Known-plaintext attack to two cryptosystems based on the BB equation

G. Alvarez, L. Hernández Encinas, and J. Muñoz Masqué

Abstract—Recently, Rama Murthy and Swamy proposed a symmetric cryptosystem based on the Brahmagupta-Bhāskara (BB) equation. The BB equation is the quadratic Diophantine equation $nx^2 + k = y^2$, where k is an integer and n is a positive integer such that $\sqrt{n}$ is irrational. For the particular case $k = 1$, the equation is called the Pell equation. The proposed cryptosystem was modified later by the same authors in order to avoid the cryptanalysis given by Youssef. Below, a known-plaintext attack to both cryptosystems is presented.

Index Terms—Brahmagupta-Bhāskara equation, Cryptanalysis, Cryptography, Pell equation.

I. PRELIMINARIES AND NOTATIONS

THE Brahmagupta-Bhāskara (BB) equation is the quadratic Diophantine equation

$$nx^2 + k = y^2,$$

where k is an integer and n is a positive integer such that $\sqrt{n}$ is irrational. For the particular case $k = 1$, the equation is called the Pell equation.

The goal of this paper is to show that the two cryptosystems proposed by Rama Murthy and Swamy in [2], [3] are vulnerable to the known-plaintext attack and hence, they are insecure.

First of all, we summarize the cryptosystem proposed in [2], which was subsequently improved in [3] in order to avoid the cryptanalysis appeared in [4]. The authors proposed a symmetric cryptosystem based on the Pell equation in the finite field $GF(p)$, so that the Pell equation can be written as

$$nX + 1 \equiv Y \pmod{p}, \quad (1)$$

where p is a prime number, and X and Y are quadratic residues modulo p . In the proposed symmetrical cryptosystem there are two classes of keys: The primary key is a prime number p , with $t = \lfloor \lg p \rfloor$ the bitlength of p ; the secondary key is a couple of prime numbers, a and b in the interval $(2^t, p)$. The plaintext corresponding to a block of the original message of t bits, B , is the integer n defined as $n = (B)_{10} + 1 < p$, where $(B)_{10}$ is the decimal expression of the block B . The

ciphertext, with bitlength $2(t + 1)$, corresponding to n is the pair of integers, (s, d) , given by

$$\begin{aligned} s &\equiv aX + bY \pmod{p}, \\ d &\equiv aX - bY \pmod{p}. \end{aligned} \quad (2)$$

The deciphering process to obtain n from (s, d) takes into account the following expressions

$$\begin{aligned} X &\equiv (2a)^{-1}(s + d) \pmod{p}, \\ Y &\equiv (2b)^{-1}(s - d) \pmod{p}, \\ n &\equiv X^{-1}(Y - 1) \pmod{p}. \end{aligned}$$

obtained from (2) and (1).

The authors claim the security of this cryptosystem is based on the fact that there are infinitely many prime numbers greater than 2^t , and the keys are only limited by practical hardware/software considerations. Nevertheless, Youssef presented in [4] a known-plaintext attack to the previous cryptosystem by using a system of four linear equations over $\mathbb{Q}$. The equations are deduced from the knowledge of two plaintexts, n_1, n_2 , and their corresponding ciphertexts, $(s_1, d_1), (s_2, d_2)$, and the four unknowns a^{-1}, b^{-1}, X_1 , and X_2 :

$$\begin{pmatrix} s_1 + d_1 & 0 & -2 & 0 \\ s_2 + d_2 & 0 & 0 & -2 \\ 0 & s_1 - d_1 & 2n_1 & 0 \\ 0 & s_2 - d_2 & 0 & -2n_2 \end{pmatrix} \begin{pmatrix} a^{-1} \\ b^{-1} \\ X_1 \\ X_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 2 \\ 2 \end{pmatrix}. \quad (3)$$

Once the cryptanalyst has obtained the values $a^{-1} = \frac{a_{11}}{a_{12}}, b^{-1} = \frac{b_{11}}{b_{12}}, X_1, X_2 \in \mathbb{Q}$, a similar system is solved by using the equations derived from n_1 or n_2 and a new pair of plaintext-ciphertext: $(n_3, (s_3, d_3))$. Then new rational values for $a^{-1} = \frac{a_{21}}{a_{22}}$ and $b^{-1} = \frac{b_{21}}{b_{22}}$ are obtained, and from them the value of

$$\gcd(a_{11}a_{22} - a_{12}a_{21}, b_{11}b_{22} - b_{12}b_{21}) = p'$$

provides a multiple of the prime p .

In order to improve the previous cryptosystem, Rama Murthy and Swamy proposed in [3] a modification in the ciphering process. With the same notations as above, the ciphertext (s, d) , corresponding to a given plaintext n , is computed as follows:

$$\begin{cases} s \equiv aX + bY \pmod{p} \\ d \equiv bX - aY \pmod{p} \end{cases} \quad (4)$$

With such a modification, the attack proposed by Youssef is

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not effective as the linear system of equations (3) becomes

$$\begin{pmatrix} -1 & 1 & (a+b) - n_1(a-b) & 0 \\ -1 & 1 & 0 & (a+b) - n_2(a-b) \\ 1 & 1 & (a-b) + n_1(a+b) & 0 \\ 1 & 1 & 0 & (a-b) - n_2(a+b) \end{pmatrix} \begin{pmatrix} a \\ b \\ X_1 \\ X_2 \end{pmatrix} = \begin{pmatrix} s_1 + d_1 \\ s_2 + d_2 \\ s_1 - d_1 \\ s_2 - d_2 \end{pmatrix},$$

which cannot be solved in the unknowns a , b , X_1 , and X_2 , when $s_1 + d_1$, $s_2 + d_2$, $s_1 - d_1$, $s_2 - d_2$, n_1 , and n_2 are known.

II. CRYPTANALYSIS

In this section we present a cryptanalysis to the cryptosystem introduced in [3] by using a known-plaintext attack. Suppose that four distinct plaintext-ciphertext pairs $(n_i, (s_i, d_i))$, $i = 1, \dots, 4$, are known.

The cryptanalysis is based on the fact that the system of equations

$$\left. \begin{aligned} s_1 &= aX_1 + b(n_1X_1 + 1) \\ d_1 &= bX_1 - a(n_1X_1 + 1) \\ s_2 &= aX_2 + b(n_2X_2 + 1) \\ d_2 &= bX_2 - a(n_2X_2 + 1) \\ s_3 &= aX_3 + bY_3 \\ d_3 &= bX_3 - aY_3, \end{aligned} \right\} \quad (5)$$

obtained from (1) and (4), admits a unique solution “in $\mathbb{Q}$ ” whenever the following inequalities hold:

$$\begin{aligned} D_1 &= (s_1n_1 - s_2n_2 + d_1 - d_2)^2 + (d_1n_1 - d_2n_2 - s_1 + s_2)^2 \neq 0, \\ D_2 &= (n_1n_2 + 1)(s_1d_2 - s_2d_1) + (n_2 - n_1)(s_1s_2 + d_1d_2) \neq 0. \end{aligned}$$

In fact, as a simple computation shows, we have

$$a = \frac{(n_1d_1 - n_2d_2 + s_2 - s_1)D_2}{D_1}, \quad (6)$$

$$b = \frac{(n_1s_1 - n_2s_2 + d_1 - d_2)D_2}{D_1}, \quad (7)$$

$$X_1 = \frac{s_1(s_1 - s_2) + d_1(d_1 - d_2) + n_2(s_1d_2 - s_2d_1)}{D_2}, \quad (8)$$

$$X_2 = \frac{s_2(s_1 - s_2) + d_2(d_1 - d_2) + n_1(s_1d_2 - s_2d_1)}{D_2}, \quad (9)$$

$$X_3 = \frac{n_1(s_1d_3 + s_3d_1) - n_2(s_2d_3 - s_3d_2) + s_3(s_1 - s_2) + d_3(d_1 - d_2)}{D_2}, \quad (10)$$

$$Y_3 = \frac{n_1(s_3s_1 + d_1d_3) - n_2(s_2s_3 + d_2d_3) + s_3(d_1 - d_2) + d_3(s_2 - s_1)}{D_2}. \quad (11)$$

Notice that the running time for solving the system (5) is negligible as explicit formulas are used. Once a , b , X_1 , X_2 , X_3 , and Y_3 are known, X_4 and Y_4 can be directly obtained as follows:

$$X_4 = \frac{as_4 + bd_4}{a^2 + b^2}, \quad (12)$$

$$Y_4 = \frac{bs_4 - ad_4}{a^2 + b^2}. \quad (13)$$

As the equations $Y_i = n_iX_i + 1$ for $i = 3, 4$ have not been used in the system (5), they allow one to compute $q = \gcd(N_3, N_4)$, where N_i is the numerator of the fraction $n_iX_i + 1 - Y_i$ for $i = 3, 4$, and q is a multiple of p , as follows from the very definition of the BB equation modulo p . Furthermore, the bitlength of N_i is less than $4(t+1)$, as N_i is the sum of three summands each of which being, at most, a product of two factors of bitlength $\leq 2(t+1)$, and other summands of less bitlength, as it is showed in equations (14) and (15).

$$N_3 = (n_1n_2 + 1)(-s_1d_2 + s_2d_1) + (n_2n_3 + 1)(s_2d_3 - s_3d_2) + n_1n_3(-s_1d_3 - s_3d_1) - s_1d_3 + s_3d_1 \quad (14)$$

$$+ (n_1 - n_2)(s_1s_2 + d_1d_2) + (n_1 - n_3)(s_1s_3 + d_1d_3) + (n_3 - n_2)(s_2s_3 + d_2d_3) \quad (15)$$

$$N_4 = (n_1n_2 + 1)(-s_1d_2 + s_2d_1) + (n_1n_4 - 1)(-s_1d_4 - s_4d_1) + (n_2n_4 - 1)(s_2d_4 + s_4d_2) + (n_1 - n_2)(s_1s_2 + d_1d_2) + (n_1 + n_4)(s_1s_4 - d_1d_4) + (n_2 + n_4)(-s_2s_4 + d_2d_4).$$

Hence the value of p is derived from q directly, without using any sophisticated factorization algorithm. Once the value of p is known, the values of a and b are obtained by reducing modulo p the rational values for a and b in (6) and (7).

Next, we show how to perform the cryptanalysis either when D_1 or D_2 vanishes. If $D_1 = 0$, then n_1 and n_2 are completely determined, namely,

$$\begin{aligned} n_1 &= \frac{s_2(s_2 - s_1) + d_2(d_2 - d_1)}{s_1d_2 - s_2d_1}, \\ n_2 &= \frac{s_1(s_2 - s_1) + d_1(d_2 - d_1)}{s_1d_2 - s_2d_1}. \end{aligned}$$

Hence, it suffices to change one of the two plaintexts and to keep the other.

Moreover, the expressions $n_2(s_2d_1 - s_1d_2) + s_1s_2 + d_1d_2$ and $n_1(s_2d_1 - s_1d_2) - s_1s_2 - d_1d_2$ cannot vanish simultaneously as the vanishing of both expressions imply $n_1 = n_2$, which contradicts the assumption. Consequently, if $D_2 = 0$, then either

$$n_1 = \frac{s_1d_2 - s_2d_1 - n_2(s_1s_2 - d_1d_2)}{n_2(s_2d_1 - s_1d_2) + s_1s_2 + d_1d_2}$$

or

$$n_2 = \frac{s_1d_2 - s_2d_1 - n_1(s_1s_2 + d_1d_2)}{n_1(s_2d_1 - s_1d_2) - s_1s_2 - d_1d_2}$$

and we proceed as in the first case.

The cryptanalysis to the original cryptosystem as formulated in [2] is similar to the previous one, by using an analogous system to (5), which provides the following expressions:

$$\begin{aligned} a &= \frac{1}{2} \frac{E_1}{E_2}, \\ b &= \frac{1}{2} \frac{E_1}{n_1(s_1 - d_1) - n_2(s_2 - d_2)}, \\ X_1 &= \frac{(s_1 + d_1)E_2}{E_1}, \\ X_2 &= \frac{(s_2 + d_2)E_2}{E_1}, \\ X_3 &= \frac{(s_3 + d_3)E_2}{E_1}, \\ Y_3 &= \frac{(s_3 - d_3)}{2} \frac{n_1(s_1 - d_1) - n_2(s_2 - d_2)}{E_1}, \end{aligned}$$

where

$$\begin{aligned} E_1 &= n_1(s_1 + d_1)(s_2 - d_2) - n_2(s_1 - d_1)(s_2 + d_2) \neq 0, \\ E_2 &= (s_1 - d_1) - (s_2 - d_2) \neq 0. \end{aligned}$$

III. NUMERICAL EXAMPLES

A numerical example with artificially small values is presented below.

Suppose the secret keys are $p = 88039103041$, $a = 82444068284$, $b = 74691386336$, and let us consider the following plaintext-ciphertext pairs:

$$\begin{aligned} n_1 &= 62889964901, (s_1 = 23448928782, d_1 = 20575584876), \\ n_2 &= 46618259518, (s_2 = 3834180954, d_2 = 17146327109), \\ n_3 &= 58426687172, (s_3 = 20527763839, d_3 = 26549794294), \\ n_4 &= 34101981540, (s_4 = 63745825525, d_4 = 34573143446). \end{aligned}$$

Here, $t = 36$. The values of the expressions (6)–(13) are

$$\begin{aligned} a &= \frac{18747516258436659290398932073219973648213388624567314364961044}{76968210889714969471714448330965363014221}, \\ b &= -\frac{4911602345504680036831831412408261019387618474005089177373658}{76968210889714969471714448330965363014221}, \end{aligned}$$

$$\begin{aligned} X_1 &= -\frac{100438259311184512896958295880}{6316558602527484445611550502004901984559}, \\ X_2 &= -\frac{20324306402445929963309585029649}{947483790379122666841732575300735297683850}, \\ X_3 &= -\frac{4042178796281286417593480091964}{15791396506318711140288762550122549613975}, \\ Y_3 &= -\frac{39736427305481927516794719785387}{947483790379122666841732575300735297683850}, \\ X_4 &= -\frac{6636253689492581216074514587646}{473741895189561333420866287650367648841925}, \\ Y_4 &= -\frac{33238053506208383143782735065227}{315827930126374222280577525100245099227950}, \\ p &= \gcd(N_3, N_4) = 88039103041. \end{aligned}$$

By using the software package MAPLE over a Intel-Pentium D PC, CPU 3.20 GHz, 2.00 GB RAM, under Windows XP (SP 2), the running time to compute these values, was 0.062 seconds.

For a prime number with bitlength $t = 64$, say

$$\begin{aligned} p &= 1901227095 \ 9876747001, \\ a &= 1871225235 \ 8876810813, \\ b &= 1888741918 \ 2566409020, \end{aligned}$$

and using the following four random plaintexts:

$$\begin{aligned} n_1 &= 1227087084 \ 8434884224, \\ n_2 &= 1687726605 \ 2631596458, \\ n_3 &= 3896750374 \ 601963646, \\ n_4 &= 1399200033 \ 9463305695, \end{aligned}$$

the running time to compute the values of a , b , X_1 , X_2 , X_3 , Y_3 , X_4 , Y_4 , and $\gcd(N_3, N_4) = 3p$, was 0.065 seconds.

For a larger prime of 128 bits

$$p = 3964446899 \ 6042275986 \ 5633386972 \ 192728973,$$

the running time to break the cryptosystem was 0.078 seconds.

For the secret primary key of 192 bits

$$\begin{aligned} p &= 1173414387 \ 5675490285 \ 0423315885 \ 2300218332 \\ &\quad 0750233346 \ 767633509, \end{aligned}$$

and secondary keys

$$\begin{aligned} a &= 6766631271 \ 3194092704 \ 6048021365 \ 0943954694 \\ &\quad 3733334474 \ 30258181, \\ b &= 7684478404 \ 6912415337 \ 2335671884 \ 2908013378 \\ &\quad 1976475131 \ 09338017, \end{aligned}$$

the running time to break the cryptosystem was 0.094 seconds. In this case, we obtained $\gcd(N_3, N_4) = 11p$.

IV. CONCLUDING REMARKS

- 1) As shown above, it usually suffices to consider four distinct plaintext-ciphertext pairs to recover the primary and the secondary keys of the both proposed symmetric cryptosystems in a computationally feasible way.
- 2) Although the values X , Y in $nX + 1 \equiv Y \pmod{p}$ are assumed to be quadratic residues in [2], [3], which is justified by the authors (personal communication) in order to avoid vulnerabilities, this property seems to play no role in the cryptanalysis proposed.
- 3) The cryptanalysis proposed does not involve general factorization algorithms, but only to compute the greatest common divisor of several pairs of integers and few factors much smaller than the modulus p in these greatest common divisors. As is well known (e.g. see [1, §2.4]), the $\gcd(u, v)$ can be computed efficiently in polynomial time, $O((\lg(\max(u, v)))^2)$, by using the Extended Euclidean Algorithm.

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