Hybrid quantization of the polarized Gowdy $T^3$ model with matter

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Some context

- This talk is about results in Loop Quantum Gravity (LQG) and more specifically in Loop Quantum Cosmology (LQC).

What is Loop Quantum Gravity?

- LQG is one of the most promising roads for a quantum theory of gravitation.
- It is a non-perturbative and background independent theory with a geometrical perspective. Its goal is to quantize General Relativity (GR) in 4D.
- In canonical LQG: Hamilton formulation of GR. GR \equiv \text{constrained system} \Rightarrow \text{Dirac quantization procedure.}
- Ashtekar–Barbero variables \Rightarrow \text{Holonomies} and fluxes.
- Problems in completing the quantization program: Dynamics and observables.
What is loop quantum cosmology?

- LQC applies the LQG methods and ideas to the quantization of reduced models.
- Quantization can be completed and **predictions** can be extracted.
- Well defined kinematical structures from the complete theory (LQG).
- Kinematical and dynamical resolution of cosmological singularities.

- Big bas singularity replaced by a **Big Bounce** mechanism.
  [A. Ashtekar, P. Singh, T. Pawłowski (2006)]
- Effective dynamics: absence of strong singularities.
Here we are going to present the hybrid quantization of the linearly polarized Gowdy $T^3$ model with min. coupled a massless scalar field.

The hybrid quantization is a quantization method for inhomogeneous systems. It combines:
- a LQC quantization for the homogeneous degrees of freedom
- a standard Fock quantization for the inhomogeneities.

It assumes a hierarchy in the quantum geometry phenomena such that the most important effects are those encoded in the global d.o.f.

Inhomogeneous cosmological models are completely quantize, and includes the most relevant effects of the (loop) quantum geometry.

Successfully applied for the quantization of realistic inflationary models. Possible window to obtain observable predictions from LQC.
Gowdy Model with matter

- Gowdy cosmologies are globally hyperbolic spacetimes with two axial commuting Killing vectors field. Compact spatial topology.

- We consider the model with $\mathbb{T}^3$ and linear polarization and include a minimally coupled massless scalar field with the same symmetries.

- After a symmetry reduction and a partial gauge fixing the model is described by
  - Four global degrees of freedom: Bianchi I cosmology $[a_j, j = \theta, \sigma, \delta]$ with a homogeneous scalar field $\phi$.
  - Two fields $\xi(\theta)$ (grav. waves) and $\varphi(\theta)$ (matter). $\theta \in S^1$.
    - Both fields take part in the system identically.
    - To obtain a proper QFT limit with unitary evolution the fields must be properly scaled.

- Two global constraints remain to be imposed:
  - (Desitized) scalar constraint: $\mathcal{C} = \mathcal{C}_{\text{BI+M}} + \mathcal{C}_{\text{inh}}$
  - Diffeomorphism constraint: $\mathcal{C}^\theta$ (rigid rotations on $\theta$)
Hybrid Quantization

- Phase space “splitting”: \( \Gamma = \Gamma_{\text{hom}} \oplus \Gamma_{\text{inh}} \)
- The representation chosen is such that the kinematical Hilbert space is given by:

\[
\mathcal{H}_{\text{kin}} = \mathcal{H}_{\text{hom}} \otimes \mathcal{H}_{\text{inh}}
\]

where:
- \( \mathcal{H}_{\text{hom}} = \mathcal{H}_{\text{hom}}^{\text{grav}} \otimes \mathcal{H}_{\text{hom}}^{\text{matt}} \)
- \( \mathcal{H}_{\text{inh}} = \bigotimes_{\alpha} \mathcal{F}^\alpha \)

- Construction of the global constraint operators.
- This quantization is not trivial because the constraints coupled the different kinematical Hilbert subspaces.
Kinematics

- Homogeneous geometry: Bianchi I
  - Ashtekar–Barbero variables:
    \[
    su(2)\text{-connection } A^i_a = \frac{c_i}{2\pi} \delta^i_a : (c_\theta, c_\sigma, c_\delta) \]
    \[
    \text{densitized triad } E^a_i = \frac{p_i}{4\pi} \delta^a_i : (p_\theta, p_\sigma, p_\delta)
    \]
  - Connection holonomies:
    \[
    h^\mu_i(c_i) = e^{\mu_i c_i \tau_i}, \mu_i \in \mathbb{R} \Rightarrow N_{\mu_i}(c_i) = e^{i\mu c_i/2}
    \]
  - Triad fluxes: \( E(S_i, f) = \frac{p_i}{4\pi^2} A_{S_i,f} \Rightarrow \) determined by \( p_i \)
  - LQC Polymeric Representation
    - \( \hat{p}_i | p_\theta, p_\sigma, p_\delta \rangle = p_i | p_\theta, p_\sigma, p_\delta \rangle \) with \( p_i \in \mathbb{R} \) (discrete spectrum).
    - Discrete inner product: \( \langle p_\theta, p_\sigma, p_\delta | p'_\theta, p'_\sigma, p'_\delta \rangle = \delta_{p_\theta, p'_\theta} \delta_{p_\sigma, p'_\sigma} \delta_{p_\delta, p'_\delta} \).
  - Improved Dynamics:
    - \( \hat{\mu}_i \Rightarrow \hat{\rho}^{ij}_a \Rightarrow F^{ij}_{ab} : \)
    \[
    h^i_j p_j p_k \tilde{\mu}_j \tilde{\mu}_k \Rightarrow \tilde{\mu}_i = \sqrt{\Delta p_i / p_j p_k}.
    \]
    - Basis relabeling: \( |\lambda_\theta, \lambda_\sigma, \nu\rangle, \lambda_i \propto \sqrt{p_i}, \nu = 2\lambda_\theta \lambda_\sigma \lambda_\delta \).
    - Actuation of \( \hat{N}_{\pm\mu_i} \): shifts \( \lambda_i \) such that \( \nu \) is shift in \( \pm 1 \).
Kinematics

- Homogeneous matter $\Rightarrow$ Standard Schrödinger quantization

\[
\hat{\phi} \psi(\phi) = \phi \psi(\phi), \quad \hat{p}_\phi = -i\hbar \partial_\phi, \quad \mathcal{H}_{\text{hom}}^{\text{matt}} = L^2(\mathbb{R}, d\phi)
\]

- Inhomogeneous sector: Privileged Fock quantization for both fields.
  - Unicity criterion: $S^1$ invariance of the vacuum $+$ unitary evolution.
  - Fourier decomposition in $\theta$ of the fields $\xi, P_\xi, \varphi, P_\varphi$.
  - Annihilation and creation variables: massless representation.

\[
\left( a^{(\alpha)}_m, a^{(\alpha)\dagger}_m \right), \quad \alpha = \xi, \varphi; \quad m \in \mathbb{Z} - \{0\}:
\]

\[
a^{(\xi)}_m = \sqrt{\frac{1}{2|m|}} (\xi_m + iP_{\xi_m}), \quad a^{(\xi)\dagger}_m = \sqrt{\frac{1}{2|m|}} (\xi_m - iP_{\xi_m})
\]

- \[
\left( a^{(\alpha)}_m, a^{(\alpha)\dagger}_m \right) \rightarrow \left( \hat{a}^{(\alpha)}_m, \hat{a}^{(\alpha)\dagger}_m \right).
\]

- Fock spaces: $\mathcal{F}^{\alpha} \quad m \longrightarrow \mathcal{H}_{\text{inh}} = \mathcal{F}_\xi \otimes \mathcal{F}_\varphi$.

- $n$-particle states: $|n^{\alpha}\rangle = |\ldots, n_{-m}^{\alpha}, \ldots, n_{m}^{\alpha}, \ldots\rangle$. 

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Inhomogeneous operators

Once we have defined a representation on a kinematical Hilbert space we construct the operators representing the constraints. First we construct the operator that only act non-trivially on $\mathcal{H}_{\text{inh}}$:

- **Diffeomorphims constraint operator:**
  \[
  \hat{C}_\theta = \sum_{n=1}^{\infty} n \left( \hat{X}^\xi_n + \hat{X}^\phi_n \right), \quad \text{where} \quad \hat{X}_n^\alpha = \hat{a}_n^{(\alpha)^\dagger} \hat{a}_n^{(\alpha)} - \hat{a}_{-n}^{(\alpha)^\dagger} \hat{a}_{-n}^{(\alpha)}.
  \]

- **Inhomogeneous operator that take part on $\hat{C}_{\text{inh}}$:**
  1. **Free field contribution:**
     \[
     \hat{H}_0 = \sum_n \left( \hat{N}^\xi_n + \hat{N}^\phi_n \right), \quad \text{where} \quad \hat{N}_n^\alpha = \hat{a}_n^{(\alpha)^\dagger} \hat{a}_n^{(\alpha)} + \hat{a}_{-n}^{(\alpha)^\dagger} \hat{a}_{-n}^{(\alpha)}.
     \]
  2. **Self-interaction of the inhomogeneities:**
     \[
     \hat{H}_\text{int} = \sum_{\alpha \in \{\xi, \phi\}} \sum_{n=1}^{\infty} \frac{1}{n} \left( \hat{N}_n^\alpha + \hat{a}_n^{(\alpha)^\dagger} \hat{a}_{-n}^{(\alpha)^\dagger} + \hat{a}_n^{(\alpha)^\dagger} \hat{a}_{-n}^{(\alpha)} \right).
     \]

- The actuation of $\hat{C}_\theta$ and $\hat{H}_0$ is diagonal in the n-particle states, whereas $\hat{H}_\text{int}$ creates and annihilates pair of particles.
Homogeneous operators

- \( c_j p_j \xrightarrow{\text{hol.}} \text{sgn}(p_j) \frac{p_j \sin(\bar{\mu}_j c_j)}{\bar{\mu}_j} \xrightarrow{\text{quant.}} \hat{\Theta}_j \)

\[
\hat{\Theta}_j = \frac{1}{2\sqrt{\Delta}} \sqrt{V} \left[ \sin(\bar{\mu}_j c_j) \text{sgn}(p_j) + \text{sgn}(p_j) \sin(\bar{\mu}_j c_j) \right] \sqrt{V}
\]

where: \( \sin(\bar{\mu}_j c_j) = \frac{1}{2i} \left[ \hat{N}_{2\bar{\mu}_j} - \hat{N}_{-2\bar{\mu}_j} \right] \), and \( \sqrt{V} = \otimes_j \sqrt{p}_j \).

- Regularize inverse triad: Thiemann’s trick

\[
\left[ \frac{1}{|p_\theta|^{1/4}} \right]^2 |\lambda_\theta, \lambda_\sigma, v\rangle \propto \frac{1}{\lambda_\theta} D(v) |\lambda_\theta, \lambda_\sigma, v\rangle
\]

where \( D(v) = |v| (\sqrt{v + 1} - \sqrt{|v - 1|})^2 \):
\[ \hat{C} = \sum_{i,j\neq i} \frac{\hat{\Theta}_i \hat{\Theta}_j}{16\pi G \gamma^2} - \frac{\hbar^2}{2} \frac{\partial^2 \phi}{\partial \phi^2} + 2\pi \hbar |p_{\theta}| \hat{H}_0 + \hbar \left[ \frac{1}{|p_{\theta}|^{\frac{1}{4}}} \right]^2 \frac{16\pi G \gamma^2}{2} \left[ \frac{1}{|p_{\theta}|^{\frac{1}{4}}} \right]^2 \hat{H}_{\text{int}}. \]

- The actuation of the Hamiltonian constraint on the homogeneous basis states \( |\lambda_{\theta}, \lambda_{\sigma}, v\rangle \) has the following properties:
  - It decouples the zero volume state \( v = 0 \), i.e. the singularity is kinematically resolved.
  - It also decouples the subspaces with different sign of the labels \( \lambda_{\theta}, \lambda_{\sigma} \) and \( v \). We can restrict the study to, e.g. \( \lambda_{\theta}, \lambda_{\sigma}, v > 0 \).
  - It only relates states with labels belonging to the sets: (allows to define superselection sectors)
    - for \( v \), semilattices of step four: \( \mathcal{L}_\epsilon = \{ v | v = \epsilon + 4k, k \in \mathbb{N} \} \)

\[ -16 -12 -8 -4 0 4 8 12 16 \]

- for \( \Lambda_a = \log \lambda_a \), \( \{ \Lambda_a | \Lambda_a = \Lambda_a^* + \omega_\epsilon, \Lambda_a^* \in \mathbb{R}, \omega_\epsilon \in \mathcal{W}_\epsilon \} \), where \( \mathcal{W}_\epsilon \) is a numerable dense set in the real line (that depends on \( \epsilon \)
The physical states are those annihilated by both constraint operators:

- \( \hat{C}_\theta \) only imposes a mild condition on the occupation number of the \( n \)-particle states,

\[
\sum_{\alpha \in \{\xi, \varphi\}} \sum_{m=1}^{\infty} m(n^\alpha_m + n^{-\alpha}_m) = 0 \Rightarrow \mathcal{F}_p \subset \mathcal{F}_\xi \otimes \mathcal{F}_\varphi
\]

- \( (\Psi | \hat{C}^\dagger = 0 \) is much more involved.
  - It is a difference equation on the label \( v \).
  - It can be shown that solutions are completely determined by the initial data on \( v = \epsilon \).
  - One can give a Hilbert space structure to the space of initial data by defining a complete set of classical observables with suitable conjugation relations and demanding that those relations convert into adjoint relations between the corresponding operators.

\[
\mathcal{H}_{\text{phy}} = \mathcal{H}_{\text{phy}}^{\text{BI}} \otimes L^2(\mathbb{R}, d\phi) \otimes \mathcal{F}_p
\]
• We have presented the quantization of the $\mathbb{T}^3$ Gowdy model with linear polarization and a massless scalar field by using a hybrid quantization technique.

• The hybrid techniques combines the polymeric quantization of LQC with the Fock quantization to obtain a complete quantization of cosmological inhomogeneous models.

• The operator representing the constraints have a well defined action in the kinematical Hilbert space.

• The LQC quantization of the global degrees of freedom leads to the kinematical resolution of the classical singularity.

• The Fock quantization allows to recover the QFT description when the “homogeneous” quantum gravity effects are negligible.

• The physical Hilbert space for the system can be built from the space of initial conditions in the section of minimum volume by imposing reality conditions.
Thanks for your attention!