Feasibility of a unitary quantum dynamics in the Gowdy T^3 cosmological model

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It has been pointed out that it is impossible to obtain a unitary implementation of the dynamics for the polarized Gowdy T^3 cosmologies in an otherwise satisfactory, nonperturbative canonical quantization proposed for these spacetimes. By introducing suitable techniques to deal with deparametrized models in cosmology that possess an explicit time dependence (as it is the case for the toroidal Gowdy model), we present in this paper a detailed analysis about the roots of this failure of unitarity. We investigate the impediments to a unitary implementation of the evolution by considering modifications to the dynamics. These modifications may be regarded as perturbations. We show in a precise manner why and where unitary implementability fails in our system, and prove that the obstructions are extremely sensitive to modifications in the Hamiltonian that dictates the time evolution of the symmetry-reduced model. We are able to characterize to a certain extent how far the model is from unitarity. Moreover, we demonstrate that the dynamics can actually be approximated as much as one wants by means of unitary transformations.

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I. INTRODUCTION

Symmetry-reduced models have been used over the past 30 years as an appropriate arena to test strategies aimed for the quantization of the full theory of gravity within the canonical approach, as well as toy models which can provide us with insights about the kind of phenomena that should be expected in quantum general relativity. Most of the examples of symmetry-reduced models studied so far are minisuperspace models [1], namely, simple systems where the reduction leaves only a finite number of physical degrees of freedom. There is another class of models which has more interest inasmuch as they retain the field complexity of general relativity. These are the so-called midisuperspace models (for a recent review see Ref. [2]), which after reduction possess an infinite number of degrees of freedom. Thus, their quantization would lead to a true quantum field theory.

Within this class, and together with the Einstein-Rosen waves [3], the model that has deserved more attention lately is the Gowdy T^3 cosmological model [4–9]. This model was introduced by Gowdy during the seventies in a systematic search for all spacetimes with two commuting spacelike Killing vector fields and compact spatial hypersurfaces [10]. Apart from the T^3 topology of a three-torus, the other two possible spatial topologies for the Gowdy spacetimes are the three-handle $S^1 \times S^2$ and the three-sphere S^3 (or the topology of a manifold covered by one of the above). The interest in the Gowdy T^3 model can thus be easily understood, since it provides the simplest of all the inhomogeneous, empty, spatially closed cosmological systems. The genuine field-theory character of this model and its possible applications in cosmology make it a natural candidate to study fundamental questions about canonical

quantum gravity and quantum field theory in curved spacetimes.

Its quantization was already considered in the seventies [11], and revisited both in the eighties [12–14] and more recently [4–8]. The first preliminary attempts to define a quantum theory and extract physics from the Gowdy T^3 model [11,12] were followed by more detailed analysis [4,14] that discussed the nonperturbative quantization of the system employing the Ashtekar formulation of Lorentzian general relativity [15]. Considerable progress has been achieved lately in defining a complete quantization of the (sub-)model with linear polarization, in which both Killing vectors are hypersurface orthogonal [5]. The proposed quantization is based on the fact that the polarized model can be equivalently treated as 2 + 1 gravity coupled to a massless scalar field, defined on a manifold whose topology is $T^2 \times \mathbb{R}$.

One important aspect in the study of quantum cosmological models is their dynamical evolution. For the polarized Gowdy T^3 model with the particular quantization performed in Ref. [5], it has been recently shown that the dynamics is not implementable at the quantum level as a unitary transformation [6,7]. From the point of view of canonical quantum gravity, this result does not represent a serious drawback for the simple reason that (owing to the compact nature of the spatial slices) time evolution is pure gauge in the Hamiltonian description. Hence, there is no time evolution and no dynamics. The system is endowed with a fictitious dynamics via a “deparametrization” procedure, and there is no apparent reason to select a preferred deparametrization.

Nevertheless, if one accepts that unitary evolution is a key ingredient in conventional (field) quantum theory, necessary in order to pose physically meaningful questions for issues like those concerning the initial singularity in cosmology, the lack of a unitary time evolution is a drawback for the kind of quantization put forward in Ref. [5].

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From that quantum theory, one would not be able to extract predictions for different instants of time, because probability is not conserved. In this sense, the quantization is not fully consistent [6]. Thus, restoration of unitarity in the evolution seems a fundamental issue in order to achieve a satisfactory quantization of the polarized Gowdy T^3 model. A rigorous quantization along these lines will provide us with a specific example of midisuperspace that can be very helpful, as a point of reference for comparisons, for a future quantization starting from loop quantum gravity (where impressive progress has been made, but exclusively for minisuperspaces [16]) or for implementations of the “consistent discretization” approach [17] (which has recently been applied to the Gowdy model [18]).

In this work, we will explore the reasons behind the failure in the unitary implementability of the dynamics as a first step in proposing solutions to it or introducing alternative descriptions for the quantum evolution. Although the lack of unitarity certainly follows from the absence of square summability for the antilinear part of the Bogoliubov transformation which relates the annihilation (creation) operator at, say, the “final” time t_f with the annihilation and creation operators at an “initial” time t_i [6,7], we want to analyze in detail the roots of this failure. We will show in a precise manner why unitary implementability fails in our system and, in a certain sense, we will be able to characterize how far the model is from unitarity. In doing so, we will introduce suitable techniques to deal with deparametrized models in cosmology that possess an explicit time dependence, like the Gowdy one.

The structure of the paper is as follows. In Sec. II we obtain the reduced phase space of the system by performing the symmetry reduction and introducing a deparametrization procedure in order to (partially) fix the gauge freedom¹ and select a Hamiltonian vector field to represent the dynamics. This reduction and gauge-fixing process had not been presented before starting with the spacetime metric of the model in general relativity, although the resulting description of the vacuum Gowdy spacetimes is essentially the same that was discussed in Refs. [5,6] (except for the remark on footnote ¹) and Ref. [7]. Therefore, the quantum theory on which we will base our analysis is that constructed by Pierri [5] (or, equivalently, that analyzed in Ref. [7]).

Section III is divided into three parts. In the first one, we review the dynamics of the scalar field that represents

(most of) the true degrees of freedom of the theory. In this part, a crucial remark is that the coordinates of the covariant phase space (namely the coefficients that determine the field in terms of an orthonormal basis of solutions—defined essentially by the negative of the complex structure given in Ref. [5]) do not evolve in time. In these coordinates, the generator of the evolution is obviously the zero Hamiltonian. The dynamics will be introduced through a time-dependent map from the covariant to the canonical phase space. In the remaining parts of the section, we will investigate the impediments for a unitary implementation of the evolution by considering modifications to the dynamics that may be regarded as *small* perturbations. We will be able to identify where the failure of unitarity comes from and, in the last subsection, prove that the severity of the problem is greatly ameliorated by the fact that small corrections to the dynamics can be implemented in a unitary way. In addition, our analysis makes clear the fact that the diagonalization of the Hamiltonian performed in Refs. [6,8] is just an instantaneous diagonalization which ignores the change in time of the Bogoliubov coefficients. Here, these time variations are explicit and rigorously taken into account. Finally, the conclusions and some further comments are presented in Sec. IV. One Appendix is added which contains a proof about the behavior of the coefficients employed in the main discussion. In the following, lower case Latin indices on a tensor will denote its purely spatial components, whereas capital case Latin indices will be used to denote the tensor itself (abstract index notation).

II. THE POLARIZED GOWDY MODEL

The polarized Gowdy T^3 model describes globally hyperbolic four-dimensional vacuum spacetimes, $(\mathcal{M}, g_{AB})$, with two commuting hypersurface orthogonal spacelike Killing fields and compact spacelike hypersurfaces homeomorphic to a three-torus. Since global hyperbolicity implies that we can foliate $(\mathcal{M}, g_{AB})$ by Cauchy surfaces, Σ_t , parametrized by a global time function t , then a $3 + 1$ decomposition is available and the line element can be written

$$ds^2 = -N^2 dt^2 + h_{ij}(dx^i + N^i dt)(dx^j + N^j dt), \quad (1)$$

where we choose $t \in \mathbb{R}^+$, the coordinates in the sections of constant time are $\{x^i\} := \{x^1 = \theta, x^2 = \sigma, x^3 = \delta\}$ with $x^i \in S^1$, N and $\{N^i\}$ are, respectively, the lapse function and the components of the shift vector N^A , and $\{h_{ij}\}$ are the components of the induced spatial metric h_{AB} .

In addition, we will impose that $(\partial/\partial x^a)^A$ ($a = 2, 3$) are the two spacelike Killing vector fields. Thus, the metric must be independent of the coordinates x^a . Moreover, performing a (partial) gauge fixing along the lines explained in Ref. [19] (for pure gravitational plane waves) and Ref. [20] (for cylindrical spacetimes), remembering that the metric functions must be periodic in θ , and using

¹The “ q number” nature of the time variable which occurs in Refs. [5,6] will be avoided here by introducing a gauge condition that is slightly different to the one imposed in those references; in this way the physical degrees of freedom will be neatly disentangled from the time variable. This type of mixing is also absent in Ref. [7]; however, the disentanglement is attained there thanks to the introduction of an appropriate (partially) reduced line element since the very beginning, rather than to its construction by gauge fixing.

that $(\partial/\partial x^a)^A$ are commuting hypersurface orthogonal vector fields, one gets a line element for the reduced model of the form

$$ds^2 = -N^2 dt^2 + h_{\theta\theta}[d\theta + N^\theta dt]^2 + \sum_{a=2}^3 h_{aa}(dx^a)^2. \quad (2)$$

Let us consider now the following change of metric variables $\{h_{ij}\} \mapsto \{Q^\alpha\} := \{\psi, \gamma, \tau\}$, defined by

$$h_{\theta\theta} = e^{\gamma-\psi}, \quad h_{\sigma\sigma} = e^{-\psi}\tau^2, \quad h_{\delta\delta} = e^\psi. \quad (3)$$

Since this change is just a point transformation, the momenta P_α canonically conjugate to Q^α are

$$P_\alpha = p^{ij} \frac{\partial h_{ij}}{\partial Q^\alpha}, \quad (4)$$

where $p^{ij} = \sqrt{h}(K^{ij} - Kh^{ij})$ are the momenta canonically conjugate to h_{ij} , we have set the Newton constant G equal to $\pi/4$, h is the determinant of the induced metric, and $\{K^{ij}\}$ are the components of its extrinsic curvature, with trace equal to K . Substituting in Eq. (2) our new set of variables and introducing the densitized lapse factor $\underline{N} := N/\sqrt{h}$, we arrive at the line element

$$ds^2 = e^{\gamma-\psi}(-\tau^2 \underline{N}^2 dt^2 + [d\theta + N^\theta dt]^2) + e^{-\psi}(\tau^2 d\sigma^2 + e^{2\psi} d\delta^2), \quad (5)$$

whose Einstein-Hilbert action is then given by

$$\begin{aligned} S &= \int_{t_i}^{t_f} dt \oint d\theta [P_\alpha \dot{Q}^\alpha - \mathcal{H}(Q^\alpha, P_\alpha)] \\ &= \int_{t_i}^{t_f} dt \oint d\theta [P_\tau \dot{\tau} + P_\gamma \dot{\gamma} + P_\psi \dot{\psi} - (\underline{N} \tilde{C} + N^\theta C_\theta)]. \end{aligned} \quad (6)$$

The presence of the remaining first class constraints $\tilde{C}$ and C_θ reflects the fact that the gauge has been only partially fixed. These (densitized) Hamiltonian constraints and momentum constraints are, respectively,

$$\begin{aligned} \tilde{C} &:= \frac{1}{2}(P_\psi^2 - 2\tau P_\tau P_\gamma) + \frac{\tau}{2}(4\tau'' - 2\gamma'\tau' + \tau\psi'^2), \\ C_\theta &:= P_\tau \tau' + P_\gamma \gamma' + P_\psi \psi' - 2P_\gamma'. \end{aligned} \quad (7)$$

The prime denotes the derivative with respect to θ .

Note that, since the spatial slices are compact, there exist no boundary contributions to the Hamiltonian. Therefore, the total Hamiltonian vanishes on the constraint surface and there is no distinction between gauge and dynamics. It is then necessary to carry out a deparametrization in order to introduce dynamics. This deparametrization is accomplished as part of a gauge fixing of the model: one imposes suitable conditions which, together with the constraints (7), form a set of second class constraints, allowing the reduction of the system. Explicitly, we demand the following gauge-fixing conditions (which are a slight modification of

those introduced in Refs. [4,5]):

$$g_1 := P_\gamma + p = 0, \quad g_2 := \tau - tp = 0. \quad (8)$$

The first of these conditions requires the momentum canonically conjugate to γ to be homogeneous (independent of θ). Furthermore, this homogenous part is a constant (of motion) p . In this sense, it is worth pointing out that the Poisson brackets of $\oint P_\gamma/(2\pi)$ (i.e. $-p$) with all the first class constraints (7) vanish weakly, so that it is indeed a Dirac observable. On the other hand, the second of our conditions fixes the metric function τ equal to the global time function t except for a rescaling that is constant on shell, though can vary on different solutions. Modulo constraints and gauge-fixing conditions, a straightforward calculation shows then that

$$\left\{g_1, \oint d\theta GC_\theta\right\} = -pG', \quad \left\{g_2, \oint d\theta F\tilde{C}\right\} = tp^2F, \quad (9)$$

where the smearing functions F and G on S^1 are, respectively, a density of weight -1 and a scalar. Therefore, if F and G' are different from zero, these Poisson brackets do not vanish provided that $p \neq 0$. Thus, we have to restrict all considerations to the sector of solutions with nonzero p in order to get a well-posed fixation.

The next step in this procedure consists in demanding the compatibility of the gauge-fixing conditions with dynamics: the total time derivative of g_1 and g_2 must vanish for some choice of N and N^θ . This derivative is the sum of the Poisson bracket with the total Hamiltonian $\oint \mathcal{H}$ and the partial derivative with respect to the explicit t dependence. Modulo constraints and gauge-fixing conditions, we have

$$\dot{g}_1 = -p(N^\theta)', \quad \dot{g}_2 = -p + tp^2\underline{N}. \quad (10)$$

The requirements $\dot{g}_1 = 0$ and $\dot{g}_2 = 0$ are then satisfied if N^θ is any function of t and $\underline{N} = (tp)^{-1}$. It is worth noticing that, while the densitized lapse function is completely determined in this process, the shift function is not fully fixed. There remains some diffeomorphism gauge freedom, generated by the homogenous part of the constraint C_θ (after reduction). Besides, note that we have to further restrict p to be positive in order to ensure the positivity of the lapse function.²

In order to extract the true degrees of freedom, one solves the set of second class constraints $\{\tilde{C}, C_\theta, g_1, g_2\}$, obtaining

$$pP_\tau = -\frac{1}{2}\left(\frac{P_\psi^2}{tp} + tp\psi'^2\right), \quad (11)$$

²Otherwise, one should consider $t \in \mathbb{R}^-$ instead of $t \in \mathbb{R}^+$ for the time flow vector field to be future directed.

$$p\gamma' = P_\psi\psi' := \Lambda. \quad (12)$$

By performing a Fourier expansion in θ of the functions γ and Λ (which is possible given the smoothness of the fields on S^1), it is not difficult to see that identity (12) allows us to solve for all modes of γ but the zero mode. More precisely, the Fourier coefficients γ_n are determined in terms of Λ_n by $in p \gamma_n = \Lambda_n$. Thus, there is still an undetermined coefficient, namely γ_0 , and consequently we are left with a global degree of freedom.

Furthermore, note that integration over S^1 of Eq. (12) leads to the global constraint

$$\Lambda_0 = \frac{1}{\sqrt{2\pi}} \oint d\theta P_\psi \psi' = 0, \quad (13)$$

which is essentially the homogenous part of the constraint C_θ . Therefore, the diffeomorphism gauge freedom has not been entirely removed and the θ component of the shift vector cannot be completely fixed. However, as we have already seen, the only allowed dependence of N^θ is that on t . This type of shift can always be absorbed by redefining our angular coordinate θ [4]. After our gauge fixing and the absorption of the shift, the metric becomes

$$ds^2 = e^{\gamma-\psi}(-dt^2 + d\theta^2) + e^{-\psi}t^2 p^2 d\sigma^2 + e^\psi d\delta^2, \quad (14)$$

$$\gamma = \frac{q}{2\pi} - i \sum_{n \neq 0} \frac{\Lambda_n}{np} \frac{e^{in\theta}}{\sqrt{2\pi}}, \quad (15)$$

where $q := \sqrt{2\pi}\gamma_0$ is the coordinate canonically conjugate to $-p$ (the zero mode of $P_\gamma/\sqrt{2\pi}$).

The reduced action for the system (modulo a spurious boundary term $-pq|_{t_i}^{t_f}$) is

$$\begin{aligned} S_r &= \int_{t_i}^{t_f} dt \left\{ \dot{p} \left(q + \oint d\theta t P_\tau \right) + \oint d\theta [P_\psi \dot{\psi} + p P_\tau] \right\} \\ &= \int_{t_i}^{t_f} dt \left\{ \dot{p} \eta + \oint d\theta \left[P_\psi \dot{\psi} - \frac{1}{2} \left(\frac{P_\psi^2}{tp} + tp\psi'^2 \right) \right] \right\}, \\ \eta &:= q - \oint d\theta \frac{t}{2p} \left(\frac{P_\psi^2}{tp} + tp\psi'^2 \right). \end{aligned} \quad (16)$$

In the first equality, P_τ denotes the solution given in Eq. (11). Thus, S_r is a functional on the reduced phase space Γ_r , which is coordinatized by (η, p, ψ, P_ψ) , and where the (only nonvanishing) basic Poisson brackets are $\{p, \eta\} = 1$ and $\{\psi(t, \theta), P_\psi(t, \tilde{\theta})\} = \delta(\theta - \tilde{\theta})$. Note that, owing to the presence of the global constraint (13), the space of physical states does not correspond to Γ_r but to a submanifold of it. However, since this submanifold is nonlinear, the reduction by the constraint is usually postponed to the quantum theory, where it is imposed as an operator condition on quantum states.

Let us now perform the canonical transformation

$$\begin{aligned} \phi &= \sqrt{p}\psi, & P_\phi &= \frac{P_\psi}{\sqrt{p}}, \\ Q &= -\eta + \frac{1}{2p} \oint d\theta P_\psi \psi, & P &= p. \end{aligned} \quad (17)$$

In terms of this new set of phase space variables the reduced action reads

$$S_r = \int_{t_i}^{t_f} dt \left(P \dot{Q} + \oint d\theta [P_\phi \dot{\phi} - \mathcal{H}_r] \right) - PQ|_{t_i}^{t_f}, \quad (18)$$

where the (reduced) Hamiltonian density is

$$\mathcal{H}_r = \frac{1}{2} \left(\frac{P_\phi^2}{t} + t\phi'^2 \right). \quad (19)$$

Thus, our midisuperspace model consists of a phase space $\tilde{\Gamma}_r$ coordinatized by the canonical pairs (Q, P) and (ϕ, P_ϕ) , which we will call the global and local degrees of freedom, respectively. Remember that P is strictly positive. To arrive at a true canonical pair of real variables, we could always replace (Q, P) with $(QP, \ln P)$. There also remains a global constraint on the system ($\Lambda_0 = 0$) which restricts the physical states to lie in a submanifold of $\tilde{\Gamma}_r$. Note that, given the (Q, P) independence of the Hamiltonian density, these “point particle” degrees of freedom are constants of motion. Hence a nontrivial evolution may only take place in the field sector $\Gamma = \{(\phi, P_\phi)\}$. Since the time evolution affects only the local degrees of freedom, we will focus on them in our analysis.

Varying action (18) with respect to ϕ and P_ϕ one gets the field equations

$$P_\phi = t\dot{\phi}, \quad \dot{P}_\phi = t\phi''. \quad (20)$$

Hence, we only have to consider all smooth solutions to the second-order differential equation

$$\ddot{\phi} + \frac{1}{t}\dot{\phi} - \phi'' = 0 \quad (21)$$

in order to specify the classical spacetime metric. Using the method of separation of variables, it is not difficult to see that these solutions, that we will generically denote by φ , adopt the form [7]

$$\begin{aligned} \varphi(t, \theta) &= \frac{1}{2\sqrt{2}} \sum_{n \in \mathbb{Z}, n \neq 0} [A_n H_0(|n|t) e^{in\theta} + A_n^* H_0^*(|n|t) e^{-in\theta}] \\ &\quad + \frac{1}{\sqrt{2\pi}} (\bar{q}_0 + \bar{p}_0 \ln t), \end{aligned} \quad (22)$$

where the symbol $*$ denotes complex conjugation, $\bar{q}_0$ and $\bar{p}_0$ are constants, H_0 is the zeroth-order Hankel function of the second kind [21] and, in order to guarantee pointwise convergence, the sequence of constant coefficients $\{A_n\}$ has to decrease faster than the inverse of any polynomial in n as $n \rightarrow \pm\infty$. Expression (22) determines the metric (14)

(with $\psi = \varphi/\sqrt{p}$) except for the values of (q, p) . One can show that its Kretschmann scalar blows up at $t = 0$, so that there is an initial singularity and the global time function t must be strictly positive.

In the field sector, the physical phase space can be alternatively described by the submanifold obtained by imposing the constraint Λ_0 in Γ , or by that submanifold of the space V of the smooth solutions (22) defined by the constraint

$$\tilde{\Lambda}_0 := \sum_{n \in \mathbb{Z}, n \neq 0} n A_n^* A_n = 0. \quad (23)$$

In addition notice that, for the field sector, the reduced action (18) can be viewed as that corresponding to an axisymmetric, massless, free scalar field propagating in the fictitious flat background in three dimensions:

$${}^{(f)}g_{AB} = -(dt)_A(dt)_B + (d\theta)_A(d\theta)_B + r^2(d\sigma)_A(d\sigma)_B. \quad (24)$$

Thus, we can identify Γ with the canonical phase space of the scalar field in this background, $(\mathcal{M} \simeq T^2 \times \mathbb{R}^+, {}^{(f)}g_{AB})$, whereas the space V of smooth solutions can be considered as the covariant phase space of such a Klein-Gordon field. Namely, ϕ and P_ϕ are the configuration and momenta on the constant-time section Σ_t of the scalar field φ propagating in $(\mathcal{M}, {}^{(f)}g_{AB})$. Besides, since the fictitious background is globally hyperbolic, given a smooth Cauchy surface Σ_{t_0} there will be a natural isomorphism $I[\Sigma_{t_0}]$ between the linear spaces Γ and V . In this framework, the analysis of the dynamics of the polarized Gowdy T^3 model becomes equivalent to the study of the time evolution of the free scalar field. In the next section we review this dynamics and discuss the obstructions to its unitary quantum implementation.

III. DYNAMICS

It has recently been shown that the dynamical evolution generated by the reduced Hamiltonian $H_r = \oint \mathcal{H}_r$ [see Eq. (19)] cannot be implemented as a unitary transformation, neither on the kinematical Fock space [6] constructed from V with the complex structure associated with the field decomposition (22), nor in the physical Hilbert space of states [7] determined by the kernel of the operator version of the constraint (23). We want to analyze in detail the reasons behind this lack of a unitary implementation and discuss how severe the problem is, studying whether small corrections (coming e.g. from quantum or perturbative modifications) to the dynamics may suffice to restore the unitarity.

For the space V , we will employ as coordinates the constant coefficients of the field decomposition (22), whereas for Γ we will use a different set that absorbs in its (implicit) time dependence all the evolution of the field. We will see that the dynamics in Γ is dictated by H_r ,

whereas that in V is frozen, because the considered coefficients are constants of motion.³

For the sake of completeness and clarity, let us remember some definitions and make a few remarks that will be useful in our analysis. First, we recall that given two field decompositions in different orthonormal bases of solutions, namely $\varphi = \sum_n A_n f_n(t, \theta) + A_n^* f_n^*(t, \theta)$ and $\varphi = \sum_n \tilde{A}_n \tilde{g}_n(t, \theta) + \tilde{A}_n^* \tilde{g}_n^*(t, \theta)$, their coefficients are related by a Bogoliubov transformation. That is, $\tilde{A}_n = \sum_m \alpha_{mn} A_m + \beta_{mn}^* A_m^*$ with $\sum_k (\alpha_{ik} \alpha_{jk}^* - \beta_{ik} \beta_{jk}^*) = \delta_{ij}$ and $\sum_k (\alpha_{ik} \beta_{jk} - \beta_{ik} \alpha_{jk}) = 0$ [22].

Second, as in the classical case, the quantum theory for the scalar field can be formulated either in a covariant or in a canonical approach. In fact, it is generally known that, by endowing the space of (smooth) solutions $V = \{\varphi\}$ to the Klein-Gordon equation with a complex structure J compatible with the symplectic structure, one can construct in a canonical way the Hilbert space of the quantum theory as a symmetric Fock space $\mathcal{F}$ on which the basic observables of the theory are represented as annihilation and creation operators [23]. On the other hand, if the scalar field propagates in a globally hyperbolic spacetime, e.g. $\mathcal{M} \approx \Sigma \times \mathbb{R}^+$, given an embedding Σ_{t_0} of Σ as a Cauchy surface in $\mathcal{M}$ one gets, from the covariant complex structure J on V , the induced complex structure J_0 on the canonical phase space Γ [24]. Once we have J_0 , we know how to construct the Schrödinger representation which is unitarily equivalent to the Fock one [24] and how to pass from one representation to the other [25]. In particular, we have the analog of the one-particle Hilbert space and, by applying the creation operator, one can construct the n (functional) particle states.

In addition, as a consequence of this unitary relation between the covariant and canonical approaches, if a symplectic transformation is unitarily implementable with respect to the Fock representation, so is it with respect to the (unitarily equivalent) Schrödinger representation (and vice versa). Recall that a symplectic transformation on V (Γ) will be unitarily implementable with respect to the Fock (Schrödinger) representation if the antilinear part of its quantum counterpart is Hilbert-Schmidt on the one-particle Hilbert space [26]. For a quantum transformation of the Bogoliubov type, the Hilbert-Schmidt condition reduces to $\sum_{n,m} |\beta_{nm}|^2 < \infty$.

Consider now the Schrödinger representation constructed from J_0 (the complex structure induced by the covariant one, J). Given a symplectic transformation T on Γ , we obtain an induced complex structure $J'_0 = T J_0 T^{-1}$

³Let us emphasize that the coefficients in Eq. (22) do not display any time dependence, not only explicitly, but also implicitly. Accordingly, the total Hamiltonian in V indeed vanishes. Properly speaking, the nonunitarity proved in Ref. [6] is that of the transformation generated by H_r on V , which in turn can be seen to imply the nonunitary character of the dynamics in Γ , rather than in V .

and, associated with it, a new Schrödinger representation. The annihilation and creation operators in this new representation are related with the annihilation and creation operators of the former one through a Bogoliubov transformation. The representations corresponding to J'_0 and J_0 are then unitarily equivalent if the antilinear part of this Bogoliubov transformation is square summable.

In particular, if $T = T_{(t_1, t_2)}$ represents the time evolution from t_1 to t_2 on Γ , then the Bogoliubov transformation relates the annihilation and creation operators at the instants t_1 and t_2 . With an appropriate choice of coordinates in Γ , the symplectic transformation $T_{(t_1, t_2)}$ acts on the elements of this space exactly as the Bogoliubov transformation on their quantum counterparts. Thus, in order to elucidate whether the symplectic transformation is unitarily implementable it suffices to analyze the square summability of the antilinear part of $T_{(t_1, t_2)}$. Let us emphasize that this procedure is equivalent to determining whether the Schrödinger representations constructed from the same Γ (associated with the embedding Σ_{t_1}) but with the distinct complex structures J_1 and $J'_2 = T_{(t_1, t_2)} J_1 T_{(t_1, t_2)}^{-1}$ are unitarily related.

In the following, we will focus our discussion on analyzing the failure of unitary implementability of the symplectic transformation that determines the dynamics in Γ , rather than examining the complex structures induced by it, since both procedures are equivalent, as we have just seen.

A. Evolution in the canonical and covariant approaches

We start by expanding in Fourier series our canonical variables ϕ and P_ϕ :

$$\phi = \sum_{n \in \mathbb{Z}} \frac{1}{\sqrt{2\pi}} \phi_n e^{in\theta}, \quad P_\phi = \sum_{n \in \mathbb{Z}} \frac{1}{\sqrt{2\pi}} P_\phi^n e^{in\theta}, \quad (25)$$

where the coefficients ϕ_n and P_ϕ^n are (implicit) functions of the global time coordinate. From the basic Poisson bracket between ϕ and P_ϕ , it is easily shown that $\{\phi_n, P_\phi^m\} = \delta_n^m$. Therefore, we can equivalently consider as our canonical phase space that whose coordinates are the set of (complex) canonical pairs $\{(\phi_n, P_\phi^n)\}_{n \in \mathbb{Z}}$. We will call it $\Gamma_{(\phi, P_\phi)_n}$. Note that this space can be decomposed as the direct sum of Γ' and Γ_0 , where Γ' is the subspace of vectors with $\phi_0 = P_\phi^0 = 0$ and Γ_0 is the span of those vectors whose only nonvanishing components are precisely those corresponding to ϕ_0 and P_ϕ^0 . For all positive integers $m \in \mathbb{N} - \{0\}$, let us now consider the transformations

$$(\phi_m, P_\phi^m, \phi_{-m}, P_\phi^{-m}) \mapsto (a_m, a_m^*, a_{-m}, a_{-m}^*), \quad (26)$$

where

$$a_m = \frac{|m| \phi_m + i P_\phi^{-m}}{\sqrt{2|m|}}, \quad a_m^* = \frac{|m| \phi_{-m} - i P_\phi^m}{\sqrt{2|m|}}. \quad (27)$$

One can check that these transformations are canonical, so that $\{a_n, i a_m^*\} = \delta_{nm}$. Hence, the canonical phase space can be alternatively described by the symplectic vector space $\Gamma_a = \Gamma_0 \oplus \bar{\Gamma}$, where the coordinates for $\bar{\Gamma}$ are the (complex conjugate pairs of) annihilation and creationlike variables $\{(a_m, a_m^*, a_{-m}, a_{-m}^*)\}_{m \in \mathbb{N} - \{0\}}$. The dynamics on Γ_a [as well as in Γ and $\Gamma_{(\phi, P_\phi)_n}$] is dictated by the reduced Hamiltonian for the polarized Gowdy T^3 model, $H_r = \oint \mathcal{H}_r$, where $\mathcal{H}_r$ is the reduced Hamiltonian density (19). Using expressions (25) and (27), and defining

$$\begin{aligned} H_0 &:= \frac{(P_\phi^0)^2}{2t}, \\ H_m &:= \frac{t^2 + 1}{2t} m(a_m a_m^* + a_{-m} a_{-m}^*) \\ &\quad + \frac{t^2 - 1}{2t} m(a_m^* a_{-m}^* + a_m a_{-m}), \end{aligned} \quad (28)$$

the Hamiltonian can be rewritten

$$H_r = H_0[P_\phi^0] + \sum_{m \in \mathbb{N}, m \neq 0} H_m[a_m, a_m^*, a_{-m}, a_{-m}^*]. \quad (29)$$

Notice that the Hamiltonian vector field $X_{H_r}^A$ on Γ_a is just the sum of the Hamiltonian vector fields $X_{H_0}^A$ on Γ_0 and X_H^A on $\bar{\Gamma}$, with $H := \sum_{m \in \mathbb{N} - \{0\}} H_m$. In other words, for any state in Γ_a , the time evolution can be deduced by composing the evolution of the projections in Γ_0 and in $\bar{\Gamma}$, where the dynamics are dictated, respectively, by H_0 and H . In particular, we see that the unitary implementability of the dynamics in Γ_a at the quantum level depends only on whether the finite transformations generated by H on $\bar{\Gamma}$ can be unitarily implemented.

The covariant phase space V , on the other hand, can be described using as coordinates the canonical pair $(\bar{q}_0, \bar{p}_0)$ and the set of (complex conjugate) annihilation and creationlike variables $\{(A_n, A_n^*)\}_{n \in \mathbb{Z} - \{0\}}$. Let us denote the covariant phase space, in such a coordinate system, by V_A . We can now separate the zero modes exactly as before, namely, V_A can be viewed as the direct sum of the subspace $\bar{V}$ for which $\bar{q}_0 = \bar{p}_0 = 0$ and the subspace V_0 whose vectors have $\bar{q}_0$ and $\bar{p}_0$ as the only nonzero components.

In the following, we will, respectively, denote states in Γ_0 and V_0 by $u_0 := (\phi_0, P_\phi^0)$ and $v_0 := (\bar{q}_0, \bar{p}_0)$. Similarly, states in $\bar{\Gamma}$ and $\bar{V}$ will be denoted by $\{\gamma_m\} := \{(a_m, a_m^*, a_{-m}, a_{-m}^*)\}$ and $\{\mathcal{A}_m\} := \{(A_m, A_m^*, A_{-m}, A_{-m}^*)\}$, with $m \in \mathbb{N} - \{0\}$.

Since ϕ and P_ϕ are the configuration and momenta at Σ_t of φ , we can express the coefficients ϕ_n and P_ϕ^n in terms of A_n and A_{-n}^* for all $n \in \mathbb{Z} - \{0\}$, and in terms of $\bar{q}_0$ and $\bar{p}_0$ for the zero mode, getting in this way a map $\tilde{M}$ from V_A to $\Gamma_{(\phi, P_\phi)_n}$. In addition, Eq. (27) defines a map $\bar{M}$ from $\Gamma_{(\phi, P_\phi)_n}$ to Γ_a . Then the composition $\bar{M} \tilde{M}$ provides us with a map $M: V_A \rightarrow \Gamma_a$. A straightforward calculation shows that for the zero modes

$$u_0 = \begin{pmatrix} 1 & \ln t \\ 0 & 1 \end{pmatrix} v_0, \quad (30)$$

where u_0 and v_0 are treated as column vectors. For the rest of the modes ($m \in \mathbb{N} - \{0\}$), one obtains $\gamma_m = M_m(t) \mathcal{A}_m$ with

$$M_m(t) = \begin{pmatrix} c_m(t) & 0 & 0 & d_m(t) \\ 0 & c_m^*(t) & d_m^*(t) & 0 \\ 0 & d_m(t) & c_m(t) & 0 \\ d_m^*(t) & 0 & 0 & c_m^*(t) \end{pmatrix} \quad (31)$$

and

$$\begin{aligned} c_m(t) &= \sqrt{\frac{\pi m}{8}} [H_0(mt) - itH_1(mt)], \\ d_m(t) &= \sqrt{\frac{\pi m}{8}} [H_0^*(mt) - itH_1^*(mt)]. \end{aligned} \quad (32)$$

Here, H_1 is the first-order Hankel function of the second kind [21]. Note that the map M is such that $M(V_0) = \Gamma_0$ and $M(\bar{V}) = \bar{\Gamma}$. Besides, the determinant of the linear transformation (30), as well as that of M_m , is equal to the unity. It hence follows that the M_m 's are Bogoliubov transformations. Thus, we get a time-dependent canonical transformation from V_A to Γ_a .

A generating function for this transformation (that depends on some appropriately chosen complete sets of compatible components—under Poisson brackets—both for V_A and Γ_a) is

$$\begin{aligned} \mathcal{F}_0(t) &= \frac{1}{2} (\bar{p}_0)^2 \ln t - \bar{p}_0 \phi_0, \\ \mathcal{F}_m(t) &= ia_{-m}^* [c_m(t) A_{-m} + d_m(t) A_m^*] \\ &\quad - ia_m [d_m^*(t) A_{-m} + c_m^*(t) A_m^*], \end{aligned} \quad (33)$$

for $m = 0$ and $m \in \mathbb{N} - \{0\}$, respectively. After a straightforward calculation we find that the partial derivative of this generating function with respect to its explicit dependence on the time coordinate t has the following form when expressed exclusively in terms of the components of the states in Γ_a :

$$\partial_t \mathcal{F}_0 = \frac{1}{2t} (P_\phi^0)^2, \quad \partial_t \mathcal{F}_m = H_m[\gamma_m]. \quad (34)$$

Therefore, we get

$$\partial_t \mathcal{F}[u, \gamma] = \sum_{m \in \mathbb{N}} \partial_t \mathcal{F}_m[u, \gamma] = H_r[u, \gamma], \quad (35)$$

where H_r is precisely the Hamiltonian (29).

At this point of the discussion, it is worth recalling that, given a canonical transformation from certain symplectic vector space $E_1 := \{(q_i, p_i)\}$ to another one $E_2 := \{(Q_i, P_i)\}$ which is determined by a generating function F that is explicitly time dependent [27], and assumed that the dynamics in E_1 is dictated by the Hamiltonian $H_1[q, p]$, the corresponding Hamiltonian in E_2 is $H_2[Q, P] =$

$H_1[q(Q, P), p(Q, P)] + \partial_t F[Q, P]$. Taking into account Eq. (35), we then see that the dynamical evolution in Γ_a , generated by H_r , arises entirely from the time dependence of the canonical transformation. As we have pointed out before, the total Hamiltonian in V_A is identically zero and there is no time evolution for the states $(v_0, \{\mathcal{A}_m\})$. Obviously, this vanishing of the Hamiltonian applies as well to the restrictions to the subspaces V_0 and $\bar{V}$. In particular, while the states in $\bar{\Gamma}$ evolve along the integral curves of the Hamiltonian vector field X_H^A , the states in $\bar{V}$ are “frozen.” Hence, an initial state $\{\gamma_m(t_0)\}$ in $\bar{\Gamma}$ will evolve to the state $\{\gamma_m(t)\}$ determined by the transformation $\gamma_m(t) = U_{H_m[\gamma]}^{(m)}(t, t_0) \gamma_m(t_0)$, where⁴ $U_{H_m[\gamma]}^{(m)}(t, t_0) := M_m(t) M_m(t_0)^{-1}$. In contrast, the corresponding states in $\bar{V}$, specified by $\mathcal{A}_m(t_0) = M_m(t_0)^{-1} \gamma_m(t_0)$ and $\mathcal{A}_m(t) = M_m(t)^{-1} \gamma_m(t)$, will be related via the identity map, so that they actually coincide.

In coordinates $\mathcal{A}_m$ rather than γ_m , the finite transformation $U_{H_m[\gamma]}^{(m)}$ is given by $U_{H_m[\mathcal{A}]}^{(m)}(t, t_0) = M_m(t_0)^{-1} M_m(t)$. As a result, the complex antilinear part of the finite transformation generated by $H = H_r - H_0$ in $\bar{V}$ is given (for each $m \in \mathbb{N} - \{0\}$) by

$$D_m(t, t_0) = \frac{i\pi m}{4} [t_0 H_1^*(mt_0) H_0^*(mt) - t H_0^*(mt_0) H_1^*(mt)], \quad (36)$$

which is not square summable in m , as has been proved in Ref. [6]. Therefore, the finite transformation provided by $M_m(t_0)^{-1} M_m(t)$ (with m running in $\mathbb{N} - \{0\}$) cannot be unitarily implemented. Moreover, since the antilinear part of $U_{H_m[\gamma]}^{(m)}(t, t_0)$ differs from that of $U_{H_m[\mathcal{A}]}^{(m)}(t, t_0)$ just by a sign in the phase of the coefficient $c_m(t)$, as one can easily check, we see that the finite transformation generated by H in $\bar{\Gamma}$ is not unitarily implementable. It is worth emphasizing that, however, this is not the case for the dynamics in $\bar{V}$; indeed, since such a dynamics is generated by the zero Hamiltonian, the evolution is described by the identity transformation, which is of course unitary.

Actually, since $U_{H_m[\gamma]}^{(m)}(t, t_0)$ is just a composition of the Bogoliubov transformations $M_m(t)$ and $M_m(t_0)$, the lack of a unitary implementation of the dynamics in $\bar{\Gamma}$ follows from the fact that the antilinear part of $M_m(t)$ fails to be square summable for generic $t > 0$. Hence, whether or not the dynamics can be unitarily implemented depends entirely on the behavior of $d_m(t)$ for large integers m . This depends in turn on the Hamiltonian via Eq. (34), which relates the generating function of $M_m(t)$ with the generator of the dynamics after the canonical transformation has been performed. Our analysis about the lack of unitarity will therefore focus on the identification of those character-

⁴ Actually $U_{H_m[\gamma]}^{(m)}(t_0, t_0) = I$ and $U_{H_m[\gamma]}^{(m)}(t_3, t_1) = U_{H_m[\gamma]}^{(m)}(t_3, t_2) \times U_{H_m[\gamma]}^{(m)}(t_2, t_1)$.

istics of the Hamiltonian that are in the origin of the failure of square summability. In doing so, we will be able to establish how critical this problem is and whether it can be corrected with small modifications to the dynamics. Roughly speaking, we will be able to determine how far the considered evolution is from being unitarily implementable. With this aim, in the next subsection we introduce a correction to the Hamiltonian H , which might be viewed as a perturbation or a quantum modification, and discuss the square summability of the antilinear part of the transformation generated by the new Hamiltonian.

B. Modified Hamiltonian

Motivated by our previous analysis, let us assume now that a certain linear (free) field theory can be described by either of the two symplectic vector spaces $E_A := \{\mathcal{A}_m\}$ or $E_B := \{\mathcal{B}_m\}$, where $\mathcal{A}_m := (A_m, A_m^*, A_{-m}, A_{-m}^*)$, $\mathcal{B}_m := (B_m, B_m^*, B_{-m}, B_{-m}^*)$, and $m \in \mathbb{N} - \{0\}$. Here, (A_m, A_m^*) and (B_m, B_m^*) are annihilation and creationlike pairs. In addition, let us suppose that the canonical map $M_m(t)$ from E_A to E_B , which in general may depend on the time coordinate t , is a Bogoliubov transformation of the form (31), with $|c_m(t)|^2 - |d_m(t)|^2 = 1$. Besides, we assume that the total Hamiltonian in E_A is zero, so that the Hamiltonian in E_B is given by $\tilde{H} = \sum_{m \in \mathbb{N} - \{0\}} \partial_t \mathcal{F}_m$, where $\mathcal{F}_m$ is a generating function of the transformation $M_m(t)$. Furthermore, we admit that (in coordinates $\mathcal{A}_m$) the partial derivative of the generating function $\mathcal{F}_m$ with respect to its explicit time dependence is

$$\begin{aligned} \partial_t \mathcal{F}_m &= \tilde{H}_m[\mathcal{A}] \\ &:= 2m[\mu a(x)A_m A_{-m} + \mu^* a^*(x)A_m^* A_{-m}^*] \\ &\quad + 2m\lambda b(x)[A_m^* A_m + A_{-m}^* A_{-m}], \end{aligned} \quad (37)$$

where $x := mt$ (strictly speaking we should write x_m , however we drop out the subindex to simplify the notation). Besides, λ and μ are a real and a complex constant,⁵ respectively, and

$$\begin{aligned} a(x) &= \frac{\pi}{8} x \{[H_0(x)]^2 + [H_1(x)]^2\}, \\ b(x) &= \frac{\pi}{8} x \{|H_0(x)|^2 + |H_1(x)|^2\}. \end{aligned} \quad (38)$$

At this stage, it is convenient to point out the analogy with our symmetry-reduced model. The Bogoliubov transformation $M_m(t)$ has the same form as that in Eq. (31). The symplectic vector spaces E_A and E_B play the role of $\bar{V}$ and $\bar{\Gamma}$, respectively. In this sense, note that the total Hamiltonian in E_A is zero. Moreover, setting $\lambda = \mu = 1$ in Eq. (37) one merely gets the H_m contribution to the total

Hamiltonian H in $\bar{\Gamma}$, expressed in coordinates $\mathcal{A}_m$ [7]. Hence, we can think in terms of the spaces $\bar{V}$ and $\bar{\Gamma}$, and regard the phase space function $\tilde{H}$ as a modification of the Hamiltonian H . Defining $\rho := \lambda - 1$ and $\epsilon := \mu - 1$, one may view the case $|\rho| \ll 1$, $|\epsilon| \ll 1$ as a perturbation of the Hamiltonian, arising from certain (classical or quantum) corrections to the dynamics.

As we already know, the unitary implementability of the dynamics dictated by $\tilde{H}$ in E_B depends on the square summability of the antilinear part of $M_m(t)$. By analyzing this summability we will relate the failure of unitarity with the precise form of $\tilde{H}$.

We first determine the relations that Eq. (37) imposes on the complex functions $c_m(t)$ and $d_m(t)$ that specify the Bogoliubov transformation $M_m(t)$. A generating function for this transformation is

$$\begin{aligned} \mathcal{F}_m(t) &= iB_{-m}^*[c_m(t)A_{-m} + d_m(t)A_m^*] \\ &\quad - iB_m[d_m^*(t)A_{-m} + c_m^*(t)A_m^*]. \end{aligned} \quad (39)$$

Taking the (explicit) time derivative and using then the inverse of $M_m(t)$ [which is easily calculated from Eq. (31)], one arrives at the following expression, exclusively in terms of the coordinates $\mathcal{B}_m$:

$$\begin{aligned} \partial_t \mathcal{F}_m &= iB_{-m}^*B_{-m}(\dot{c}_m c_m^* - \dot{d}_m d_m^*) \\ &\quad + iB_m B_{-m}(\dot{c}_m^* d_m^* - \dot{d}_m^* c_m^*) \\ &\quad + iB_m^* B_m(\dot{d}_m^* d_m - \dot{c}_m^* c_m) \\ &\quad + iB_m^* B_{-m}^*(\dot{d}_m c_m - \dot{c}_m d_m). \end{aligned} \quad (40)$$

The dot denotes the (total) derivative with respect to the time coordinate t . From now on, we do not generally display the dependence of c_m and d_m on this coordinate in order to simplify the notation.

By translating also into coordinates $\mathcal{B}_m$ the Hamiltonian $\tilde{H}_m$, the condition (37) that the dynamics in E_B arise entirely from the time derivative of our canonical transformation can be seen to reduce to the following system of first-order (complex) differential equations for c_m and d_m :

$$\begin{aligned} 0 &= \frac{i}{2}(\dot{c}_m^* c_m - \dot{d}_m^* d_m) + m\lambda b(x)(|c_m|^2 + |d_m|^2) \\ &\quad - m[\mu a(x)c_m^* d_m + \mu^* a^*(x)c_m d_m^*], \end{aligned} \quad (41)$$

$$\begin{aligned} 0 &= \frac{i}{2}(\dot{d}_m c_m - \dot{c}_m d_m) + 2m\lambda b(x)c_m d_m \\ &\quad - m[\mu a(x)d_m d_m + \mu^* a^*(x)c_m c_m]. \end{aligned} \quad (42)$$

Let us call Y_m the ratio d_m/c_m . Since $|c_m|^2 - |d_m|^2 = 1$, we have that $|Y_m|$ is strictly smaller than the unity and, in terms of it, the (complex) norms of c_m and d_m are

$$|c_m|^2 = \frac{1}{1 - |Y_m|^2}, \quad |d_m|^2 = \frac{|Y_m|^2}{1 - |Y_m|^2}. \quad (43)$$

Realizing that $(\dot{d}_m c_m - \dot{c}_m d_m)/c_m^2$ is just the time deriva-

⁵We might allow for an x dependence in λ and μ , but this would unnecessarily complicate our discussion. Some comments about this generalization of the analysis are presented in Sec. III C.

tive of Y_m and performing the change of variable $t \mapsto x = mt$ (so that $\dot{Y}_m = m dY_m/dx$), it is easy to see that Eq. (42) can be rewritten as

$$\frac{i}{2} \frac{dY_m}{dx}(x) + 2\lambda b(x)Y_m(x) - \mu a(x)Y_m^2(x) - \mu^* a^*(x) = 0. \quad (44)$$

Remarkably, this differential equation for Y_m is independent of the positive integer m (regarding x as the relevant variable). Using this universal character of the equation, valid for all values of m , we can drop out the subindex in the function Y_m and consider it as a single function Y for all the modes of our system. With the convenient redefinitions

$$z(x) := \exp(-2ix)Y(x), \quad \Delta(x) := \exp(2ix)a(x), \quad (45)$$

we then arrive at the following equation for z :

$$\frac{dz}{dx}(x) = 2iz(x)[2\lambda b(x) - 1] - 2i\mu\Delta(x)z^2(x) - 2i\mu^*\Delta^*(x). \quad (46)$$

In addition, given a function z satisfying Eq. (46) and remembering that $|c_m|^2 - |d_m|^2 = 1$, it is straightforward to see that the differential Eq. (41) is equivalent to

$$\frac{d \ln c_m}{dx}(x) = 2i[\mu\Delta(x)z(x) - \lambda b(x)] \quad (47)$$

which again is a universal equation for all modes with $m \in \mathbb{N} - \{0\}$. We suppress the subindex m and consider only one function $c(x)$, which can be obtained by direct integration of Eq. (47) (except for a multiplicative constant that can be fixed with an initial condition for c). Finally, the function $d(x) := \exp(2ix)z(x)c(x)$ provides the missing coefficient of our Bogoliubov transformation, namely, $d_m(t) = d(x = mt)$.

Using Eq. (43) and $|Y(x)| = |z(x)|$, we conclude

$$|d_m(t)|^2 = |d(x = mt)|^2 = \frac{|z(x = mt)|^2}{1 - |z(x = mt)|^2}. \quad (48)$$

Therefore, an important consequence of the observed universality is that the square summability of the coefficients $d_m(t)$ at any fixed positive value of t , which is only sensitive to the behavior for large m , turns out to depend exclusively on the behavior of the function $z(x)$ when x approaches infinity (because $x = mt$ grows linearly with m for all $t > 0$). Thus, to discuss the square summability of d_m , we only need to consider Eq. (46) and exploit our knowledge about the asymptotic behavior of the functions $\Delta(x)$ and $b(x)$.

From Hankel's asymptotic expansions of H_0 and H_1 [21] one gets that, for $x \gg 1$, $\Delta(x)$ and $b(x)$ are given by the asymptotic series⁶

⁶In general, we will say that a function f admits an asymptotic series at infinity if there exists a series of the form $\sum_{k=0}^{\infty} f_k/x^k$ such that $\lim_{x \rightarrow \infty} |x^N \{f(x) - \sum_{k=0}^N f_k/x^k\}| = 0$ for all $N \geq 0$ (see e.g. Ref. [28]).

$$\Delta(x) = \frac{i}{4} \sum_{k,n=0}^{\infty} \frac{(-1)^{k+n}}{(2x)^{2(k+n)}} \left[\xi_{k,n} - i \frac{\xi_{k,n+1/2}}{x} - \frac{\xi_{k+1/2,n+1/2}}{4x^2} \right], \quad (49)$$

$$b(x) = \frac{1}{4} \sum_{k,n=0}^{\infty} \frac{(-1)^{k+n}}{(2x)^{2(k+n)}} \left[\sigma_{k,n} + \frac{\sigma_{k+1/2,n+1/2}}{4x^2} \right],$$

where

$$\begin{aligned} \xi_{k,n} &:= (0, 2k)(0, 2n) - (1, 2k)(1, 2n), \\ \sigma_{k,n} &:= (0, 2k)(0, 2n) + (1, 2k)(1, 2n), \end{aligned} \quad (50)$$

and (k, n) is the so-called Hankel symbol:

$$(k, n) := \frac{\Gamma(k + n + \frac{1}{2})}{n! \Gamma(k - n + \frac{1}{2})}. \quad (51)$$

The asymptotic series representation for $b(x)$ contains only even powers in $1/x$, while the corresponding asymptotic series for Δ contains, in principle, both even and odd powers. With the change of variable $y = 1/x$, these asymptotic expansions can thus be written in the form

$$\Delta(y) = \sum_{k=0}^{\infty} \Delta_k y^k, \quad b(y) = \sum_{k=0}^{\infty} b_k y^k, \quad (52)$$

where $b_{2k+1} = 0$ for all $k \in \mathbb{N}$. From Eqs. (49), we get in particular

$$\begin{aligned} \Delta_0 &= i \frac{\xi_{0,0}}{4} = 0, & \Delta_1 &= \frac{\xi_{0,1/2}}{4} = -\frac{1}{4}, \\ b_0 &= \frac{\sigma_{0,0}}{4} = \frac{1}{2}. \end{aligned} \quad (53)$$

Employing the asymptotic expressions for $\Delta(x)$ and $b(x)$, a formal asymptotic series for z can be constructed. More precisely, introducing expansions (52) in the differential equation (46) (with the change $y = 1/x$) and writing z as an asymptotic series in y , namely $z = \sum_k z_k y^k$, one obtains

$$z_0^2 \mu \Delta_0 + z_0(1 - 2\lambda b_0) + \mu^* \Delta_0^* + \sum_{k=1}^{\infty} y^k S_k = 0, \quad (54)$$

where, for each $k \geq 1$,

$$\begin{aligned} S_k &:= \mu \sum_{m=0}^k \Delta_{k-m} \left(\sum_{j=0}^m z_{m-j} z_j \right) - 2\lambda \sum_{m=0}^k b_m z_{k-m} \\ &\quad + z_k + \mu^* \Delta_k^* + i \frac{k-1}{2} z_{k-1} + \mu \Delta_k z_0^2. \end{aligned} \quad (55)$$

In particular, the term independent of y in Eq. (54) must vanish. Since $b_0 = 1/2$ and $\Delta_0 = 0$, one gets (with $\rho = \lambda - 1$)

$$z_0 \rho = 0. \quad (56)$$

Hence, provided that ρ does not vanish, the coefficient z_0 must be zero. As a consequence, the resulting function z will tend to zero as $x \rightarrow \infty$, which in turn implies that $|d|$ vanishes in the limit of large values of x .

The rest of the terms in Eq. (54) require the vanishing of S_k for all $k \geq 1$. Substituting $z_0 = 0$ and the values of b_0 and Δ_0 , it is straightforward to derive the following recurrence relation for the complex coefficients of the asymptotic series of z :

$$z_k = \frac{1}{\rho} \left[\mu \sum_{m=0}^{k-1} \Delta_{k-m} \left(\sum_{j=0}^m z_{m-j} z_j \right) - 2(1 + \rho) \times \sum_{m=1}^k b_m z_{k-m} \right] + \frac{1}{\rho} \left[\mu^* \Delta_k^* + i \frac{k-1}{2} z_{k-1} \right]. \quad (57)$$

For the first coefficient, we get $z_1 = \mu^* \Delta_1^* / \rho = -\mu^* / (4\rho)$. Thus, for sufficiently large values of x , the differential equation (46) should admit a solution z such that⁷

$$z(x) = -\left(\frac{\mu^*}{4\rho}\right) \frac{1}{x} + o\left(\frac{1}{x}\right). \quad (58)$$

As we will see, this result suffices to prove the square summability of the coefficients d_m . Of course, the above behavior is not allowed for z when $2b_0\lambda - 1 = \rho$ vanishes, as is the case for the Hamiltonian H . This explains the breakdown of unitarity for that specific case.

C. Unitarity of the modified dynamics

Let us finally show that the deduced asymptotic behavior for $z(x)$, together with relation (48), guarantees the square summability of the sequence $\{d_m(t)\}$ for all fixed, strictly positive values of the time coordinate t . From Eq. (58), we see that, at infinity, $\lim_{x \rightarrow \infty} |xz + \mu^* / (4\rho)| = 0$. So, given any constant $\varepsilon > 0$ there exists a positive number $x_0(\varepsilon)$ such that $|xz + \mu^* / (4\rho)| < \varepsilon$ for all $x > x_0(\varepsilon)$. One can see that this inequality implies that, for all $x > x_0(\varepsilon)$,

$$|z|^2 < \frac{R(\varepsilon, \rho, \mu)}{x^2}, \quad (59)$$

$$R(\varepsilon, \rho, \mu) := 2\varepsilon^2 + \frac{\varepsilon|\mu|}{|\rho|} + \frac{|\mu|^2}{16\rho^2}.$$

Let us now choose a number $\tilde{x}_0$ in the interval $(x_0(\varepsilon), \infty)$ such that $\eta_0 := R(\varepsilon, \rho, \mu) / \tilde{x}_0^2 < 1$, which is clearly always possible. Then, for all $x > \tilde{x}_0$,

$$|z|^2 < \frac{R(\varepsilon, \rho, \mu)}{x^2} < \eta_0 < 1. \quad (60)$$

Therefore, for all $x > \tilde{x}_0$ we have

⁷A function $f(x)$ is $o(1/x^k)$ at infinity if $\lim_{x \rightarrow \infty} x^k f(x) = 0$.

$$\frac{|z|^2}{1 - |z|^2} < \frac{|z|^2}{1 - \eta_0} < \frac{R_0}{x^2}, \quad R_0 := \frac{R(\varepsilon, \rho, \mu)}{1 - \eta_0}. \quad (61)$$

Notice that R_0 is a finite and strictly positive constant. Employing Eq. (48) and this inequality, we obtain that for all $m > M_0 := \text{int}\{\tilde{x}_0/t\} + 1$ (where $\text{int}\{x\}$ is the integer part of x)

$$|d(mt)|^2 < \frac{R_0}{(mt)^2}. \quad (62)$$

Therefore, we conclude that, for every fixed $t > 0$,

$$\sum_{m \in \mathbb{N}, m \neq 0} |d(mt)|^2 < \sum_{m=1}^{M_0} |d(mt)|^2 + \frac{R_0}{t^2} \sum_{m=M_0+1}^{\infty} \frac{1}{m^2} < \sum_{m=1}^{M_0} |d(mt)|^2 + \frac{\pi^2 R_0}{6t^2} < \infty. \quad (63)$$

Thus, d_m is square summable, and the dynamics generated by $\tilde{H}$ in E_B is unitarily implementable. The proof, which makes use of Eq. (58), fails when $\rho = 0$ and, in particular, for the Hamiltonian H . We hence see that unitary implementability is extremely sensitive to the value of ρ . For instance, for every nonvanishing real ρ in any neighborhood of zero (in fact, for all $\rho \in \mathbb{R} - \{0\}$), $\tilde{H}_m[\mathcal{A}] = H_m[\mathcal{A}] + 2m\rho b(x)(A_m^* A_m + A_{-m}^* A_{-m})$ gives rise to a unitarily implementable transformation.

On the other hand, it is worth emphasizing that, to prove the square summability of d_m , we have not actually employed the existence of an asymptotic series for a solution z of Eq. (46). What we have used in fact is a weaker property, namely, the existence of a solution with the behavior (58). A rigorous proof of this existence is given in the Appendix.

In our analysis, we have assumed a specific form for the Hamiltonian, in particular that λ and μ are a real and a complex constant. It is nonetheless possible to generalize our discussion to other cases. Suppose, e.g., that the function $\lambda b(x)$ is replaced by a new real function $\bar{b}(x)$ and that $\mu a(x)$ is changed into $\bar{a}(x) := \exp[-2i\psi(x)]\bar{\Delta}(x)$ (and similarly for its complex conjugate), with ψ and $\bar{\Delta}$ being a real and a complex function, respectively. Let us then call $\bar{\beta}(x) := 2\bar{b}(x) - d\psi(x)/dx$ and assume that $\bar{\beta}(x)$ and $\bar{\Delta}(x)$ admit an asymptotic series at infinity such that $\bar{\Delta}_0 = 0$. Defining now $\bar{z}(x) = \exp[-2i\psi(x)]Y(x)$, it is straightforward to see that the same line of reasoning presented in the previous subsection leads to an equation analogous to Eq. (46), but with the replacements of z by $\bar{z}$, $[2\lambda b - 1]$ by $\bar{\beta}$, and $\mu\Delta$ by $\bar{\Delta}$. The counterpart of Eq. (56) is then $\bar{z}_0 \bar{\beta}_0 = 0$, which implies that $\bar{z}_0$ vanishes unless so does $\bar{\beta}_0$, which plays now the role of ρ . In addition, the analog of Eq. (57) is

$$\begin{aligned} \bar{z}_k = & \frac{1}{\bar{\beta}_0} \left[\sum_{m=0}^{k-1} \bar{\Delta}_{k-m} \left(\sum_{j=0}^m \bar{z}_{m-j} \bar{z}_j \right) - \sum_{m=1}^k \bar{\beta}_m \bar{z}_{k-m} \right] \\ & + \frac{1}{\bar{\beta}_0} \left[\bar{\Delta}_k^* + i \frac{k-1}{2} \bar{z}_{k-1} \right]. \end{aligned} \quad (64)$$

Therefore, we get $\bar{z}(x) = \bar{\Delta}_1^*/(\bar{\beta}_0 x) + o(1/x)$. Again, this asymptotic behavior suffices to guarantee the square summability of the antilinear part of the map $M_m(t)$ for all $t > 0$ provided that $\bar{\beta}_0$ differs from zero.

As a final comment, let us consider the formal quantum expression for $\hat{H}$:

$$:\hat{H}: = :\hat{H}: + 2\rho \sum_{m \in \mathbb{N}, m \neq 0} mb(mt) [\hat{A}_m^\dagger \hat{A}_m + \hat{A}_{-m}^\dagger \hat{A}_{-m}]. \quad (65)$$

Because of the unitary implementability of $M_m(t)$, we know that (65) generates the unitary evolution operator through which the basic operators $\hat{B}_n$ and $\hat{B}_n^\dagger$ evolve (when these basic operators are represented as the annihilation and creation operators on the Hilbert space $\mathcal{H}_B$ constructed from E_B and its corresponding complex structure J_B). On the other hand, even though the operator (65) generates a map which acts unitarily on the Hilbert space $\mathcal{H}_{\bar{\Gamma}}$ (constructed from $\bar{\Gamma}$ and its associated complex structure $J_{\bar{\Gamma}}$), we know that this map does not correspond to the actual time evolution of the basic operators $\hat{B}_n$ and $\hat{B}_n^\dagger$ (now represented as the annihilation and creation operators on $\mathcal{H}_{\bar{\Gamma}}$). The quantum generator comes from a phase space function which certainly can be considered as close as one wants to H (the generator of the dynamics in $\bar{\Gamma}$), but does not coincide with it. Nevertheless, it is interesting to note that if ρ is regarded as a constant of quantum origin, e.g. by setting ρ proportional to $\hbar$, then the classical limit of $\hat{H}$ would be just H . That is, in spite of the lack of a unitary implementation for H , if we consider that the modification of the Hamiltonian arises from a quantum correction, then we will get a unitary map whose generator, in the naive limit $\hbar \rightarrow 0$, provides the classical dynamics in $\bar{\Gamma}$.

IV. CONCLUSIONS AND FURTHER COMMENTS

We have analyzed the impossibility of obtaining a unitary implementation of the dynamics in the polarized Gowdy T^3 model with the quantization put forward in Ref. [5], a problem that has recently been pointed out in Refs. [6,7]. With this aim, we have first presented a complete derivation of the model starting with general relativity and introducing a symmetry-reduction and gauge-fixing procedure. Employing then a time-dependent map from the covariant phase space to the canonical phase space of the system, we have been able to reformulate the issue of unitary implementability of the evolution as a question about the square summability of the antilinear part of such a map. In this process, it is important to realize that

the total Hamiltonian in the covariant phase space vanishes, whereas the considered map includes in an explicit manner all of the time variation of the system. Exploiting this reformulation of the unitarity problem, we have considered (certain types of) modifications to the dynamics and analyzed whether the symplectic maps associated with them are unitarily implementable. In this way, we have traced back the failure of unitarity to the presence of some specific contributions in the Hamiltonian that generates the dynamics. In addition, we have seen that negligibly small modifications of these contributions suffice to restore unitarity. In the rest of the section, we present some comments about the main results of the work.

In our analysis, two facts have played a particularly relevant role. First, as we have noticed, there is some kind of universality in the behavior of the Bogoliubov coefficients. This has allowed us to consider just one equation [namely Eq. (46)] in order to examine the square summability of these coefficients, rather than investigating an infinite number of differential equations, one for each mode. Second, to know whether the modified Hamiltonian is unitarily implementable, instead of solving the universal equation (46), it actually suffices to study the leading term of the function z in the asymptotic limit of large values of its argument. In this sense, one does not need to explicitly integrate the dynamical equations.

On the other hand, we note that solving Eq. (42) amounts to “diagonalizing” the total Hamiltonian by means of a time-dependent canonical transformation, namely, to requiring that the terms proportional to $B_m B_{-m}$ and $B_m^* B_{-m}^*$ vanish in the phase space function $\hat{H}_2[\mathcal{B}] := \hat{H}[\mathcal{B}] - \partial_t \mathcal{F}[\mathcal{B}]$. Had we ignored the term containing the time derivative of the Bogoliubov coefficients, we would have obtained from Eq. (42) an algebraic quadratic equation for the ratio $Y(x)$ of the coefficients that leads to an instantaneous diagonalization (i.e., at a fixed instant of time) of the Hamiltonian $\hat{H}$. In fact, for $\lambda = \mu = 1$ so that $\hat{H}$ reduces to H , one can see that using this algebraic equation and the relation $|c|^2 = 1 + |d|^2$, it is possible to recover the instantaneous diagonalization given in Ref. [6]. In addition, we emphasize that diagonalizing the Hamiltonian is equivalent to the resolution of the dynamics. Indeed, as we have seen, if we solve the universal equation (46), which is equivalent to the diagonalization condition (42), then $c(x)$ can be found by simple integration of the first-order differential equation (47), whereas $d(x)$ is determined as $d(x) = \exp(2ix)z(x)c(x)$.

Our discussion can be extended to Hamiltonians for which the functions $\mu a(x)$ and $\lambda b(x)$ in Eq. (37) are replaced by more general functions $\bar{a}(x)$ and $\bar{b}(x)$. We have seen that this is the case at least if $\bar{a}(x)$ is of the form $\exp[-2i\psi(x)]\bar{\Delta}(x)$ with $\psi(x)$ real, and $\bar{\Delta}(x)$ and (the real function) $\bar{\beta}(x) := 2\bar{b}(x) - d\psi(x)/dx$ admit asymptotic series with a vanishing coefficient $\bar{\Delta}_0$. More precisely, we have proved that the dynamics generated by those

Hamiltonians can be implemented as a unitary transformation as far as the coefficient $\tilde{\beta}_0$ differs from zero. In this sense, our study provides a general treatment for Hamiltonians of the form (37), quadratic in the coordinates $\mathcal{A}_m$.

We have proved that the obstructions to unitarity are extremely sensitive to perturbations of the Hamiltonian. In the quantization performed in Ref. [5], one can in fact approximate the dynamics as much as desired by means of unitary transformations. Thus, although a thorough study of the influence of the choice of deparametrization and complex structure must be performed in order to elucidate if the failure of unitarity is a phenomenon inherent to the polarized Gowdy T^3 model, our result suggests that it may be actually possible to restore unitarity by considering a different choice.

Finally, employing Eq. (38) and the asymptotic expansions of the Hankel functions, it is not difficult to check that, for large values of t , the m th contribution to the total Hamiltonian of the polarized Gowdy T^3 model [see Eq. (37) with $\lambda = 1$] becomes $H_m[\mathcal{A}] \approx m(A_m^* A_m + A_{-m}^* A_{-m})$. So, the Hamiltonian for asymptotically large values of t is given by $H[\mathcal{A}] \approx \sum_{n \in \mathbb{Z} - \{0\}} |n| A_n^* A_n$, which is a combination of harmonic oscillators. Therefore, we could approximate the dynamics in a unitary way by means of some modified Hamiltonian $\tilde{H}$ and, for large values of t , by a sum of harmonic oscillators. Actually, corrections to the dynamics are not unexpected, e.g., from a quantization in the framework of loop quantum gravity. It might happen that some kind of (fictitious) effective dynamics could take place as a result of the smearing of the initial singularity by quantum effects (like it occurs, in fact, for a flat isotropic universe, where the cutoff for curvatures provides us with an effective Friedmann equation [29]). This effective dynamics could correspond to a perturbation of the evolution generated by H which might be unitarily implementable.

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APPENDIX: PROOF OF THE ASYMPTOTIC BEHAVIOR

In this Appendix, we want to prove that Eq. (46) admits one solution which, at infinity, has the asymptotic behavior (58), provided that $\rho \neq 0$. Let us start by defining a new function $w(x)$ by means of the relation

$$z(x) = -\frac{\mu^*}{4\rho x} + \frac{w(x)}{x}. \quad (\text{A1})$$

Substituting this expression in Eq. (46) we obtain an equivalent nonlinear differential equation of the Riccati type

$$\frac{dw}{dx}(x) = w(x)\beta(x) - 2i\mu \frac{\Delta(x)}{x} w^2(x) + \alpha(x), \quad (\text{A2})$$

where

$$\begin{aligned} \beta(x) &:= 2i[2\lambda b(x) - 1] + \frac{1}{x} + i\frac{|\mu|^2}{\rho} \frac{\Delta(x)}{x}, \\ \alpha(x) &:= -2i\mu^* x[\Delta^*(x) - \Delta_1^*] - i\frac{\mu^* \lambda}{\rho} [b(x) - b_0] \\ &\quad - \frac{\mu^*}{4\rho x} - i\frac{\mu^* |\mu|^2}{8\rho^2} \frac{\Delta(x)}{x}. \end{aligned} \quad (\text{A3})$$

The constants Δ_1 and b_0 are given in Eq. (53) and we have used $\lambda = 1 + \rho$.

In order to arrive at the desired result about the asymptotic behavior of $z(x)$, we only have to demonstrate that Eq. (A2) admits a solution that tends to zero at infinity.

Employing the asymptotic expansions (52) of the functions $\Delta(x)$ and $b(x)$, recalling that $\Delta_0 = b_1 = 0$, and making use of Eq. (49) to compute the coefficient $\Delta_2 = i/16$, one can rewrite

$$\beta(x) = 2i\rho + \frac{1}{x} + \tilde{\beta}(x), \quad \alpha(x) = \left(\frac{1}{2} - \frac{1}{\rho}\right) \frac{\mu^*}{4x} + \tilde{\alpha}(x), \quad (\text{A4})$$

with $\tilde{\alpha}(x)$ and $\tilde{\beta}(x)$ being $O(1/x^2)$ at infinity [we say that a function $f(x)$ is $O(1/x^n)$ at infinity if $|x^n f(x)|$ admits a finite limit when $x \rightarrow \infty$]. Explicitly, these functions are

$$\begin{aligned} \tilde{\beta}(x) &:= 4i\lambda[b(x) - b_0] + i\frac{|\mu|^2}{\rho} \frac{\Delta(x)}{x}, \\ \tilde{\alpha}(x) &:= -2i\mu^* x\left[\Delta^*(x) - \Delta_1^* - \frac{\Delta_2^*}{x}\right] \\ &\quad - i\frac{\mu^* \lambda}{\rho} [b(x) - b_0] - i\frac{\mu^* |\mu|^2}{8\rho^2} \frac{\Delta(x)}{x}. \end{aligned} \quad (\text{A5})$$

On the other hand, the function $\Delta(x)/x$ that multiplies $w^2(x)$ in Eq. (A2) is also $O(1/x^2)$ asymptotically. For solutions $w(x)$ that are small at infinity, we then expect the quadratic term in our Riccati equation to be negligible. We will hence approximate our equation by a linear one that can be explicitly solved, find for it a solution that tends to zero at infinity, and prove that, for that solution, the removed quadratic term can in fact be neglected in the original differential equation.

For the linear differential equation

$$\frac{dw_l}{dx}(x) = w_l(x)\beta(x) + \alpha(x) \quad (\text{A6})$$

all solutions can be constructed starting with those of the

associated homogeneous equation. Using Eq. (A4), the homogeneous solutions can be found to be proportional to

$$w_l^h(x) = x \exp \left[2i\rho x + \int_{\infty}^x d\bar{x} \tilde{\beta}(\bar{x}) \right]. \quad (\text{A7})$$

Note that the asymptotic behavior of $\tilde{\beta}$ guarantees that the integral that appears in this expression is well defined. Solutions to Eq. (A6) [modulo the possible addition of $w_l^h(x)$ times a complex constant] are then of the form

$$w_l(x | x_0) = w_l^h(x) \int_{x_0}^x d\bar{x} \frac{\alpha(\bar{x})}{w_l^h(\bar{x})}, \quad (\text{A8})$$

where x_0 is a constant. A convenient integration by parts leads then to

$$\begin{aligned} w_l(x | x_0) = w_l^h(x) \int_{x_0}^x d\bar{x} \frac{\bar{x}}{w_l^h(\bar{x})} \frac{d}{d\bar{x}} \left[\frac{\alpha(\bar{x})}{\{2i\rho + \tilde{\beta}(\bar{x})\}\bar{x}} \right] \\ + w_l^h(x) \left[\frac{\alpha(\bar{x})}{w_l^h(\bar{x})\{2i\rho + \tilde{\beta}(\bar{x})\}} \right]_{x_0}^x. \end{aligned} \quad (\text{A9})$$

Remembering that $\alpha(x)$, $\tilde{\beta}(x)$, and $w_l^h(x)/x$ are respectively $O(1/x)$, $O(1/x^2)$ and $O(1)$ at infinity, it is possible to see that the integrand in the above expression is $O(1/x^3)$. Therefore, the integral converges when x_0 tends to infinity.

In that limit, one gets the particular solution

$$\begin{aligned} w_l^P(x) = w_l^h(x) \int_{\infty}^x d\bar{x} \frac{\bar{x}}{w_l^h(\bar{x})} \frac{d}{d\bar{x}} \left[\frac{\alpha(\bar{x})}{\{2i\rho + \tilde{\beta}(\bar{x})\}\bar{x}} \right] \\ - \frac{\alpha(x)}{2i\rho + \tilde{\beta}(x)}. \end{aligned} \quad (\text{A10})$$

One can see (e.g. using l'Hôpital's rule for the term containing the integral) that this solution tends to zero when $x \rightarrow \infty$. Furthermore, repeating the explained procedure of integration by parts, one can show that the total contribution to $w_l^P(x)$ coming from the factor that includes the integral is $O(1/x^2)$. Using Eq. (A4), one then concludes that the asymptotic behavior of $w_l^P(x)$ is

$$w_l^P(x) = -\left(\frac{1}{2} - \frac{1}{\rho}\right) \frac{\mu^*}{8i\rho x} + o\left(\frac{1}{x}\right). \quad (\text{A11})$$

It is now a simple exercise to check that, in the Riccati equation (A2), the quadratic term is $o(1/x^3)$ at infinity for the solution $w_l^P(x)$, which is in fact negligible when compared with the rest of the terms in the equation (in particular with dw/dx), which are at least of order $1/x^2$. This concludes our proof.

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