

Unitary time evolution in the Gowdy T^3 model

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Abstract. The polarized Gowdy T^3 model is a classically solvable cosmological model with local degrees of freedom. This explains the increasing interest in its quantization. After gauge fixing, the system can be described in terms of a point particle and a massless scalar field propagating in an expanding torus. A Fock space representation has been proposed for the quantization of the model, but the classical dynamics for the scalar field cannot be unitarily implemented in this quantum theory. We show that, nevertheless, unitarity can be regained by slightly modifying the dynamics with quantum corrections. Moreover, a time-dependent rescaling of the scalar field leads to new evolution equations that can be unitarily implemented.

1. Introduction

In the search for a quantum theory of gravity, symmetry-reduced models have received considerable attention as a suitable arena in which one can discuss conceptual and technical issues inherent to the quantization of general relativity. Symmetry reductions that allow the existence of an infinite number of gravitational degrees of freedom are of special relevance, since one expects them to capture the field complexity of general relativity [1]. Among this type of models, the Gowdy T^3 spacetimes with linear polarization [2, 3, 4, 5] occupy a particularly prominent position. These are vacuum spacetimes that possess two commuting and hypersurface orthogonal Killing vectors and whose spatial topology is that of a three-torus. They provide the simplest of all possible spacetimes with local gravitational degrees of freedom that admit cosmological solutions [6].

Another reason for the interest in the quantization of the polarized Gowdy model is that, after an adequate gauge fixing, the system is classically equivalent to 2+1 gravity coupled to an axially symmetric, massless scalar field [5]. The quantization of this scalar field in the associated fictitious background provides then a quantum description of the local degrees of freedom of the Gowdy cosmologies. This line of reasoning was recently followed by Pierri [5] to propose a quantum theory for this gravitational model. However, it was soon realized that this particular proposal leads to a quantization in which the classical dynamics cannot be unitarily implemented [7, 8]. This obstruction poses a serious drawback, preventing the availability of a standard probabilistic interpretation that is preserved in the evolution. It is worth noting that a satisfactory quantization of the model, consistent with unitarity, would serve as a valuable reference for comparison with other approaches to quantum cosmology, such as e.g. loop quantum cosmology [9], which has been applied with success to homogenous models but whose generalization to scenarios with physical local degrees of freedom is still to be developed.

The aim of this work is to investigate the reasons behind the observed failure of unitarity and discuss possible solutions. We will show that unitarity can be regained by allowing negligible

quantum corrections to the evolution. More importantly, we will also see that it is possible to achieve a unitary evolution, without modifying the classical dynamics, by introducing a time-dependent canonical transformation that results in a rescaling of the basic scalar field.

2. The polarized Gowdy T^3 model

The linearly polarized Gowdy T^3 spacetimes possess two commuting, axial and hypersurface orthogonal Killing vectors. After a partial gauge fixing, their metric can be written [10]:

$$ds^2 = e^{\gamma-\psi} \left(-\tau^2 \tilde{N}^2 dt^2 + [d\theta + N^\theta dt]^2 \right) + e^{-\psi} \left(\tau^2 d\sigma^2 + e^{2\psi} d\delta^2 \right), \quad (1)$$

where all functions depend only on the time t and on the angular coordinate $\theta \in S^1$. The Killing vectors are given by ∂_σ and ∂_δ [with $\sigma, \delta \in S^1$], N^θ is the only nonvanishing component of the shift and \tilde{N} is the densitized lapse. So, the system is still subject to the momentum constraint for the θ -coordinate and to the Hamiltonian constraint. The associated gauge freedom can be almost entirely fixed by imposing the conditions:

$$P_\gamma = -p := \oint \frac{d\theta}{2\pi} P_\gamma, \quad \tau = tp. \quad (2)$$

Here, P_γ is the momentum canonically conjugate to γ . The gauge fixing is well posed provided that p does not vanish. Since p turns out to be a constant of motion, because it commutes with all the constraints under Poisson brackets, one can consistently restrict all considerations to the sector of the phase space with positive p (the sector with negative p is related with this one by means of a time reversal).

After the above fixation, there is only one homogeneous constraint left, which corresponds to the zero mode of the θ -momentum constraint. Subsequently, the shift N^θ is not totally fixed, but is allowed to be any function of time. Nevertheless, a shift of this type can always be absorbed by redefining the θ -coordinate [10], so that in practice one can obviate it.

In this way, one arrives at a reduced system with the following action and Hamiltonian, modulo the homogenous momentum constraint:

$$S_r = \int_{t_i}^{t_f} dt \left(P\dot{Q} + \oint d\theta [P_\phi \dot{\phi} - \mathcal{H}_r] \right), \quad \mathcal{H}_r = \frac{1}{2t} [P_\phi^2 + t^2(\phi')^2]. \quad (3)$$

Here, the dot and prime denote the derivatives with respect to t and θ , respectively, we have set $4G = \pi$ and we have introduced the canonical variables

$$Q := \oint d\theta \left(-\gamma p + t\mathcal{H}_r + \frac{P_\psi \psi}{2} \right), \quad P := \ln p, \quad \phi := \sqrt{p}\psi, \quad P_\phi := \frac{P_\psi}{\sqrt{p}}. \quad (4)$$

The variables Q and P are constants of motion, and describe a ‘‘point particle’’ degree of freedom. On the other hand, the field ϕ satisfies the wave equation $\ddot{\phi} + (\dot{\phi}/t) - \phi'' = 0$. This is precisely the equation corresponding to a massless scalar field with axial symmetry propagating in a three-dimensional background with metric $ds_0^2 = -dt^2 + d\theta^2 + t^2 d\sigma^2$ and the spatial topology of T^2 . Hence, as we had anticipated, the reduced Gowdy model is essentially equivalent to such a scalar field. Smooth real solutions to the above equation have the form

$$\phi = \frac{1}{2\sqrt{2}} \sum_{n=-\infty, n \neq 0}^{\infty} \left[A_n H_0(|n|t) e^{in\theta} + A_n^* H_0^*(|n|t) e^{-in\theta} \right] + \frac{\bar{q}_0 + \bar{p}_0 \ln t}{\sqrt{2\pi}}, \quad (5)$$

where the symbol $*$ stands for complex conjugation and H_0 is the zeroth-order Hankel function of the second kind [11]. In addition, the A_n 's (with n any nonzero integer) are constants of motion that must decrease with n faster than the inverse of any polynomial [8]. It is possible to check that A_n and A_n^* behave like pairs of annihilation and creation-like variables under Poisson brackets. Furthermore, in terms of them, the homogenous θ -momentum constraint that remains on the system adopts the simple expression $\sum n A_n^* A_n = 0$.

3. Evolution in the covariant and canonical approaches

The covariant phase space of the system can be identified with the space of smooth solutions. This space V can be decomposed as a direct sum $V = V_0 \oplus \bar{V}$ where V_0 is the subspace of homogeneous smooth solutions, which contains the “point particle” degree of freedom and the zero mode of the field ϕ [determined by the constants \bar{q}_0 and \bar{p}_0 , see Eq. (5)]. \bar{V} is the subspace of nonzero modes of ϕ . From Eq. (5), this subspace can be coordinatized by the complex conjugate constants $\{A_m, A_m^*, A_{-m}, A_{-m}^*\}$ with $m \in \mathbb{N} - \{0\}$. It is important to point out that the dynamics on the covariant phase space is in fact frozen, because V is described by coordinates that are constants of motion and therefore do not evolve.

On the other hand, the canonical phase space is given by the configuration and momentum of the field ϕ and of the “point particle” degree of freedom. Recalling the periodicity in θ , we can expand in Fourier series (at any instant of time) the field and its momentum,

$$\phi = \sum_{n=-\infty}^{\infty} \phi_n \frac{e^{in\theta}}{\sqrt{2\pi}}, \quad P_\phi = \sum_{n=-\infty}^{\infty} P_\phi^{-n} \frac{e^{in\theta}}{\sqrt{2\pi}}. \quad (6)$$

Since the field is real, $\phi_n^* = \phi_{-n}$ and $(P_\phi^n)^* = P_\phi^{-n}$. Besides, ϕ_n and P_ϕ^n are canonically conjugate. For all nonzero integers n , one can then construct the annihilation and creation-like variables

$$a_n := \frac{|n|\phi_n + iP_\phi^{-n}}{\sqrt{2|n|}}, \quad a_n^* := \frac{|n|\phi_{-n} - iP_\phi^n}{\sqrt{2|n|}}. \quad (7)$$

Similar to the situation explained for V , the canonical phase space Γ can be decomposed as a direct sum $\Gamma = \Gamma_0 \oplus \bar{\Gamma}$ where Γ_0 contains the zero modes [the “point particle” and the pair (ϕ_0, P_ϕ^0)] and $\bar{\Gamma}$ can be coordinatized by the complex conjugate variables $\{a_m, a_m^*, a_{-m}, a_{-m}^*\}$ with $m \in \mathbb{N} - \{0\}$. On Γ , the dynamics is generated by the reduced Hamiltonian $H_r := \oint d\theta \mathcal{H}_r$ given in Eq. (3). The problem with this dynamics [7, 8] is that the evolution generated by H_r cannot be implemented as a unitary transformation on the Fock space constructed from V with the complex structure associated with the field decomposition (5) (i.e., with the choice of $\{A_m, A_m^*, A_{-m}, A_{-m}^*\}$ as annihilation and creation-like variables), nor in the physical Hilbert space determined by the kernel of the operator version of the constraint $\sum n A_n^* A_n = 0$.

4. Modified dynamics

Motivated by our previous discussion, we will now consider t -dependent maps $M(t)$ from \bar{V} to a symplectic space $\bar{\Gamma}$ with annihilation and creation-like variables \tilde{a}_n and \tilde{a}_n^* . We will restrict our analysis to maps with the following form on each subspace of modes with wave number m :

$$\begin{pmatrix} \tilde{a}_m \\ \tilde{a}_m^* \\ \tilde{a}_{-m} \\ \tilde{a}_{-m}^* \end{pmatrix} = M_m(t) \begin{pmatrix} A_m \\ A_m^* \\ A_{-m} \\ A_{-m}^* \end{pmatrix}, \quad M_m(t) = \begin{pmatrix} c_m(t) & 0 & 0 & d_m(t) \\ 0 & c_m^*(t) & d_m^*(t) & 0 \\ 0 & d_m(t) & c_m(t) & 0 \\ d_m^*(t) & 0 & 0 & c_m^*(t) \end{pmatrix}. \quad (8)$$

This particular form of $M(t)$ is inspired by the map to the canonical phase space $\bar{\Gamma}$, for which \tilde{a}_n can be replaced with a_n (for all $n \neq 0$) and the functions $c_m(t)$ and $d_m(t)$ respectively become

$$c_m(t) = \sqrt{\frac{\pi m}{8}} [H_0(mt) - itH_1(mt)], \quad d_m(t) = \sqrt{\frac{\pi m}{8}} [H_0^*(mt) - itH_1^*(mt)]. \quad (9)$$

Here, H_1 denotes the first-order Hankel function of the second kind.

Note that $M_m(t)$ is a Bogoliubov transformation provided that $|c_m(t)|^2 - |d_m(t)|^2 = 1$. We will assume from now on that this relation is satisfied, as it happens in the case of the map

(9) for the standard dynamics of the Gowdy model. As a consequence, it turns out that the symplectic map $M(t)$ can be unitarily implemented on \tilde{V} (and hence on V) if and only if the sequence $\{d_m(t)\}$ is square summable [10, 12], i.e. if and only if $\sum |d_m(t)|^2 < \infty$.

The considered transformation is generated by the t -dependent function

$$\mathcal{F}(t) = \sum_{m=1}^{\infty} \mathcal{F}_m(t), \quad i\mathcal{F}_m(t) = \tilde{a}_m[d_m^*(t)A_{-m} + c_m^*(t)A_m^*] - \tilde{a}_{-m}^*[c_m(t)A_{-m} + d_m(t)A_m^*]. \quad (10)$$

Since there is no evolution on \tilde{V} , the dynamics on $\tilde{\Gamma}$ is then generated by the derivative of \mathcal{F} with respect to its explicit t -dependence, namely, by the Hamiltonian $\tilde{H} = \partial_t \mathcal{F}$. This dynamics is nothing but the symplectic transformation $M(t)M^{-1}(t_0)$ on $\tilde{\Gamma}$ from a given time t_0 to t .

Let us therefore consider a more general Hamiltonian \tilde{H} than that obtained from Eq. (3) for the Gowdy model. Specifically, we assume that $\tilde{H} = \sum_{m>0} \tilde{\mathcal{H}}_m$ with

$$\tilde{\mathcal{H}}_m[\mathcal{A}] := m [\mathcal{V}(mt)A_m A_{-m} + \mathcal{V}^*(mt)A_m^* A_{-m}^*] + mW(mt) [A_m^* A_m + A_{-m}^* A_{-m}]. \quad (11)$$

The function W must be real, because so is the Hamiltonian. We further suppose that the other function appearing in the Hamiltonian has the form $\mathcal{V}(x) := e^{-2i\omega(x)}2\Delta(x)$, where $\omega(x)$ is a real function such that both $\Delta(x)$ and $[W(x) - d\omega/dx]$ admit asymptotic series representations at infinity (in powers of $1/x$) and $\lim_{x \rightarrow \infty} \Delta(x) = 0$. That is, for large x ,

$$\beta(x) := W(x) - \frac{d\omega}{dx} = \sum_{k=0}^{\infty} \frac{\beta_k}{x^k}, \quad \Delta(x) = \sum_{k=1}^{\infty} \frac{\Delta_k}{x^k}. \quad (12)$$

Actually, these assumptions are satisfied by (the nonzero modes part of) the Hamiltonian $H_r = \oint d\theta \mathcal{H}_r$ that generates the standard dynamics of the Gowdy model [10]. In this case, $\omega(x) = x$, $\mathcal{V}(x) = \pi x([H_0(x)]^2 + [H_1(x)]^2)/4$ and $W(x) = \pi x(|H_0(x)|^2 + |H_1(x)|^2)/4$. Asymptotic representations like (12) are then obtained from the Hankel's asymptotic expansions [11].

Given expression (11), the identity $\tilde{H} = \sum_{m>0} \tilde{H}_m = \partial_t \mathcal{F}$ [together with Eq. (10)] allows one to deduce the functions $c_m(t)$ and $d_m(t)$ that determine the map (8). Although in principle one should obtain an infinite number of equations (for $m \in \mathbb{N} - \{0\}$), the particular form of the Hamiltonian $\tilde{\mathcal{H}}_m$ guarantees that the equations for all the nonzero modes are equivalent. In this way, one ends up with just two independent differential equations in t : a Ricatti equation for the ratio of $d_m(t)$ and $c_m(t)$ (with no further dependence on these functions) and a linear differential equation for $c_m(t)$ that can be trivially integrated once the former ratio is known [10]. As a result, the nontrivial information about the time dependence of the map $M(t)$ is encoded in a Ricatti equation. Calling $x := mt$ (for any $m > 0$), this equation can be written

$$\frac{i}{2} \frac{dz}{dx}(x) + \beta(x)z(x) - \Delta(x)z^2(x) - \Delta^*(x) = 0, \quad \text{with} \quad z(x = mt) := e^{-2i\omega(mt)} \frac{d_m(t)}{c_m(t)}. \quad (13)$$

It is possible to show that, provided that $\beta_0 \neq 0$ [see Eq. (12)], there exists a solution with the asymptotic behavior $z(x) = \Delta_1^*/(\beta_0 x) + o(1/x)$, i.e., such that $\lim_{x \rightarrow \infty} xz(x) = \Delta_1^*/\beta_0$. On the other hand, recalling that $|c_m(t)|^2 - |d_m(t)|^2 = 1$ and that $\omega(mt)$ is real, one concludes that $|d_m(t)|^2 = |z(x = mt)|^2/[1 - |z(x = mt)|^2]$. Therefore, the asymptotic behavior of the solution $z(x)$ implies that $|d_m(t)|^2$ is of order $1/m^2$ for large m (as far as $t > 0$). This suffices to ensure that the sequence $\{d_m(t)\}$ is square summable for all positive times t . According to our comments above, the symplectic transformation generated by \tilde{H} is hence unitarily implementable on V as long as the corresponding value of β_0 differs from zero.

The problems with unitarity for the standard reduced Hamiltonian $H_r = \oint d\theta \mathcal{H}_r$ can be understood on the basis that, in this case, the asymptotic coefficient β_0 turns out to vanish [10]. Nevertheless, we see that if H_r is slightly modified so that β_0 is no longer zero, for instance by allowing corrections of a quantum origin, the resulting dynamics will be implementable as a unitary transformation. The considered modification can lead to a $\beta_0 \neq 0$ as small as desired.

5. Field redefinition

A substitution of the the field ϕ and its momentum by their Fourier series in sines and cosines [similar to those in Eq. (6)] leads to a decomposition of the Hamiltonian $H_r = \oint d\theta \mathcal{H}_r$ in (a zero mode part and) an infinite number of harmonic oscillators, except for the fact that their masses and spring constants are not truly constant but proportional to t . A naive definition of annihilation and creation-like variables for those oscillators would basically give an expression of the form (7) for the nonzero modes (modulo trivial linear combinations) if one multiplies $|n|$ by a factor t . Owing to this explicit t -dependence, the total time variation of such annihilation and creation-like variables would not coincide with that generated by the Hamiltonian H_r . The best way to deal with this explicit time dependence is by means of a field redefinition in which one scales the field ϕ of the Gowdy model by \sqrt{t} [14]. This scaling can then be completed into a t -dependent canonical transformation. We demand that the new Hamiltonian that generates the dynamics after the transformation, \bar{H}_r , does not contain products of the field and its momentum. This condition essentially fixes the canonical transformation and ensures that, in the Fock quantization, the new Hamiltonian has a well-defined action on the vacuum.

In this way, one arrives at the new basic field and momentum [13]:

$$\xi = \sqrt{t}\phi, \quad P_\xi = \frac{P_\phi}{\sqrt{t}} + \frac{\phi}{2\sqrt{t}}. \tag{14}$$

This transformation is generated by the phase-space function $\mathcal{F} = -\oint d\theta [(\xi P_\phi/\sqrt{t}) + \xi^2/(4t)]$. The new Hamiltonian is then that for a free field \bar{H}_0 plus a t -dependent perturbation \bar{H}_t , interpretable as a mass term:

$$\bar{H}_r = H_r + \partial_t \mathcal{F} := \bar{H}_0 + \bar{H}_t, \quad \bar{H}_0 := \oint \frac{d\theta}{2} [P_\xi^2 + (\xi')^2], \quad \bar{H}_t := \oint d\theta \frac{\xi^2}{8t^2}. \tag{15}$$

We are now in an adequate position to show that the standard dynamics of the Gowdy model can be implemented as a unitary transformation if one adopts ξ and P_ξ as the basic field and momentum of the system. In order to do this, we first note that the smooth classical solutions for ξ are straightforwardly obtained from those for ϕ [see Eq. (5)] via multiplication by \sqrt{t} . Therefore, the variables $\{A_m, A_m^*, A_{-m}, A_{-m}^*\}$ with $m \in \mathbb{N} - \{0\}$ continue to provide good coordinates for \bar{V} (the part of the covariant phase space corresponding to the nonzero modes). In addition, one can decompose the field ξ and its momentum in Fourier modes (with coefficients ξ_n and P_ξ^n) as we did for ϕ in expression (6), and introduce annihilation and creation-like variables for the nonzero modes similar to those in Eq. (7), namely $\tilde{a}_n := (|n|\xi_n + iP_\xi^{-n})/\sqrt{2|n|}$ and their complex conjugate \tilde{a}_n^* . The variables $\{\tilde{a}_m, \tilde{a}_m^*, \tilde{a}_{-m}, \tilde{a}_{-m}^*\}$ with $m \in \mathbb{N} - \{0\}$ allow to coordinatize the subspace $\bar{\Gamma}$ of the canonical phase space.

The relation between $\{\tilde{a}_m, \tilde{a}_m^*, \tilde{a}_{-m}, \tilde{a}_{-m}^*\}$ and $\{A_m, A_m^*, A_{-m}, A_{-m}^*\}$ has the form (8) with

$$d_m(t) = \sqrt{\frac{\pi mt}{8}} \left[H_0^*(mt) \left(1 + \frac{i}{2mt} \right) - iH_1^*(mt) \right], \quad c_m(t) = \sqrt{\frac{\pi mt}{2}} H_0(mt) - d_m^*(t). \tag{16}$$

In particular, employing the Hankel's asymptotic expansions [11], it is easy to check that $|d_m(t)|^2 = 1/(4mt)^4 + o(1/[mt]^4)$ for all positive times t . Therefore, the sequence $\{d_m(t)\}$ is square summable and the considered transformation can be unitarily implemented on the Fock space constructed from V with the choice of $\{A_m, A_m^*, A_{-m}, A_{-m}^*\}$ as annihilation and creation-like variables. Furthermore, one can prove [13] that the result is also valid on the subspace of physical states determined by the kernel of the homogenous θ -momentum constraint.

6. Conclusions

We have analyzed the problem of the unitary implementation of the dynamics in the system obtained after an almost complete gauge fixing of the family of Gowdy spacetimes with the spatial topology of a three-torus and a linear polarization of the gravitational waves. We have been able to reformulate the problem in terms of the square summability of the antilinear part of a time-dependent map from the covariant to the canonical phase space for the nonzero modes of the model. In this way we have proved that, whereas the standard evolution is not unitarily implementable with the choice of basic field and Fock quantization proposed in Ref. [5], negligibly small modifications in the reduced Hamiltonian suffice to restore unitarity. These modifications might arise from perturbations of the system or just from quantum corrections.

It is worth remarking that, in our proof of the unitarity of this modified dynamics, we have not needed the explicit form of the classical solutions to the evolution generated by the modified Hamiltonian. The nontrivial information about the evolution has been encoded in a Riccati equation. Unitary implementability follows just from the asymptotic behavior of a solution to that equation.

More importantly, we have also shown that one can in fact attain a unitary evolution without modifying the classical dynamics. This can be achieved by means of a time-dependent canonical transformation that rescales the basic field of the model. In this way, a problematic part of the dynamics is re-assigned to the explicit time dependence of the system. The evolution of the rescaled field is unitarily implementable on a natural Fock space representation [13]. In addition, the associated Hamiltonian can be interpreted as that of a conventional free field in a Minkowskian background (up to topology) corrected with a potential that corresponds to a time-dependent mass, and that can be treated as a perturbation in the asymptotic region of large times [13].

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7. References

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