one monadic predicate P and take two structures over the standard Gödel chain, $\langle [0, 1]_{\mathrm{G}}, \mathbf{M}_1\rangle$ and $\langle [0, 1]_{\mathrm{G}}, \mathbf{M}_2\rangle$. The domain in both cases is the set of all natural numbers N and the interpretation of the predicate is:

$$P_{\mathbf{M_1}}(n) = \begin{cases} \frac{3}{4} - \frac{1}{n} & \text{if } n \geq 2 \\ 0 & 0 \leq n \leq 1 \end{cases} \qquad \text{and} \qquad P_{\mathbf{M_2}}(n) = \begin{cases} \frac{1}{2} - \frac{1}{n} & \text{if } n \geq 2 \\ 0 & 0 \leq n \leq 1. \end{cases}$$

On the one hand, $||(\exists x)P(x)||_{\mathbf{M_1}} = \frac{3}{4}$ but $||(\exists x)P(x)||_{\mathbf{M_2}} = \frac{1}{2}$, so the structures are not strongly elementary equivalent. On the other hand, elementary equivalence still holds. Take g as any non-decreasing bijection from $[0,1]$ to $[0,1]$ such that $g(\frac{3}{4}) = \frac{1}{2}$, $g(1) = 1$, $g(0) = 0$, and for every $n \in \mathsf{N}$ $g(\frac{3}{4} - \frac{1}{n}) = \frac{1}{2} - \frac{1}{n}$. g is a G-homomorphism preserving suprema and infima. Then we can consider the σ-mapping $\langle g, Id\rangle$ and apply [3, Proposition 8] to obtain that $\langle [0,1]_{\mathrm{G}}, \mathbf{M}_1\rangle \equiv \langle [0,1]_{\mathrm{G}}, \mathbf{M}_2\rangle$.

Definition 11. Let $S(t)$ be the set of subterms of t that are not variables. We define by induction the *nested rank* of φ, denoted by $NR(\varphi)$, as follows.

- For every n-ary predicate R of $\mathcal{P}$, $NR(R(t_1, \dots, t_n)) = |\bigcup_{1\leq i\leq n} S(t_i)|$.
- For every $n \geq 1$, every $\mathcal{P}$-formulae $\phi_1, \dots, \phi_n$ and every n-ary connective $\lambda \in \mathcal{L}$,

$$NR(\lambda(\phi_1, \dots, \phi_n) = \max\{NR(\phi_1), \dots, NR(\phi_n)\} + 1.$$

- For any 0-ary connective $\lambda \in \mathcal{L}$, $NR(\lambda) = 0$.
- For every $\mathcal{P}$-formula φ, $NR((\forall x)\varphi) = NR((\exists x)\varphi) = NR(\varphi) + 1$.

Definition 12. Given $\mathcal{P}$-structures $\langle B_1, \mathbf{M_1}\rangle$ and $\langle B_2, \mathbf{M_2}\rangle$, we write $\langle B_1, \mathbf{M_1}\rangle \equiv_n \langle B_2, \mathbf{M_2}\rangle$ whenever for every $\mathcal{P}$-sentence σ with $NR(\sigma) \leq n$, $||\sigma||^{B_1}_{\mathbf{M_1}} \in F^{B_1}$ iff $||\sigma||^{B_2}_{\mathbf{M_2}} \in F^{B_2}$.

Definition 13. A pair $\langle T, R\rangle$ is a *partial relative relation* between $\langle B_1, \mathbf{M}_1\rangle$, $\langle B_2, \mathbf{M}_2\rangle$ if

1. $T \subseteq B_1 \times B_2$ such that $dom(T) = B_1$ and $rg(T) = B_2$.
   For each n-ary λ, if $\langle a_1, b_1\rangle, \dots, \langle a_n, b_n\rangle \in T$, then $\langle \lambda^{B_1}(a_1, \dots, a_n), \lambda^{B_2}(b_1, \dots, b_n)\rangle \in T$. For every $a \in B_1$ and $b \in B_2$, such that $\langle a, b\rangle \in T$, $a \in F^{B_1}$ iff $b \in F^{B_2}$.
2. $R \subseteq M_1 \times M_2$ and if $\langle d_1, e_1\rangle, \dots, \langle d_n, e_n\rangle \in R$, then for each n-ary P, $\langle ||P(d_1, \dots, d_n)||^{B_1}_{\mathbf{M_1}}, ||P(e_1, \dots, e_n)||^{B_2}_{\mathbf{M_2}}\rangle \in T$.

Definition 14. We say that two structures $\langle B_1, \mathbf{M}_1\rangle$ and $\langle B_2, \mathbf{M}_2\rangle$ are *n-finitely relatives via* $\langle I_m \mid m \leq n\rangle$ (we write $\langle B_1, \mathbf{M}_1\rangle \sim_n \langle B_2, \mathbf{M}_2\rangle$) if

1. Every I_m is a non-empty set of partial relative relations,
2. *(Forth condition)* For any $m + 1 \leq n$, any $\langle T, R\rangle \in I_{m+1}$ and any $d \in M_1$, there is a relation $\langle T, R'\rangle \in I_m$, such that $R \subseteq R'$ and $d \in dom(R')$.
3. *(Back condition)* For any $m + 1 \leq n$, any $\langle T, R\rangle \in I_{m+1}$ and any $e \in M_2$, there is a relation $\langle T, R'\rangle \in I_m$, such that $R \subseteq R'$ and $e \in rg(R')$.
4. For any $m + 1 \leq n$, any $\langle T, R\rangle \in I_{m+1}$, and any constant c of $\mathcal{P}$,$\langle T, R \cup \{\langle c_{\mathbf{M_1}}, c_{\mathbf{M_2}}\rangle\}\rangle \in I_m$.
5. For any $m + 1 \leq n$, any $\langle T, R\rangle \in I_{m+1}$, any n-ary function symbol f of $\mathcal{P}$, and any $\langle d_1, e_1\rangle, \dots, \langle d_n, e_n\rangle \in R$, $\langle T, R \cup \{\langle f_{\mathbf{M_1}}(d_1, \dots, d_n), f_{\mathbf{M_2}}(e_1, \dots, e_n)\rangle\}\rangle \in I_m$.

Theorem 15 (Back and forth). *If $\mathcal{P}$ is finite and $\langle B_1, \mathbf{M}_1\rangle$, $\langle B_2, \mathbf{M}_2\rangle$ are witnessed, then for each $n \in \omega$, $\langle B_1, \mathbf{M}_1\rangle \equiv_n \langle B_2, \mathbf{M}_2\rangle$ iff $\langle B_1, \mathbf{M}_1\rangle \sim_n \langle B_2, \mathbf{M}_2\rangle$.*

We will also discuss characterizations in terms of back and forth of the other notions of elementary equivalence we have introduced.

Acknowledgements. Dellunde and García-Cerdaña are supported by EdeTRI (TIN2012-39348-C02-01); García-Cerdaña is also supported by the Spanish MICINN project MTM 201125745 and the grant 2009SGR 1433 from the Generalitat de Catalunya; Noguera is supported by the project GA13-14654S of the Czech Science Foundation and by the FP7-PEOPLE-2009-IRSES project MaToMUVI (PIRSES-GA-2009-247584).

[1] C. C. Chang and H. J. Keisler. *Model Theory*, volume 73 of *Studies in Logic and the Foundations of Mathematics*. North-Holland, Amsterdam, 1973.

[2] P. Cintula and C. Noguera. A general framework for mathematical fuzzy logic. In P. Cintula, P. Hájek, and C. Noguera, editors, *Handbook of Mathematical Fuzzy Logic - Volume 1*, volume 37 of *Studies in Logic, Mathematical Logic and Foundations*, pages 103–207. College Publications, London, 2011.

[3] P. Dellunde. Preserving mappings in fuzzy predicate logics. *Journal of Logic and Computation*, 22(6):1367–1389, 2012.

[4] P. Dellunde and F. Esteva. On elementary equivalence in fuzzy predicate logics. *Archive for Mathematical Logic*, 52(1):1–17, 2013.

[5] P. Hájek. *Metamathematics of Fuzzy Logic*, volume 4 of *Trends in Logic*. Kluwer, Dordrecht, 1998.

[6] P. Hájek. On witnessed models in fuzzy logic. *Mathematical Logic Quarterly*, 53(1):66–77, 2007.

[7] P. Hájek and P. Cintula. On theories and models in fuzzy predicate logics. *Journal of Symbolic Logic*, 71(3):863–880, 2006.

[8] W. Hodges. *Model Theory*, volume 42 of *Encyclopedia of Mathematics and its Applications*. Ox, Cambridge, 1993.

[9] V. Novák, I. Perfilieva, and J. Močkoř. *Mathematical Principles of Fuzzy Logic*. Kluwer, Dordrecht, 2000.

[10] A. Tarski. Der wahrheitsbegriff in den formalisierten sprachen. *Studia Philosophica*, 1:261–405, 1935.

[11] A. Tarski and R. L. Vaught. Arithmetical extensions of relational systems. *Compositio Mathematica*, 13:81–102, 1957.

▶ WOJCIECH DZIK AND PIOTR WOJTYLAK, *Admissible rules and almost structural completeness in some first-order modal logics*.
Institute of Mathematics, University of Silesia, Katowice, Poland.
E-mail: wojciech.dzik@us.edu.pl.
Institute of Mathematics and Computer Science, University of Opole, Opole, Poland.
E-mail: piotr.wojtylak@math.uni.opole.pl.

Almost Structural Completeness is proved and the form of admissible rules is found for some first-order modal logics extending S4.3. Bases for admissible rules are also investigated.

A logic is structurally complete if all (structural) rules which are admissible are also derivable in it. Many logics are not structurally complete because the only rules that are admissible but not derivable are passive. A rule $r: \quad \varphi_1, \ldots, \varphi_k/\psi$ is *passive* in a logic L if $\sigma(\varphi_1, \ldots, \varphi_k) \not\subseteq L$, for every substitution σ, that is r can not be applied to theorems of L. For example the following rule $P_2: \quad \Diamond p \wedge \Diamond\neg p/\bot$ is passive in modal logics extending S4. A logic is *almost structurally complete* if every (structural) rule which is admissible but not passive, is also derivable in it.

W.A.Pogorzelski and T.Prucnal [8] introduced substitutions for atomic formulas in first-order logic (which are homomorphisms of the language algebra modulo bounded variables). They showed that classical first-order logic (in the standard formalization: with Modus Ponens and Generalization rules) is not stucturally complete, but the system extended with a (non-structural) rule of substitution for atomic formulas is stucturally complete. It was shown in [2] that classical first-order logic in the standard formalization is almost structurally complete.

Let L be a first-order language (for simplicity: without identity, functions and constant symbols) containing infinitely many predicate symbols P_j, for each arity $n \geq 0$, with a special 0-ary predicate symbol $\bot$ that denotes syntactic falsehood . Formulas