Minimal model dynamics for twelvefold quasipatterns

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A dynamical model of the Swift-Hohenberg type is proposed to describe the formation of twelvefold quasipattern as observed, for instance, in optical systems. The model incorporates the general mechanisms leading to quasipattern formation and does not need external forcing to generate them. Besides quadratic nonlinearities, the model takes into account an angular dependence of the nonlinear couplings between spatial modes with different orientations. Furthermore, the marginal stability curve presents other local minima than the one corresponding to critical modes, as usual in optical systems. Quasipatterns form when one of these secondary minima may be associated with harmonics built on pairs of critical modes. The model is analyzed numerically and in the framework of amplitude equations. The results confirm the importance of harmonics to stabilize quasipatterns and assess the applicability of the model to other systems with similar generic properties.

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The existence of planar quasipatterns with N -fold rotational symmetry and $N \geq 5$ has been predicted and studied in a variety of nonlinear dynamical systems. They usually require resonant couplings between noncritical modes or harmonics [1–5], induced by external or boundary forcing [6–8] or interacting bifurcations [9,10]. This is the case for most of the experimental observations, where a spatiotemporal instability is submitted to external parametric forcing, which may induce quasipatterns [11–14] or superlattices [13,15–17].

However, the observation of twelvefold quasipatterns has also been reported in an autonomous optical system for a wide range of parameters [18–20]. The experimental system is built on a nonlinear medium which is extensively described in [20–22]. The medium is driven by a laser beam, and the main observation is that, on increasing the input power, bifurcations lead from a homogeneous state to hexagonal structures, and then to twelvefold quasiperiodic patterns. The last structure is based on two hexagonal triads of wave vectors rotated, the one with respect to the other, by an angle of $\pi/6$. It is also worth noting that the spectrum of the quasipatterns contains spatial harmonics resulting from the addition of pairs of fundamental wave vectors. When these harmonics are suppressed by filtering techniques, quasipatterns disappear and evolve into hexagonal ones. This suggests that quadratic interactions between harmonic and fundamental modes are able to stabilize twelvefold structures. A model is also proposed in [18] which reproduces fairly well the experimental observations. Other repressed systems may also be expected to generate quasipatterns, for example, bistable systems where, due to diffusive instabilities, the two homogeneous steady states become unstable versus spatial modes with different wave numbers. In this case, it has been theoretically shown that resonant couplings between such modes may induce quasiperiodic patterns [23].

Our aim in this paper is to show that a minimal modification of the Swift-Hohenberg dynamics, which is a paradigm for pattern formation in dissipative systems [24,25], is able to reproduce the formation of quasipatterns. We also identify the minimum ingredients for the robust observation of these spatial structures, namely, a marginal stability curve with at least two local minima, quadratic nonlinearities which favor

hexagonal triads of spatial modes, and cubic nonlinearities which favor multimode patterns. As a result, this phenomenon should be observable in other systems presenting the same generic properties.

Regular multimode patterns are well documented either experimentally (see for example [26–31]) and theoretically (see for example [32–36]). They may be described by generalized Swift-Hohenberg equations of the type

$$\partial_t \sigma = [\epsilon - (q_c^2 + \nabla^2)^2] \sigma - \nu \nabla^2 \sigma^2 + NL_3(\vec{\nabla}, \sigma), \quad (1)$$

where NL_3 is a cubic nonlinear term which couples gradients of spatial modes. This term leads to cross-coupling coefficients between the amplitudes of modes with wave vectors forming an angle θ , which, normalized by the direct coupling coefficient, are defined as $\gamma(\theta)$. For example, for Proctor-Sivashinsky dynamics [37–40], where $NL_3 = u \vec{\nabla} \cdot (|\vec{\nabla} \sigma|^2 \vec{\nabla} \sigma)$, $\gamma(\theta) = \frac{2}{3} + \frac{4}{3} \cos^2 \theta$. For von Kármán plate theory [41], $\gamma(\theta) = \nu + (1 - \nu) \cos^2 \theta$ (where ν is the Poisson ratio of the material, $0 \leq \nu \leq 0.5$). In models for thin-film deformation under laser irradiation $\gamma(\theta) = \frac{2 \cos^2 \theta}{1 + \alpha \sin^2 \theta}$ [25]. Other examples resulting from a nonlinear dependence of the cubic term on spatial gradients may be found in [10]. In these examples, $\gamma(\theta)$ may be less than 1, and supercritical square patterns are usually steady-state solutions. On the other hand, due to the quadratic nonlinearities of the dynamics, hexagonal patterns built on equilateral triads of unstable modes may also be a solution. Furthermore, in dynamics like (1), square and hexagonal patterns may usually be simultaneously stable [32–36]. It seems thus natural to consider the possible development of equilateral twelvefold quasipatterns in this model. Such patterns are built on six pairs of wave vectors. When represented on the circle, two adjacent wave vectors are separated by an angle of $\pi/6$. This set of wave vectors may also be considered as two orthogonal triads underlying two orthogonal hexagonal lattices where the wave vectors may be fixed as $\vec{q}_1 = q_c \vec{1}_x$, $\vec{k}_1 = q_c \vec{1}_y$, $\vec{q}_2 = -q_c [\frac{1}{2} \vec{1}_x - \frac{\sqrt{3}}{2} \vec{1}_y]$, $\vec{k}_2 = -q_c [\frac{\sqrt{3}}{2} \vec{1}_x + \frac{1}{2} \vec{1}_y]$, $\vec{q}_3 = -q_c [\frac{1}{2} \vec{1}_x + \frac{\sqrt{3}}{2} \vec{1}_y]$, and $\vec{k}_3 = q_c [\frac{\sqrt{3}}{2} \vec{1}_x - \frac{1}{2} \vec{1}_y]$ (see Fig. 1). The corresponding amplitude

FIG. 1. Representation of the set of wave vectors underlying twelvefold quasipatterns and examples of their corresponding harmonics.

equations, when neglecting harmonics, are

$$\begin{aligned}
 \partial_t A_i &= \epsilon A_i + \bar{v} A_{i+1}^* A_{i+2}^* \\
 &\quad - g A_i \left[|A_i|^2 + \gamma \left(\frac{\pi}{3} \right) (|A_{i+1}|^2 + |A_{i+2}|^2) \right. \\
 &\quad \left. + \gamma \left(\frac{\pi}{2} \right) |B_i|^2 + \gamma \left(\frac{\pi}{6} \right) (|B_{i+1}|^2 + |B_{i+2}|^2) \right], \\
 \partial_t B_i &= \epsilon B_i + \bar{v} B_{i+1}^* B_{i+2}^* \\
 &\quad - g B_i \left[|B_i|^2 + \gamma \left(\frac{\pi}{3} \right) (|B_{i+1}|^2 + |B_{i+2}|^2) \right. \\
 &\quad \left. + \gamma \left(\frac{\pi}{2} \right) |A_i|^2 + \gamma \left(\frac{\pi}{6} \right) (|A_{i+1}|^2 + |A_{i+2}|^2) \right], \quad (2)
 \end{aligned}$$

where A_i and B_i are the amplitudes of the modes with wave vectors $\vec{q}_i$ and $\vec{k}_i$, respectively, $\bar{v} = 2vq_c^2$ [42], and g is the direct coupling coefficient, which for Proctor-Sivashinsky dynamics is $g = 3uq_c^4$ (g can be rescaled to 1 without loss of generality).

These equations admit square, hexagonal, and twelvefold pattern solutions. Steady-state amplitudes are respectively given by

$$\begin{aligned}
 |A_1| = |B_1| = R_s &= \sqrt{\frac{\epsilon}{[1 + \gamma(\frac{\pi}{2})]g}}, \quad A_{j(\neq 1)} = B_{j(\neq 1)} = 0, \\
 |A_i| = R_h &= \frac{[\bar{v} + \sqrt{\bar{v}^2 + 4\epsilon\alpha_1 g}]}{2\alpha_1 g}, \quad B_i = 0, \\
 |A_i| = |B_i| = R_{12} &= \frac{[\bar{v} + \sqrt{\bar{v}^2 + 4g(\alpha_1 + \alpha_2)\epsilon}]}{2g(\alpha_1 + \alpha_2)}, \quad (3)
 \end{aligned}$$

with $\alpha_1 = 1 + 2\gamma(\frac{2\pi}{3})$ and $\alpha_2 = \gamma(\frac{\pi}{2}) + 2\gamma(\frac{\pi}{6})$. Subcritical hexagons may thus appear for $\epsilon > -\frac{\bar{v}^2}{4\alpha_1 g} = \epsilon_{h-}$ and quasipatterns for $\epsilon > -\frac{\bar{v}^2}{4(\alpha_1 + \alpha_2)g}$.

We discuss now the linear stability of these patterns without considering either phase variations or phase instabilities. Such instabilities have been studied in detail elsewhere [24,25,43,44] and will not affect the present analysis.

In fact, the experiments that motivate this work occur in finite geometries and the patterns are not sufficiently extended to present relevant phase variations. Usual analysis [24,25] shows that squares may exist for $\epsilon > 0$, but they are unstable for

$$\epsilon < \frac{\bar{v}^2}{g} \frac{1 + \gamma(\frac{\pi}{2})}{[\gamma(\frac{\pi}{6}) + \gamma(\frac{\pi}{3}) - 1 - \gamma(\frac{\pi}{2})]^2} = \epsilon_s. \quad (4)$$

Hexagons, on the other hand, are linearly stable for $\alpha_2 > \alpha_1$, while for $\alpha_2 < \alpha_1$ they become unstable for

$$\epsilon > \frac{\bar{v}^2}{g} \frac{\alpha_2}{(\alpha_1 - \alpha_2)^2} = \epsilon_{h+} \quad (5)$$

and

$$\epsilon > \frac{2\bar{v}^2}{g} \frac{\alpha_1 + 3}{(\alpha_1 - 3)^2}, \quad (6)$$

when $\alpha_1 > 3$. For $\alpha_2 > \alpha_1$, twelvefold quasipatterns are linearly unstable. However, if $\alpha_1 > \alpha_2$, they are stable for

$$\epsilon_{12} = \frac{\bar{v}^2}{4g} \frac{3\alpha_2 - \alpha_1}{(\alpha_1 - \alpha_2)^2} < \epsilon < \frac{2\bar{v}^2}{g} \frac{\alpha_1 + \alpha_2 + 3}{(\alpha_1 + \alpha_2 - 3)^2} = \epsilon_{12}^+. \quad (7)$$

For Proctor-Sivashinsky dynamics [37–40], $\alpha_1 = 3$ and $\alpha_2 = 4$, and twelvefold quasipatterns are unstable in the weakly nonlinear description (2). Figure 2 shows the bifurcation diagram of twelvefold quasipatterns, hexagons, and squares in this case. It has also been shown that quasipatterns are solutions of the Swift-Hohenberg equation, although they are unstable for this dynamics since $\gamma(\theta) = 2$ [45].

The previous conclusion has been obtained neglecting harmonics. In some cases, however, their role in the dynamics may considerably change the scenario. Harmonics resulting from the combinations of wave vectors underlying twelvefold quasipatterns correspond to the following wave vectors: $\pm\vec{q}_1 \pm \vec{k}_1$, $\pm\vec{q}_1 \pm \vec{k}_2$, $\pm\vec{q}_1 \pm \vec{k}_3$, $\pm\vec{q}_2 \pm \vec{k}_1$, $\pm\vec{q}_2 \pm \vec{k}_2$, $\pm\vec{q}_2 \pm \vec{k}_3$, $\pm\vec{q}_3 \pm \vec{k}_1$, $\pm\vec{q}_3 \pm \vec{k}_2$, $\pm\vec{q}_3 \pm \vec{k}_3$, and their wave numbers are $q_c\sqrt{2}$, $q_c\sqrt{2 + \sqrt{3}}$, and $q_c\sqrt{2 - \sqrt{3}}$ (see Fig. 1). Near instability, they are slaved to critical modes and may thus be adiabatically eliminated. This process leads, close to

FIG. 2. Bifurcation diagram of Eqs. (2) with $q_c = 1$, $v = 1$, $u = 1$, $\alpha_1 = 3$, and $\alpha_2 = 4$. Since $\alpha_2 > \alpha_1$ twelvefold quasipatterns are unstable. Solid (dashed) lines correspond to stable (unstable) solutions. R is plotted in arbitrary units.

FIG. 3. Marginal stability curve leading to the dynamics (10) for $\mu = 0.1$ and $\nu = 2 + \sqrt{3}$.

threshold, to a reduction of the cross-coupling coefficients, which become

$$\begin{aligned} \gamma\left(\frac{\pi}{2}\right) &\rightarrow \bar{\gamma}\left(\frac{\pi}{2}\right) = \gamma\left(\frac{\pi}{2}\right) - \frac{4\bar{v}^2}{g}, \\ \gamma\left(\frac{\pi}{6}\right) &\rightarrow \bar{\gamma}\left(\frac{\pi}{6}\right) = \gamma\left(\frac{\pi}{6}\right) - \frac{\bar{v}^2}{g}. \end{aligned} \quad (8)$$

Within the framework of Eqs. (2), this correction is small, since $\frac{\bar{v}^2}{g} \sim \epsilon \ll 1$, and it does not change the scenario qualitatively. However, as in various optical systems, the linear growth rate $\omega(\epsilon, \vec{q})$ may present several “tongues” (see Fig. 3). In this case, it remains negative outside the critical range, but may become close to zero for some sets of noncritical wave vectors. If such wave vectors correspond to harmonics with wave numbers close to $q_c\sqrt{2+\sqrt{3}}$, $q_c\sqrt{2-\sqrt{3}}$, or $q_c\sqrt{2}$, the decrease of the cross-coupling coefficient for $\theta = \pi/6$ or $\theta = \pi/6$ may become significant since, in fact,

$$\begin{aligned} \gamma\left(\frac{\pi}{6}\right) &\rightarrow \bar{\gamma}\left(\frac{\pi}{6}\right) \\ &= \gamma\left(\frac{\pi}{6}\right) + \frac{(2-\sqrt{3})4v^2}{g\omega(\epsilon, \vec{q}_1 - \vec{k}_3)} + \frac{(2+\sqrt{3})4v^2}{g\omega(\epsilon, \vec{q}_1 + \vec{k}_3)}, \\ \bar{\gamma}\left(\frac{\pi}{2}\right) &\rightarrow \bar{\gamma}\left(\frac{\pi}{2}\right) + \frac{2\bar{v}^2}{g\omega(\epsilon, \vec{q}_1 + \vec{k}_1)} + \frac{2\bar{v}^2}{g\omega(\epsilon, \vec{q}_1 - \vec{k}_1)}, \end{aligned} \quad (9)$$

where $|\vec{q}_1 \pm \vec{k}_3| = q_c\sqrt{2 \pm \sqrt{3}}$ and $|\vec{q}_1 \pm \vec{k}_1| = q_c\sqrt{2}$.

If this effect is able to reduce α_2 to values lower than α_1 , twelvefold quasipatterns become stable. If $\omega(\epsilon, \vec{q}_1 + \vec{k}_3)$ in (9) approaches zero, the corresponding renormalization diverges and this analysis breaks down. In this case, harmonics become unstable, and the amplitude equation description should be modified accordingly. Note that if ω becomes closer to zero for wave numbers near $q_c\sqrt{2}$ no quasipatterns are observed. Although this wave number corresponds to harmonics which couple two orthogonal triads of critical wave vectors, it also couples two pairs separated by a 90° angle and forming squares. In this case, the numerically selected pattern is found to be the square one.

FIG. 4. Bifurcation diagram for Eqs. (2) with $\alpha_1 = 4.4$ and $\alpha_2 = 2.4$ (ϵ is scaled by $\bar{v}^2/g$). Other parameters as in Fig. 2.

For the model describing the nonlinear optical pattern forming system studied in [20–22], $\bar{\gamma}(\theta)$ has been computed explicitly (see Fig. 9 of [46]), and it is such that $\alpha_1 \simeq 4.4$ and $\alpha_2 \simeq 2.4$. Note that, with inversion symmetry, this model produces eightfold quasipatterns [47], and in the absence of it, twelvefold ones are obtained. The corresponding bifurcation diagram given by Eqs. (2) for these values of α_1 and α_2 is shown in Fig. 4, and is consistent with experimental observations. Furthermore, in the range $\epsilon_{12} < \epsilon < \epsilon_{h+}$ mixed mode patterns with $|A_i| \neq |B_i| \neq 0$ may exist. Here also, phase or sideband instabilities of quasipatterns [48] are not considered.

The basic elements of the dynamics which lead to the formation of quasipatterns are thus as follows.

- (i) A marginal stability curve with at least two local minima as illustrated in Fig. 3.
- (ii) Quadratic nonlinearities which favor hexagonal triads of spatial modes.
- (iii) Cubic nonlinearities which favor multimode patterns.

These elements may be cast in a minimal model of the Proctor-Sivashinsky-Knobloch type:

$$\begin{aligned} \partial_t \sigma &= [\epsilon + \mu \nabla^2 - (q_0^2 + \nabla^2)^2 (v q_0^2 + \nabla^2)^2] \sigma \\ &\quad - v \nabla^2 \sigma^2 + u \vec{\nabla} \cdot (|\vec{\nabla} \sigma|^2 \vec{\nabla} \sigma). \end{aligned} \quad (10)$$

The linear part of the dynamics results from the marginal stability curve sketched in Fig. 3. This type of behavior may arise in nonlinear optical systems, such as the ones studied in [19,46,47], but also in systems with some type of nonlocal interactions [49]. In the latter case, when the nonlocality is weak, the interaction term may be expanded in a series of spatial derivatives of the dynamical variable. When limited to the lowest relevant terms needed to capture the existence of two local minima in the marginal stability curve, this expansion should give a linear growth rate analogous to the one proposed in Eq. (10). For small μ , the linear growth rate of Eq. (10) has local maxima close to $|\vec{q}|^2 = q_{c1}^2 = q_0^2$ and $q_{c2}^2 = v q_0^2$. On increasing ϵ beyond $\epsilon_1 = \mu q_0^2$, modes with $|\vec{q}|^2 = q_{c1}^2$ become first unstable. Increasing ϵ further there is a critical slowing down of modes with $|\vec{q}|^2 = q_{c2}^2$, which become eventually unstable at $\epsilon = \epsilon_2 = \mu v q_0^2$. If ν is near $2 + \sqrt{3}$, the second

set of modes includes harmonics of orthogonal triads of wave vectors with wave number near $q_c\sqrt{2+\sqrt{3}}$, which should lead to the stabilization of twelvefold quasipatterns as explained above.

Note that twelvefold quasipatterns along with other types of patterns have been obtained numerically by a generalized Swift-Hohenberg equation, with scalar quadratic and cubic couplings, where two wavelengths become simultaneously unstable on increasing the bifurcation parameter [50]. Their stability domains have been determined through the minimization of the associated Lyapunov functional. However, numerical analysis of this model does not seem to provide robust pattern selection mechanisms, possibly due to the existence of several metastable states and local minima of this functional, and the scalar nature of the nonlinear couplings.

Similar results have also been obtained in a bicritical dynamics [10]. In this model, two quadratically coupled fields become unstable for different sets of wave vectors. For appropriate relationships between the corresponding wave numbers, twelvefold quasipatterns may be expected. However, numerical evidence has only been presented when both sets of wave vectors are unstable and have the same growth rate.

In the dynamics (10) harmonics do not need to be unstable to generate quasipatterns and they always have a lower growth rate than fundamental modes. Numerical simulations of this model confirm the robust formation of twelvefold quasipatterns under these conditions (Fig. 5). Equation (10) has been integrated using a pseudospectral method where linear terms in Fourier space are treated exactly while nonlinear terms are integrated with a second-order accurate in time algorithm [51]. In the nonlinear terms, derivatives are evaluated using the Fourier transform, although the nonlinear operations are evaluated in real space. The complete nonlinear term in real space is then included in the nonlinear part of the algorithm. We use a 256×256 square grid with $\Delta_x = \Delta_y = 0.4$. Starting from random initial conditions, either hexagonal or twelvefold quasipatterns form between ϵ_1 and ϵ_{12}^+ . Once quasipatterns are formed they are stable for $\epsilon_{12} < \epsilon < \epsilon_{12}^+$. Hexagons are stable for $\epsilon_h < \epsilon$, up to values larger than 1 (not shown in the figure), where they become eventually unstable to squares. These results do not change qualitatively for small changes in μ or ν , showing the robustness of the mechanism leading to the formation of quasipatterns explained in this work.

Figure 5 has been obtained for $\nu = 1$, beyond the strict range of validity of Eqs. (2) and (9) [42]. Therefore, no quantitatively relevant values of α_1 and α_2 can be computed for Eq. (10) in this case, since the correction to $\gamma(\pi/6)$ in Eq. (9) is too large. Reducing ν in (10), however, reduces also ϵ_{12}^+ , until for $\nu \ll 1$ the region of existence of quasipatterns becomes too small to be practically observed in numerical simulations. Nevertheless, the qualitative explanation provided by Eq. (9) on the role of the harmonics and the tongues of the linear growth rate in stabilizing quasipatterns still holds. Thus we observe that changing ν to values distant from $2 + \sqrt{3}$, so that $|\omega(\epsilon, \vec{q}_1 + \vec{k}_3)|$ becomes large, destabilizes quasipatterns, recovering a bifurcation diagram equivalent to the one shown in Fig. 2.

To conclude, we have shown that the formation of twelvefold quasipatterns may be described by a generalized

FIG. 5. (Color online) (a) Bifurcation diagram for model (10) obtained numerically for $\mu = 0.1$, $q_0 = 1$, $\nu = 2 + \sqrt{3}$, $\nu = 1$, and $u = 1$. Here $R = \max[\sigma(x, y)]$. Blue (black) lines correspond to hexagonal patterns (twelvefold quasipatterns). Only stable solutions have been computed. Dashed lines corresponding to unstable solutions have been sketched to guide the eye. Panels (b) and (c) show the field $\sigma(x, y)$ (left) and its Fourier transform (right) for hexagonal patterns and twelvefold quasipatterns, respectively.

Swift-Hohenberg equation. It incorporates the few basic properties needed to generate such patterns. The model is based on quadratic and cubic mode interactions. Cubic ones depend on the relative orientation of the interacting modes and favor bimodal patterns. Quadratic ones result from the lack of inversion symmetry and favor hexagonal patterns. If, furthermore, the system presents a marginal stability curve with more than one minimum, it may generate harmonics of the basic unstable modes which are able to stabilize quasipatterns. General agreement between the results of amplitude equation and numerical analysis assesses the possibility of transitions and bistability between hexagonal and twelvefold patterns. It also confirms the essential role of harmonics in the formation of quasipatterns, as observed experimentally and conjectured theoretically.

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