University of Balearic Islands

Master Thesis

Co-evolution of networks

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Declaration of Authorship

I, Simone Loreti, declare that this thesis titled, 'Co-evolution of networks' and the work presented in it are my own. I confirm that:

- This work was done wholly or mainly while in candidature for a master degree at this University.
- Where any part of this thesis has previously been submitted for a degree or any other qualification at this University or any other institution, this has been clearly stated.
- Where I have consulted the published work of others, this is always clearly attributed.
- Where I have quoted from the work of others, the source is always given. With the exception of such quotations, this thesis is entirely my own work.
- I have acknowledged all main sources of help.
- Where the thesis is based on work done by myself jointly with others, I have made clear exactly what was done by others and what I have contributed myself.

Signed: ____________________________

Date: ____________________________

“Memento audere semper.”

Gabriele D’Annunzio

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With special thanks to Enrico, Miguel, Karin, and Ricardo.

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Abbreviations

O	"**O**riginal" network (first and second part of thesis)
L	weights **L**ower than Delta ($\delta = 0.01$, second part of thesis)
I	**I**nitial nucleus (third part of thesis)
ER	**E**rdos **R**enyi network
RERA	**RE**gular **RA**ndom network
REG	**REG**ular (not Random) network
FULL	**F**ull connected network
k	Parameter of the equations (in the original paper)
α	Parameter of the equations (in the original paper)
N	**N**odes' number of the network

Dedicated to Alessandra

Chapter 1

Introduction

1.1 Science of networks

Thanks to computational progresses and to the availability of large dataset, the "science of networks" underwent a strong growth and development phase in the last years. In this period, named the modern period (1998 – present), scientists demonstrated the universality of this science, and the possibility to use the same concepts, tools, and methods in different fields of application: social networks analysis, market competition models, control theory, physical and biological sciences, just to name a few.
Researchers found common "laws" and behaviours, for a wide range of natural and synthetic networks present in different fields. They uncovered general concepts and properties intrinsic to all of these diverse networks, and divided them in a few categories: regular, random, scale-free, and small-world networks [1].
Regular and random networks are two extremes, the first one with a purely deterministic structure, the second one with a pure random structure, and between them there are networks which share characteristics of regular graphs and random graphs: small-world (mostly structured, partly random) and scale-free (mostly random, partly structured) networks.
Each class has a peculiar feature:

Network model	Feature
random	poisson distribution
regular	zero-entropy value
scale-free	power law distribution
small-world	high average cluster coefficient

In literature other classifications of network models exist. As described in the book "Large Scale Structure and Dynamics of Complex Networks", of Guido Caldarelli and Alessandro Vespignani [2], it is possible to identify two principal belonging classes for network models: static random networks and evolving random networks.
In static random networks the number of nodes does not change and links among them are inserted with a probabilistic law. Reference network for this class is the Erdos-Renyi model, which is considered as the forefather of all network models, and it dates back to 1959.
Evolving random networks, grow and change in time by the addition of new vertices and edges, therefore the network history becomes important to understand how it acts on the final structure and on properties. Paradigm for the second class of models is the Barabasi-Albert model, which is decisively younger than the Erdos-Renyi model. According to Caldarelli and Vespignani, a special case of evolving random networks is represented by more recent new models named "weighted networks", where weights and topology co-evolve changing the structure of networks. In these cases, where the weights co-evolve with the network structure, their distribution follows power-laws.
Power law distribution for weights, is considered by authors a strong signature of the dynamical nature of the process.

In this thesis we dealt with undirected weighted networks. Following the first classification (regular, random, scale-free, small-world), we can state that the implemented networks belong to the regular and random categories. Following the Caldarelli and Vespignani classification, we used evolution random networks, but only with an "evolution" of links, without adding new nodes (so, no growth of the network appears). This kind of evolution concerns with weight of links and size of nodes, therefore it partially fulfills the description of evolving random network, and of the weighted networks, as introduced by Caldarelli and Vespignani, since new nodes are not inserted in the networks.

The following list includes all the types of networks investigated, two randoms and two regulars (four structures):

> **Definition 1.** The Erdos-Renyi random network $G(N, p)$ (abbreviated as "ER") is a random graph with N vertices, where each possible edge has probability p of existing.

The degree distribution of the Erdos-Renyi network, is a Binomial distribution:

$$P(k) = \binom{N}{K} p(1-p)^{N-K} \tag{1.1}$$

where K represents the degree of a vertex, N the total number of nodes in the network, and p is the probability to have an edge.

Definition 2. A Random k-Regular graph (abbreviated as "RERA") is a random graph, in which the elements of the correspondent adjacency matrix a_{ij} are chosen randomly, but the sum over each row and each column is always the same and it is constant, k_0. A popular method to produce this graph was first proposed by Bollobás [3] and it was called the "pairing model" (sometimes also "the configuration model"). For a graph of N nodes and m links (Nm must be even), imagine to have a urn with Nm balls, where balls represents the "open links" of nodes (we have m copies for each node i) [4]. We randomly select balls from the urn, matching pairs connecting respective nodes.
Example: consider a set of $N = 4$ nodes, $\{1, 2, 3, 4\}$, and $m = 3$ links. We will have 12 balls, and since we need of m copies of each node, we can write the set of balls in this way: $\{{m_1}^A, {m_1}^B, {m_1}^C, {m_2}^A, {m_2}^B, {m_2}^C, {m_3}^A, {m_3}^B, {m_3}^C, {m_4}^A, {m_4}^B, {m_4}^C\}$. Now, just randomly select pairs of open links ${m_{node}}^{copy}$ from the set, and connect them.
It is a very simple algorithm but, if N is low, there could be the possibility to have self-connections (loops), or multiple edges (two or more connections with the same node)

Definition 3. A 2-dimensional lattice (named "grid" as well, and abbreviated as "REG") is a regular network in which nodes are placed at integer coordinate points, in the 2-dimensional Euclidean space, and each node is connected to nodes which are exactly one unit far from it. This sort of definition can be extended to the case of n-dimensional lattice.
In this thesis we considered 2-d square lattices. Examples:
A network of $N = 4$ nodes, presents $m = 4$ links, and the coordinates for nodes are $(1, 1)$, $(1, 2)$, $(2, 1)$, $(2, 2)$. A network of $N = 9$ nodes, presents 12 links, because the corner-nodes have degree $k = 2$, the side-nodes have degree $k = 3$, and the central-nodes have degree $k = 4$. In this case the coordinates are $(1, 1)$, $(1, 2)$, $(1, 3)$, $(2, 1)$, $(2, 2)$, $(2, 3)$ $(3, 1)$, $(3, 2)$, $(3, 3)$.

Definition 4. A fully connected network (also called “all-to-all”, or “complete” network, and abbreviated “FULL”) is a regular network, where each node is connected to every other node. The number of links in a fully connected graph is $m = N(N-1)/2$.
Let us explain the reason [1]: the first node connects to $(N-1)$ nodes, the second connects to $(N-2)$ nodes, and so on, until the last node. The total number of edges is the following:

$$\begin{aligned} m &= (N-1) + (N-2) + (N-3) + ... \\ &= \sum_{i}^{(N-1)} i = N\frac{N-1}{2} \end{aligned}$$

About dynamics, two kinds of dynamical equations can describe features of evolving random networks: master-like equations and microscopic quantity equations [2].
Master-like equations are functional equations which give information on the evolution of some probability distribution. An example for these equations can be written using the degree distribution:

$$\frac{dP(k,t)}{dt} = F[\{P(k,t)\}_{k,1,\infty}] \tag{1.2}$$

where $F[\{P(k,t)\}_{k,1,\infty}]$ is a functional of all the degree probabilities.

Microscopic quantity equations are related to the evolution of some specific network property, for istance the degree of a specific site i, as in the following equation:

$$\frac{dk_i}{dt} = \prod[k_{i=1,N}, t] \tag{1.3}$$

where $\prod[k_{i=1,N}, t]$ represents the rate of change of the degree of vertex i as a function of time t and of all the degrees of the N vertices of the network.

1.2 State of the art

We developed our work basing on the paper of Takaaki Aoki and Toshio Aoyagi "Scale-Free Structures Emerging from Co-evolution of a Network and the Distribution of a Diffusive Resource on it" [5], where they propose and describe a model for coevolving network dynamics.

They assign a dynamical variable x_i to each node, calling that as "resource", i.e. x is a resource understood as a quantity of real-world networks (example: money in human networks), and the subscript i indicates the $i-th$ node of the network. Starting from an intial condition of a normal distribution for resources x_i and weights of links w_{ij}, they use the following two equations for the diffusion of resources and for a resource-driven evolution of the link weigths:

$$\begin{aligned}\triangle x_i &= x_i(t+1) - x_i(t) \qquad (1.4)\\ &= \underbrace{F(x_i(t))}_{\text{Reaction}} + \underbrace{D\sum_{j\in\mathcal{N}_i}\left(\frac{w_{ij}(t)}{s_j(t)}x_j(t) - \frac{w_{ji}(t)}{s_i(t)}x_i(t)\right)}_{\text{Diffusion}} \qquad (1.5)\end{aligned}$$

$$\begin{aligned}\triangle w_{ij} &= w_{ij}(t+1) - w_{ij}(t) \qquad (1.6)\\ &= \epsilon\left[x_i(t)^\alpha x_j(t)^\alpha - w_{ij}(t)^\alpha\right] \qquad (1.7)\end{aligned}$$

where:

$$F(x_i(t)) = -\kappa\,(x_i(t)-1) \qquad (1.8)$$

$\kappa =$ parameter for dissipation

$D =$ parameter for diffusion

$s_i =$ strength of the i-th node

$\epsilon^{-1} =$ relaxation time scale of the weigths dynamics

$t =$ time

$\mathcal{N}_i =$ set of nodes connected to the i-th node

$\alpha =$ controls the non-linearity of the resource dependence

The definition of strength is $s_i(t) \equiv \sum_{j\in\mathcal{N}_i} w_{ji}(t) = \sum_j a_{ij}w_{ij}$ (where a_{ij} = element of the Adjacency matrix), and it is important to notice that we consider a time-dependent symmetric weigth $w_{ij}(t) = w_{ji}(t)$ for each existing link.

To better understand the definition of strength for the $i-th$ node we are supported by the Figure 1.1. In this example, the node number 5 (grey) will be the $i-th$ node, the index j indicates the surrounding nodes (red), and the weights are the blue numbers

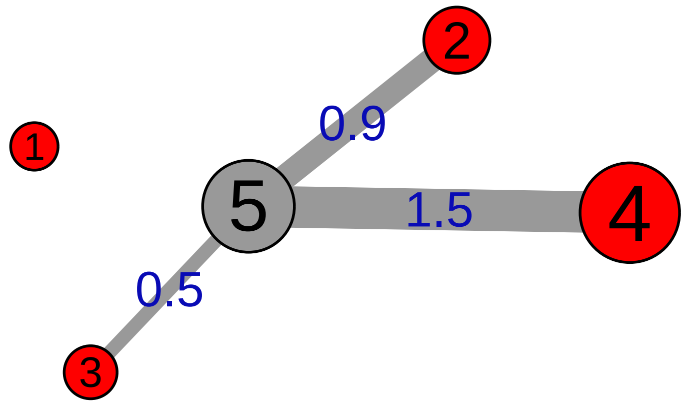


Figure 1.1: This picture is an example to explain the definition of strength s_i. The $i-th$ node is represented by the grey node 5, and the blue numbers indicate the weigths of the different links.

overlying the links. So:

$$\begin{aligned}
s_{i=5} &= \sum_j a_{ij} w_{ij} \\
&= a_{51}w_{51} + a_{52}w_{52} + a_{53}w_{53} + a_{54}w_{54} \\
&= \underbrace{a_{51}}_{=0} w_{51} + \underbrace{a_{52}}_{=1} w_{52} + \underbrace{a_{53}}_{=1} w_{53} + \underbrace{a_{54}}_{=1} w_{54} \\
&= w_{52} + w_{53} + w_{54} \\
&= 0.9 + 0.5 + 1.5 \\
&= 2.9
\end{aligned}$$

In this case, the set of nodes connected to node 5 is $\mathcal{N}_5 = \{2, 3, 4\}$.
The authors studied the co-evolving dynamics over two networks, the Regular Random Graph and the Erdos-Renyi graph, and selected two sets of parameters for their networks, showed in Table 1.1. With those choices of parameters κ, D, ϵ, α (for $\kappa = 0$, $\alpha \geq 1$),

(a) **Case $\kappa = 0.05$**

Parameter	Value
D	0.34
ϵ	0.01
α	1

(b) **Case $\kappa = 0$**

Parameter	Value
D	0.02
ϵ	0.01
α	{0.5, 1, 1.25 }

Table 1.1: Two sets of parameters chosen by the Authors T.Aoki and T.Aoyagi

they showed that the system acquires scale-free characteristics in the asymptotic state, as showed in Table 1.2.

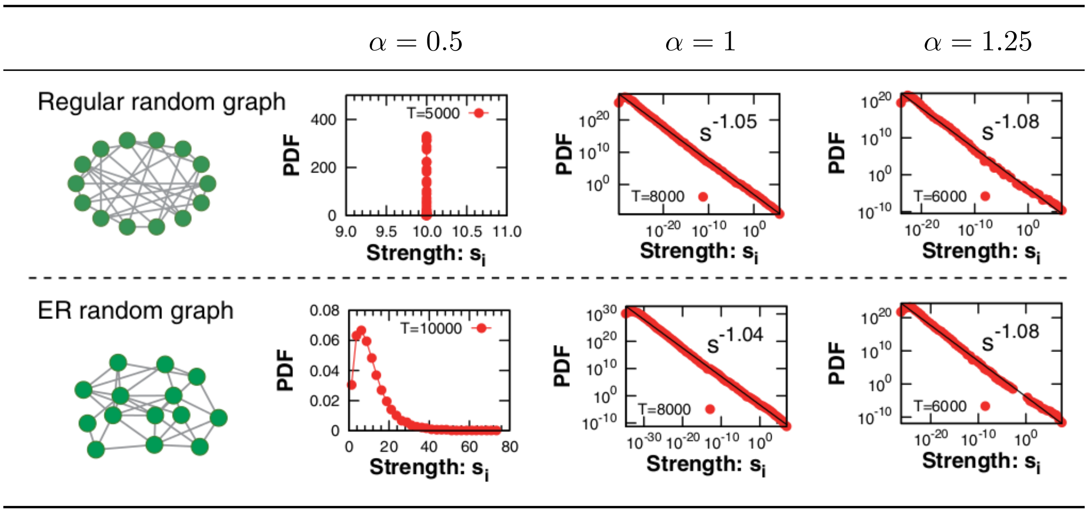


Table 1.2: This picture has been taken by the paper of Aoki and Aoyagi [5] and it shows the strength distributions of Regular Random network (RERA) and the Erdos-Renyi (ER) network, for three different cases: $\alpha = 0.5$, $\alpha = 1$ and $\alpha = 1.25$. Basically their results demonstrate the presence of a "scale-free structure" with a power-law distribution in RERA and in ER for $\alpha = 1$ and $\alpha = 1.25$. The number of nodes and the degree used in those simulations are: RERA $\rightarrow N = 512$, $k = 5$ and ER $\rightarrow N = 16384$, average degree $\langle k \rangle = 10$. The time of simulation represents the number of cycles of the simulation in which all the algorithm is performed. This time varies from $T = 5000$ to $T = 10000$. As the reader can notice , the intervals for the strengths s_i and the "PDF" are very large, from 10^{-30} to 10^{10} for s_i and from 10^{30} to 10^{10} for the "PDF". This is due to the particular technique for plotting power-law data, in which logarithmic bins are used to construct the histogram.

1.3 Thesis targets

In the previous section we presented the results obtained by T.Aoki and T.Aoyagi. They showed that scale free structures with power law distributions appear for particular conditions in the Regular Random network (RERA) and in the Erdos-Renyi network (ER). From this moment on, we will describe the novel and original part of this master thesis, and when it will be necessary, we will reference to the work of T.Aoki and T.Aoyagi. Goals of this thesis are indicated in the following list:

1. Reproduce the Erdos Renyi network (with Montecarlo method) and the random k-Regular network, and study the evolution of resources and edges.

2. Reproduce the fully connected network and the regular network (a 2-dimensional lattice), and study the evolution of resources and edges.

3. If the weight of edges is lower than delta (delta $= 0.01$) $\rightarrow$ delete the link, and study the evolution. Do the Distribution of degrees change?

4. Study the propagation of strengths in an Erdos-Renyi network, starting from an initial nucleus (differently from the normal distribution chosen by T.Aoki and T.Aoyagi).

A computational creation of the Erdos Renyi network, with the Montecarlo method, is described in Appendix A. The analysis of resources and edges evolution, of points 1 and 2, will be showed together to the results of task 3, in the Chapter 2 (degree distributions and strength distributions). A study of point 4, for a system with an initial nucleus, will be discussed in the Chapter 3.

Chapter 2

Delete links for a peculiar condition

2.1 Do the Distribution of degrees change?

As we noticed in different plots for the Erdos-Renyi network and for the Regular Random network in Table 1.2, our values for strengths reached numbers as 10^{-20} or 10^{-30}. Physically they have no sense, so we decided to set a a minimum, called $\delta = 0.01$. Basically our rule is:

if the weigth $w_{ij} < \delta$, cut the link.

This deletion of links has been performed during the simulation, in the interplay of equations 1.4 and 1.6.
It makes you wonder whether this change, affects the probability distribution of degrees and/or the strengh distribution. An answer to the above "question", is contained into the next figures. They represent the results of simulations performed for each structure of network, listed in the introduction:

1. The Erdos-Renyi random network ("ER"),
2. The 2-dimensional lattice ("REG"),
3. The Random k-Regular graph ("RERA"),
4. The fully connected network ("FULL").

First 4 pictures, Figure 2.1, Figure 2.2, Figure 2.3, Figure 2.4, show the superposition of probability degree distributions coming from the "original" networks and the respective "reduced" networks. Other 4 pictures, Figure 2.5, Figure 2.6, Figure 2.7, Figure 2.8, show

the comparison among strength distributions, correspondent to the "original" networks and the respective "reduced" networks.
The adjective "original" makes reference to the computational creation of a network, and to the simulation of equations 1.4 and 1.6 on each kind of network. Once the network is produced, no modifies will be applied during the simulation, therefore its degree distibution and its strength distribution can be named "original".
The adjective "reduced" means that, after the reproduction of a network, the above condition for removing links is applied, during the evolution of the system (using the Aoki and Aoyagi equations, 1.4 and 1.6).
As consequence, a network can have a modified degree distribution and/or a modified strength distribution.

Size and parameters used for the creation of networks are the same of Aoki and Aoyagi, for the Erdos-Renyi network (ER) and the Random k-Regular network (RERA). The main values for each network are in the following list:

1. ER $\rightarrow N = 16384$, and average degree $\langle k \rangle = 10$.
2. REG $\rightarrow N = 3600$, degree depends on node's position, $k = \{2, 3, 4\}$
3. RERA $\rightarrow N = 512$, and degree $k = 5$.
4. FULL $\rightarrow N = 400$, and degree $k = 400$.

In the present thesis, we analyzed the case $\kappa = 0$ for each value of $\alpha = \{0.5, 1, 1.25\}$, as indicated in Table 1.1 , and every simulation has been performed with the same number of time steps used by Aoki and Aoyagi, for networks studied by them, i.e. for the Erdos-Renyi network (ER) and the Random k-Regular network (RERA). Please see the Table 1.2 as reference of simulation's times. For the other structures of networks (REG and FULL), we used the number of time steps of Erdos-Renyi network, so $t = 10000$ for $\alpha = 0.5$, $t = 8000$ for $\alpha = 1$, $t = 6000$ for $\alpha = 1.25$.

2.2 Description of the results

In the first 4 pictures, (from Figure 2.1 to Figure 2.4), degree distributions correspondent to the so called "original" networks are painted with blue dots, while degree distributions for the "reduced" networks are represented by red dots. For the Erdos-Renyi network in Figure 2.1, a superposition of degree distributions does not reveal big differences among them, and the distribution's peak keeps the position on $k = 10$ (i remember that the average degree is $\langle k \rangle = 10$).

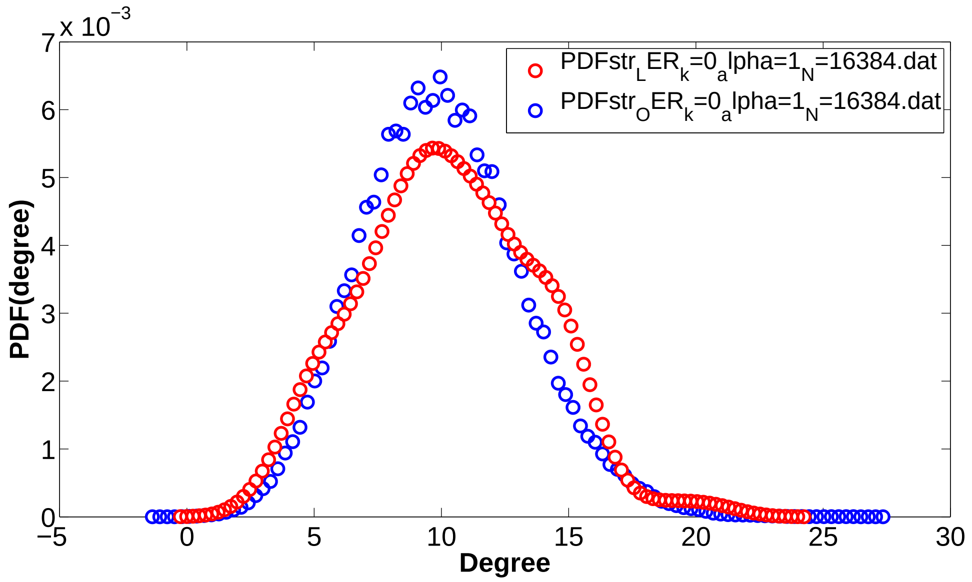
A different scenario appears for other kinds of networks, in Figure 2.2, Figure 2.3, and in Figure 2.4. In all cases when $\alpha = 0.5$, the degree distribution perfectly maintains its "shape", but a notable change turn out to be when $\alpha = 1$ and $\alpha = 1.25$, where the degree distributions decrease, due to imposed condition of links deletion. It is worth mentioning that, among degree distributions with $\alpha = 1$ and $\alpha = 1.25$, the FULL network with $\alpha = 1$ (in Figure 2.4(b)), appears not modified.

Since degree distribution and strength distributions are related, changes in one of them reflect on the other one. Comparison among strenght distributions, belonging to "original" and "reduced" networks, are showed in Figure 2.5, Figure 2.6, Figure 2.7, Figure 2.8. About the "original" Erdos-Renyi network (OER) with $\alpha = 1$, in Figure 2.5, we have a power law for the strenght distribution, with slope -1.05 and strength-range between 10^{-30} and 10^5. It is similar to what obtained by Aoky and Aoyagi in Table 1.2: they got a power law with slope of -1.04.
The correspondent "reduced" network (LER), has a power law for strenght distribution with slope of -1.06 and strength-range between 10^{-2} and 10^6.
For OER with $\alpha = 1.25$, we got a power law with slope equal to -1.09 and strength-range between 10^{-25} and 10^5. Still in Table 1.2 it is possible to read the slope obtained by Aoky and Aoyagi which is -1.08. The "reduced" case, LER, shows a change in the power law distribuition: slope of -0.9 in the strength-range between 10^{-2} and 10^5, and slope of -1.45 in the strength-range between 10^5 and 10^7. According to Milojevic in "Power Law Distributions in Information Science: Making the Case for Logarithmic Binning", [6], those distribution can be called "Double power law", which is a deviation form the perfect power law (it a power law with two different exponents).

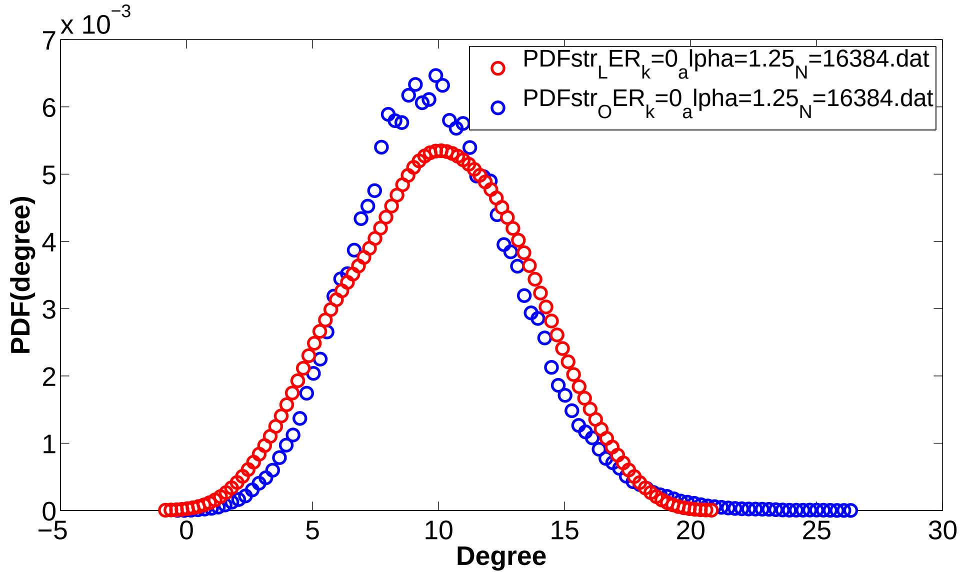
In Figure 2.6 we have comparison between OREG and LREG (as before, same meaning for letters "O", as "original", and "L", as "reduced" networks). For $\alpha = 0.5$ no change occur among OREG and LREG. We notice changes in cases of $\alpha = 1$ and $\alpha = 1.25$. In $\alpha = 1$ there is a passage from a power law for strenght distribution with slope -0.96 to a power law distribution between 10^{-2} and 10^2 with slope -0.65, and then an exponential behaviour with exponent $b = -0.01$. In agreement to Milojevic in [6], this last distribution can be considered as a "power law with an exponential cutoff".
Instead, OREG, $\alpha = 1.25$ has a power law for the strength distribution with slope -1.06, and the correspondent LREG, $\alpha = 1.25$, shows a power law distribution between 10^{-1} and 10^1 with slope 1, and in the final part an exponential behaviour with exponent $b = -0.004$. Here, the shape of the strenght distribution is not well defined.

The third structure of network here analyzed is the random k-regular graph (RERA), in Figure 2.7. For $\alpha = 0.5$ the shape of distribution is slightly different, and the distribution's peak is shifted from ORERA to LRERA for a value of about 0.07 in strenght.
In $\alpha = 1$ and $\alpha = 1.25$, the passage to "reduced" LRERA is different: the strength distributions with power law in ORERA, change "shape" in LRERA, towards values of strenght of 10^3. So in LRERA we get two slopes, for both cases.
In ORERA, $\alpha = 1$ the power law distribution for strengths has slope -1.01, and the correspondent LRERA, $\alpha = 1$, exhibits a power law distribution with two slopes: $slope = -0.94$ between 10^{-1} and 10^3, and $slope = -1.88$ between 10^3 and 10^4.
In ORERA, $\alpha = 1.25$ the power law distribution for strengths has slope -1.05, and the correspondent LRERA, $\alpha = 1.25$, has a power law distribution with two slopes: $slope = -0.8$ between 10^{-1} and 10^3, and $slope = -1.63$ about between 10^3 and 10^4. As described in the paper Milojevic, [6], it is possible to identify and call as "double power law", the strenght distributions obtained from LRERA, for $\alpha = 1$ and $\alpha = 1.25$.

Last kind of network discussed in this section is the fully connected network (FULL), , in Figure 2.8.
Distributions of "original" and respective "reduced" FULL networks, i.e. of OFULL and LFULL, are quite similar for $\alpha = 0.5$ and $\alpha = 1$. They do not considerably change their shape in the "passage" from OFULL to LFULL.
An important change occurs for the case of $\alpha = 1.25$, where a power law for strenght distribution in OFULL with slope -1.1 reduces drastically in LFULL. Indeed only three links with strength $s_i = 416484$ survive.

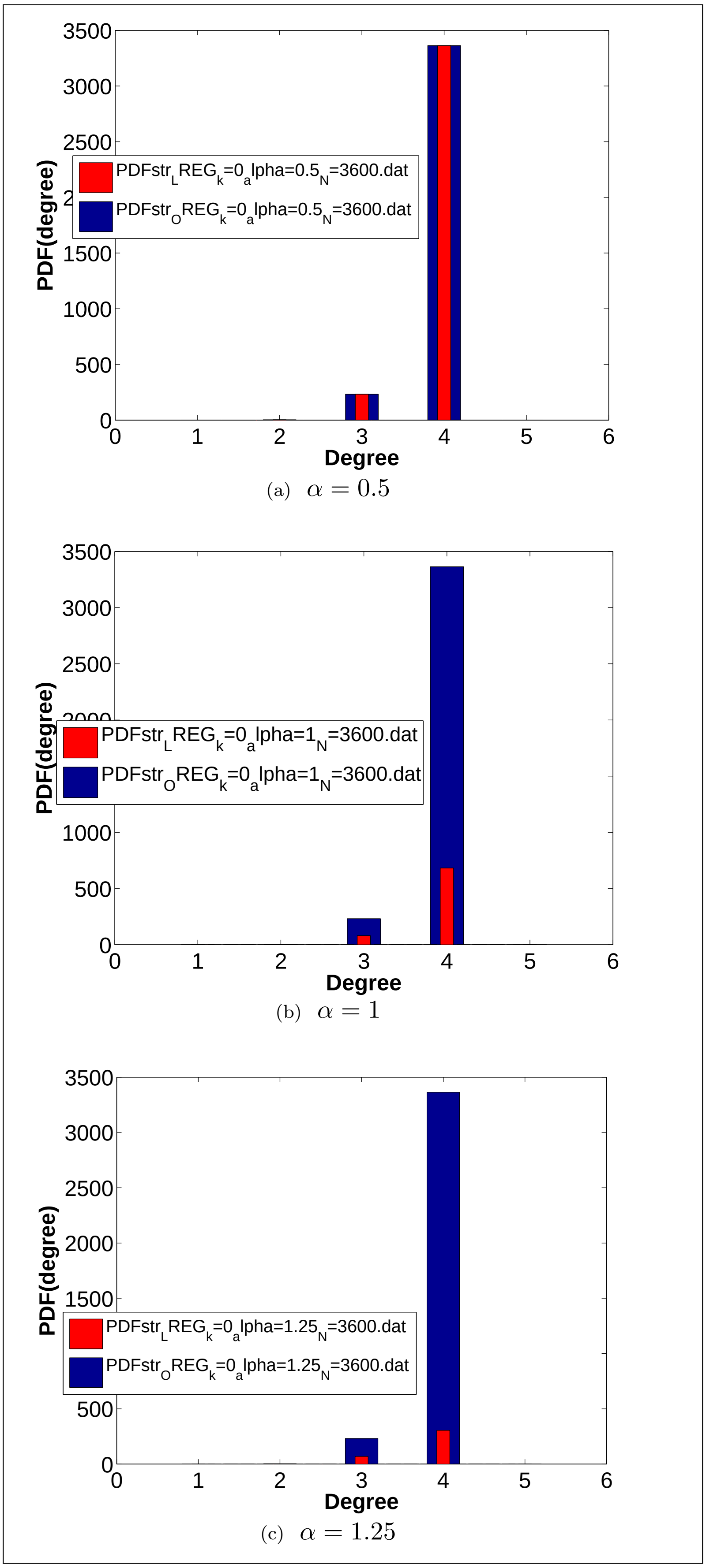


(a) $\alpha = 1$



(b) $\alpha = 1.25$

Figure 2.1: Superposition of the probability degree distributions of the "original" Erdos-Renyi network (OER), and the "reduced" Erdos-Renyi network (LER), due to the employ of our rule for the weigths. The red distribution (LER), is close enough to the original blue distribution. There is not a big difference, although the two shapes do not match in several areas. The peak of the distribution remains on the average value $\langle k \rangle = 10$.

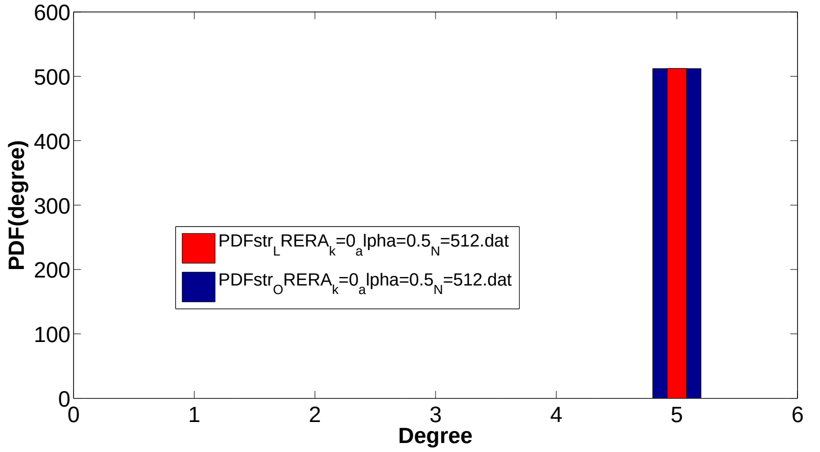


(a) $\alpha = 0.5$

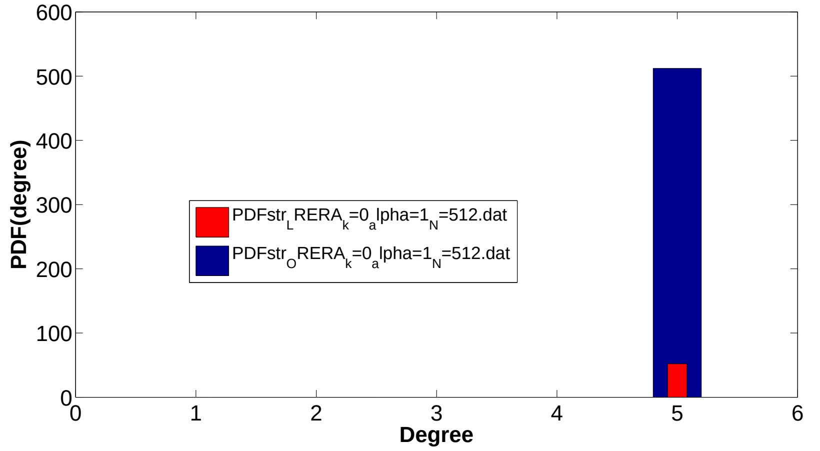
(b) $\alpha = 1$

(c) $\alpha = 1.25$

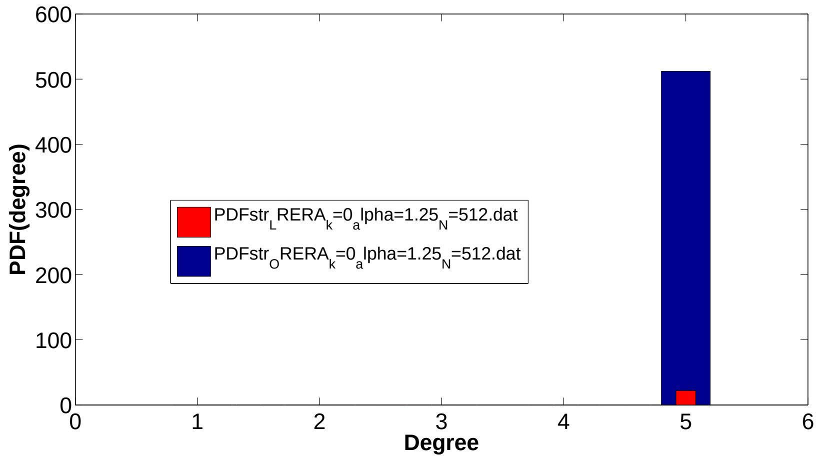
Figure 2.2: Superposition of the probability degree distributions of the “original” Regular (not random) network (OREG), and the “reduced” Regular (not random) network (LREG), due to the employ of our rule for the weigths. For the case $\alpha = 0.5$, the two distributions look the same, but for the second and third cases, $\alpha = 1$ and $\alpha = 1.25$, the red distribution of LREG is lower than the original one of OREG. It means that the large part of links in the network has been cut during the simulation.



(a) $\alpha = 0.5$

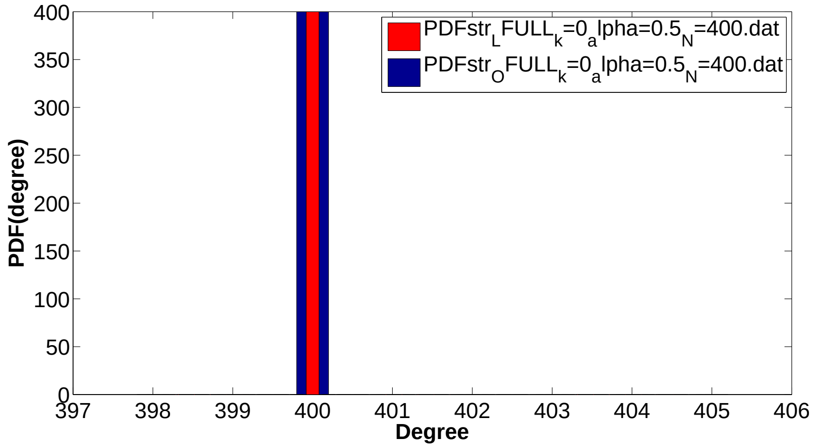


(b) $\alpha = 1$

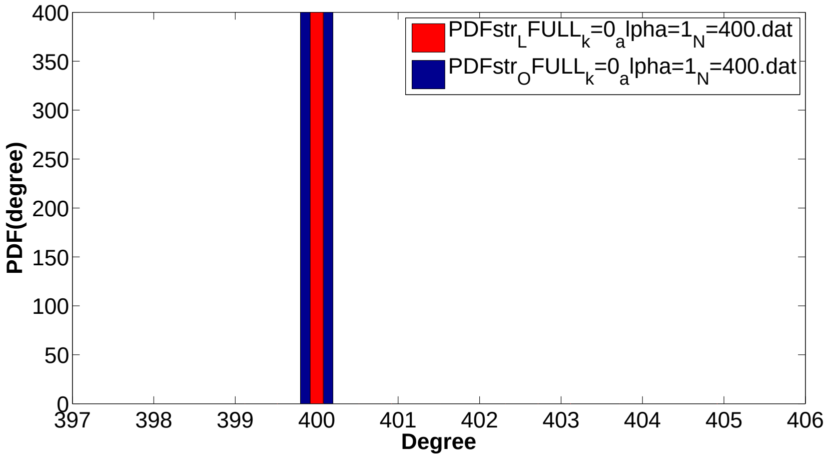


(c) $\alpha = 1.25$

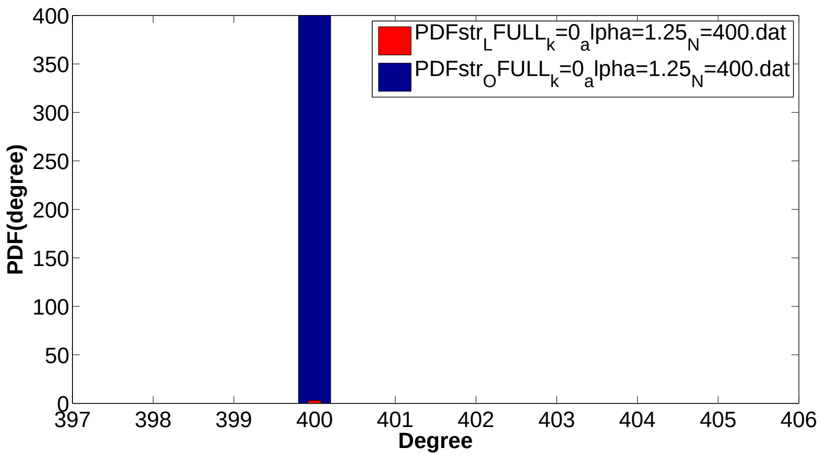
Figure 2.3: Superposition of the probability degree distributions of the "original" Regular Random network (ORERA), and the "reduced" Regular Random network (LRERA), due to the employ of our rule for the weigths. We have a similar situation to the Figure 2.2, where for the case $\alpha = 0.5$, the two distributions look the same, and for the other cases, $\alpha = 1$ and $\alpha = 1.25$, the red distribution of LRERA is lower that the original one of ORERA. For this kind of network, only the case of $\alpha = 0.5$ maintains perfectly the same "original" shape.



(a) $\alpha = 0.5$

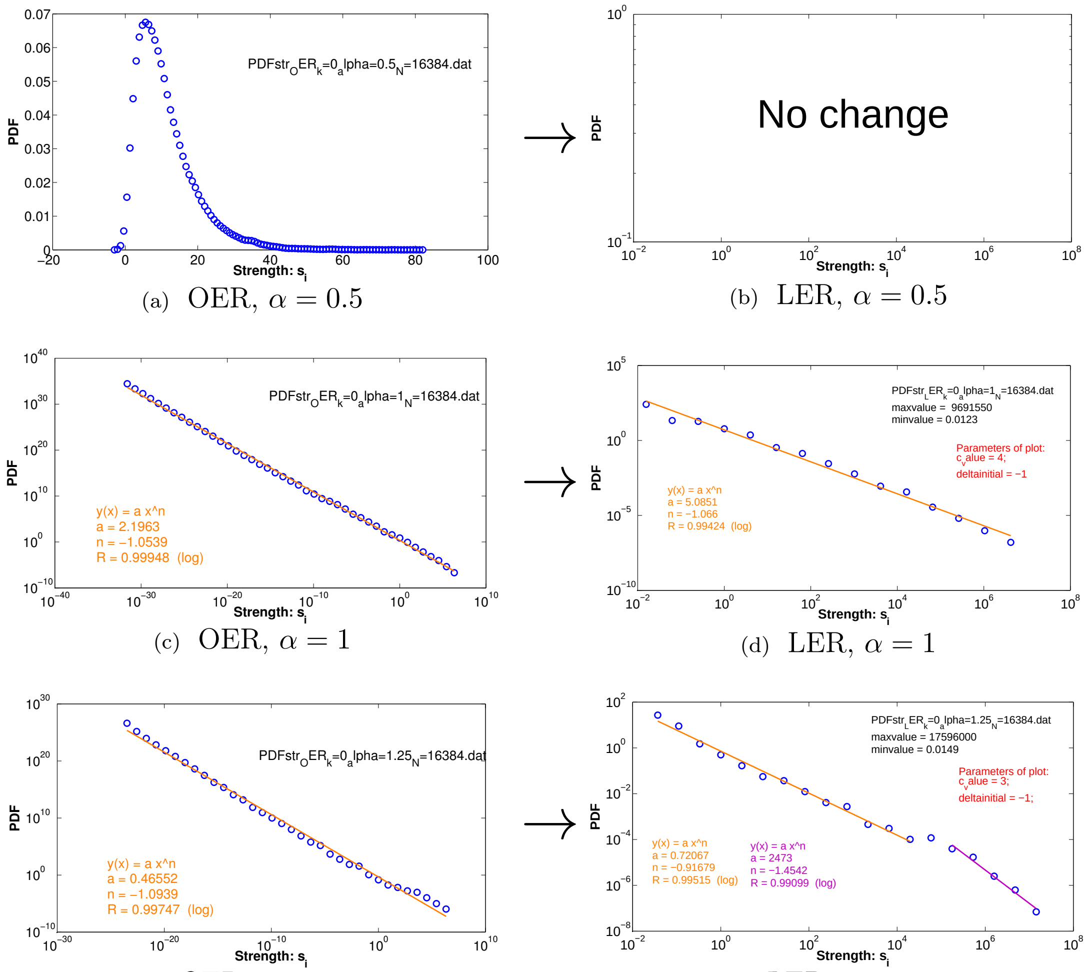


(b) $\alpha = 1$



(c) $\alpha = 1.25$

Figure 2.4: Superposition of the probability degree distributions of the “original” Full connected network (OFULL), and the “reduced” Full connected network (LFULL), due to the employ of our rule for the weigths. Comparing these degree distributions with the ones in Figure 2.2 and in Figure 2.3, we can notice a little difference: in the case of $\alpha = 1$, the degree distribution maintains its structure perfectly, while in the previous two figures our degree distributions at $\alpha = 1$ changed their “physiognomy”.

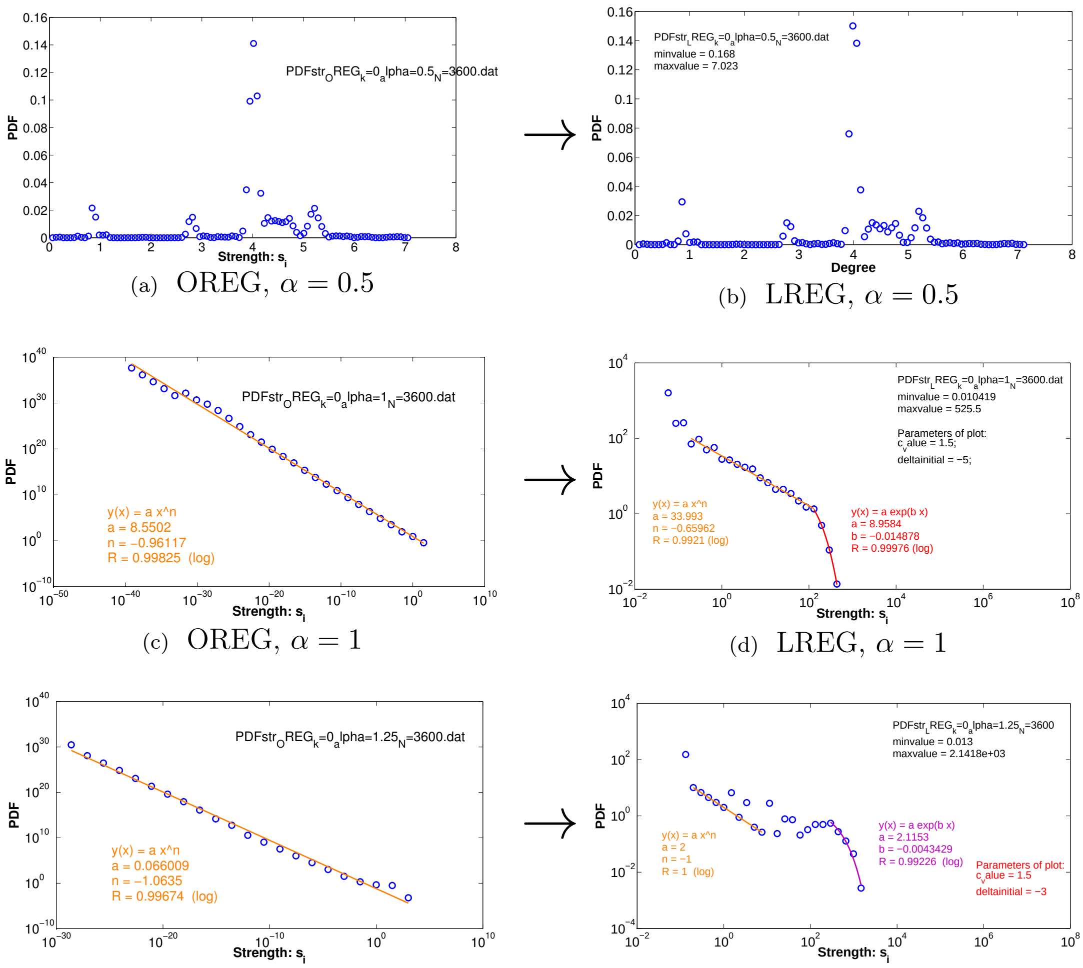


(a) OER, $\alpha = 0.5$ (b) LER, $\alpha = 0.5$

(c) OER, $\alpha = 1$ (d) LER, $\alpha = 1$

(e) OER, $\alpha = 1.25$ (f) LER, $\alpha = 1.25$

Figure 2.5: What happens to the strength distributions if we apply our rule over the weights? Here we have a comparison between the "original" Erdos-Renyi network (OER), and the "reduced" Erdos-Renyi network (LER) for three cases: $\alpha = 0.5$, $\alpha = 1$ and $\alpha = 1.25$. We grouped all these plots in one figure for a better visual impact, to quickly recognise the big differences in the distributions. About the OER, $\alpha = 0.5$ in Figure 2.5(a), we have no $w_{ij} < \delta$, so there are no cuts of links and the distribution maintains its shape. Different scenarios for OER, $\alpha = 1$ in Figure 2.5(c), and OER, $\alpha = 1.25$ in Figure 2.5(e), where the distibutions change. For $\alpha = 1$, a power law distribution with slope -1.05 and strength-range between 10^{-30} and 10^5 , becomes a power law distribuition with slope -1.06 and strength-range between 10^{-2} and 10^6. For $\alpha = 1.25$, the power law distribution with slope -1.09 and strength-range between 10^{-25} and 10^5 , becomes a power law distribuition with slope -0.9 in the strength-range between 10^{-2} and 10^5, and with slope -1.45 in the strength-range between 10^5 and 10^7.



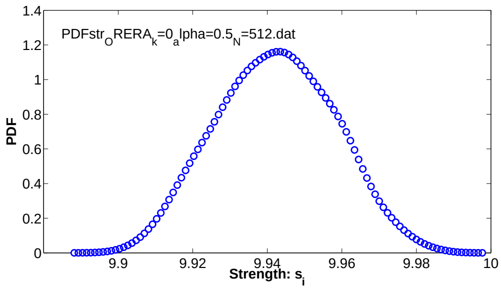
(a) OREG, $\alpha = 0.5$ (b) LREG, $\alpha = 0.5$

(c) OREG, $\alpha = 1$ (d) LREG, $\alpha = 1$

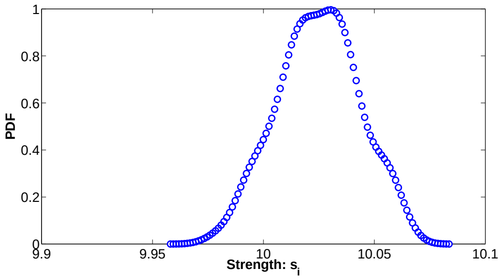
(e) OREG, $\alpha = 1.25$ (f) LREG, $\alpha = 1.25$

Figure 2.6: Here we consider the Regular not random network (REG) and the changing of its strength distributions, in the passage to the "reduced" Regular not random network (LREG). We have three cases as in Figure 2.5: $\alpha = 0.5$, $\alpha = 1$ and $\alpha = 1.25$.

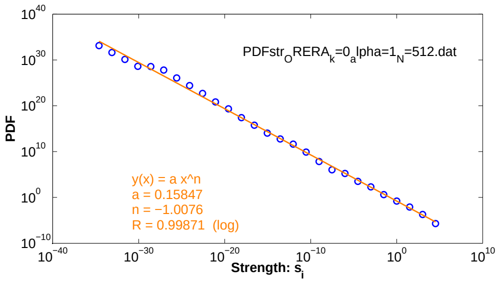
Taking a look to OREG, $\alpha = 0.5$ in Figure 2.6(a), and to LREG, $\alpha = 0.5$ in Figure 2.6(b), we observe a similar distribution with a noteworthy peak at $s_i = 4$ in both figures. In $\alpha = 1$ and $\alpha = 1.25$, the passage to "reduced" LREG is different: the strength distributions with power law in OREG, lose their "shape" in LREG, towards values of $s_i = 10^2$, showing a behaviour very close to an exponential law. Details: OREG, $\alpha = 1$ has a power law for the strength distribution with slope -0.96, and the correspondent LREG, $\alpha = 1$, exhibits a power law distribution between 10^{-2} and 10^2, and then an exponential behaviour with exponent $b = -0.01$. Instead, OREG, $\alpha = 1.25$ has a power law for the strength distribution with slope -1.06, and the correspondent LREG, $\alpha = 1.25$, shows a power law distribution between 10^{-1} and 10^1 with slope 1, and in the final part an exponential behaviour with exponent $b = -0.004$.



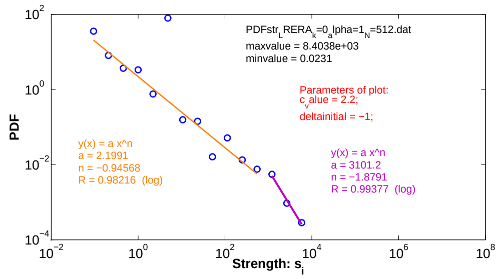
(a) ORERA, $\alpha = 0.5$



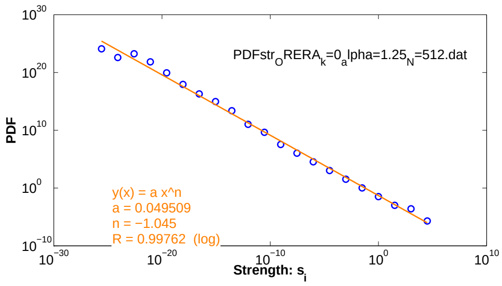
(b) LRERA, $\alpha = 0.5$



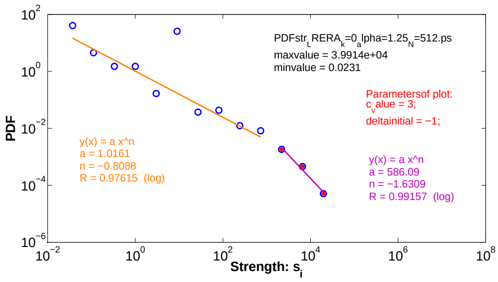
(c) ORERA, $\alpha = 1$



(d) LRERA, $\alpha = 1$

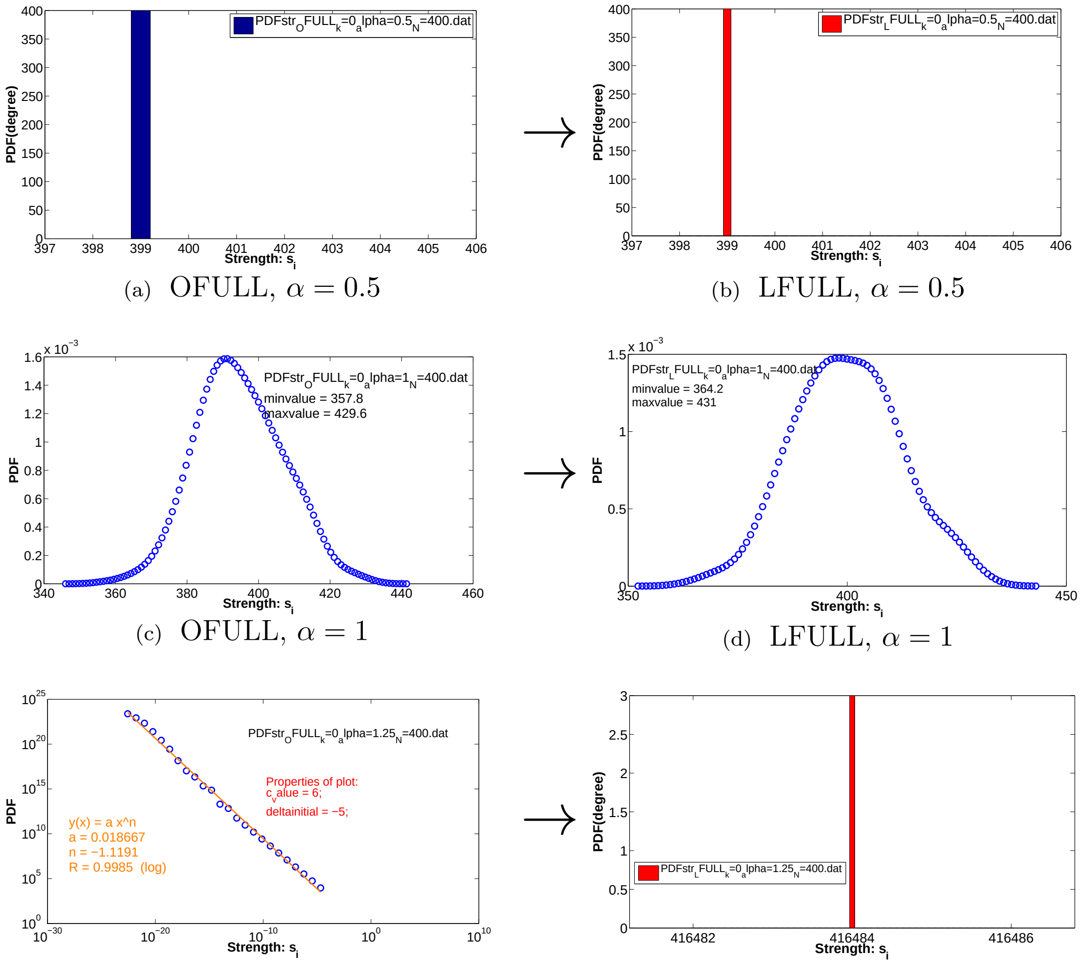


(e) ORERA, $\alpha = 1.25$



(f) LRERA, $\alpha = 1.25$

Figure 2.7: Comparison between strength distribution of the "original" Regular random network (ORERA) and of the "reduced" Regular random network (LRERA). We have three cases as for the other kind of networks: $\alpha = 0.5$, $\alpha = 1$ and $\alpha = 1.25$. Let's consider ORERA, $\alpha = 0.5$ in Figure 2.7(a), and LRERA, $\alpha = 0.5$ in Figure 2.7(b): despite what we observed for OREG, and LREG, $\alpha = 0.5$ in Figure 2.6(a), and in Figure 2.6(b), now the peak of the strength distribution moves from $s_i = 9.94$ towards $s_i = 10.03$, and the also its shape lightly modifies, becoming tighter. In $\alpha = 1$ and $\alpha = 1.25$, the passage to "reduced" LRERA is different: the strength distributions with power law in ORERA, change "shape" in LRERA, towards values of $s_i = 10^3$, showing a behaviour very close to an exponential law. So in LRERA for both cases we get two slopes. In ORERA, $\alpha = 1$ the power law distribution for strengths has slope -1.01, and the correspondent LRERA, $\alpha = 1$, exhibits a power law distribution with two slopes: *slope* $= -0.94$ between 10^{-1} and 10^3, and *slope* $= -1.88$ between 10^3 and 10^4. In ORERA, $\alpha = 1.25$ the power law distribution for strengths has slope -1.05, and the correspondent LRERA, $\alpha = 1.25$, has a power law distribution with two slopes: *slope* $= -0.8$ between 10^{-1} and 10^3, and *slope* $= -1.63$ about between 10^3 and 10^4.



(a) OFULL, $\alpha = 0.5$

(b) LFULL, $\alpha = 0.5$

(c) OFULL, $\alpha = 1$

(d) LFULL, $\alpha = 1$

(e) OFULL, $\alpha = 1.25$

(f) LFULL, $\alpha = 1.25$

Figure 2.8: Comparison between strength distribution of the "original" Full network (OFULL) and of the "reduced" FULL network (LFULL). We have three cases like in the other kind of networks: $\alpha = 0.5$, $\alpha = 1$ and $\alpha = 1.25$.

The strength distribution of OFULL, $\alpha = 0.5$ in Figure 2.8(a), is exactly identical to the distribution of LFULL, $\alpha = 0.5$ in Figure 2.8(b), and all the strength s_i take the same value. The distribution of FULL, $\alpha = 1$ roughly keeps the shape as well, but now the values lie in a wider interval (about between $s_i = 360$ and $s_i = 430$). From OFULL, $\alpha = 1$ in Figure 2.8(c) to LFULL, $\alpha = 1$ in Figure 2.8(d) the sharp strength distribution moves its peak towards higher values of s_i, and at the same time the peak smooths out. The last case concerns OFULL, $\alpha = 1.25$, which shows a power law distribution for the strengths, with a slope of -1.1, changes drastically shape: essentialy, the reduction of links due to the use of the rule over the weights leaves only three links with strength $s_i = 416484$.

Chapter 3

Propagation of strengths starting from an initial nucleus

3.1 Studying of a small Erdos-Renyi network with an intial nucleus

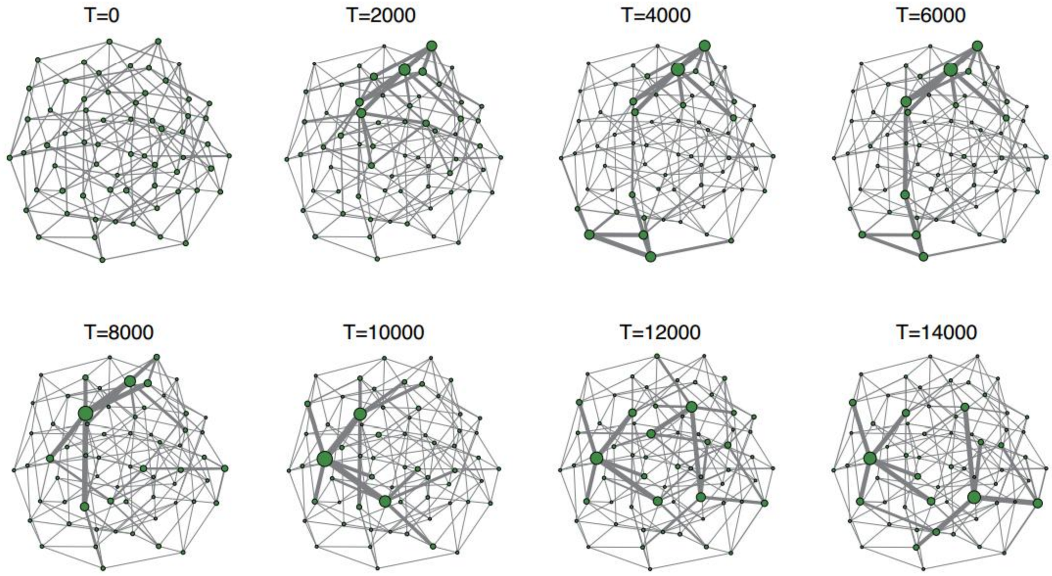
In the first chapter we considered the Erdos-Renyi (ER) network, with an intial condition of a normal distribution for resources x_i and weights w_{ij} of links.
Task of this last part of the thesis is a study on a different initial condition for the weights in ER network. This involves the formation of a simple initial structure in the network called "initial nucleus": this is a triangular structure composed by three nodes with stronger links than all the other nodes in the network.
The nuclues has been created following the criterion of minimum total degree, joining the nodes with minimum (possible) degree.

In the original paper, Takaaki Aoki and Toshio Aoyagi studied the behavior of the co-evolving dynamics for a regular random graph showing that the system changes status all the time, up to time $t = 14000$. (please see the Figure 3.1).

In our case, imposing the new initial condition to the Erdos-Renyi (ER) network with the following features,

$$\alpha = 1 \text{ and } \kappa = 0 \quad .$$

we get a different evolution than T.Aoki and T.Aoyagi.
The system reaches a steady state after a certain time.



(a) IER, $\alpha = 1$

Figure 3.1: Co-evolving dynamics for a regular random graph obtained by T.Aoki and T.Aoyagi, and showed in the "supplemental material" of the original paper [7]. Here, size $N = 64$ and degree $k = 5$, and the initial values have been generated with a normal distribution for resources x_i and links weights w_{ij}.
Other parameter values: $D = 0.35$, $\epsilon = 0.01$, $\kappa = 5$.

We can show an example of our result for a system of 100 nodes (ER network): this is depicted in Figure 3.2 for the time evolution of strengths and in Figure 3.3 for the time evolution of resources (node sizes).

Figure 3.2(a) and Figure 3.3(a) are tridimensional images: the x-axis represents the name of nodes, each one with different color, from 1 to 100, the y-axis is the time-axis, from 0 to 14000, and the z-axis is the strength-axis and size-axis, respectively.
Nodes with the highest strength and node size, are indicated in the initial state (3 nodes at $t = 0$) and in the final state (3 nodes at $t = 14000$).

As may be seen from both pictures

1. nodes with highest values at the first time, are different from nodes with highest values at the last time,
2. nodes with highest values at the last time are the same for strength and node size

Regarding the node degree, we can take a look to the top fifteen degree-list, in the first time step ($t = 0$) here below:

```
#1 --> degree 19, node 91
#2 --> degree 16, node 2
```

```
#3 --> degree 16, node 93
#4 --> degree 15, node 33
#5 --> degree 15, node 37
#6 --> degree 15, node 40
#7 --> degree 15, node 61
#8 --> degree 14, node 14
#9 --> degree 14, node 20
#10 --> degree 14, node 25
#11 --> degree 14, node 46
#12 --> degree 14, node 71
#13 --> degree 13, node 23
#14 --> degree 13, node 28
#15 --> degree 13, node 67
```

It is a descending order list from the node with highest degree to the node with lowest degree. The three highest value nodes in Figure 3.2 and in Figure 3.3 appear at the fifth, seventh and fifteenth position over 100 positions (in red).

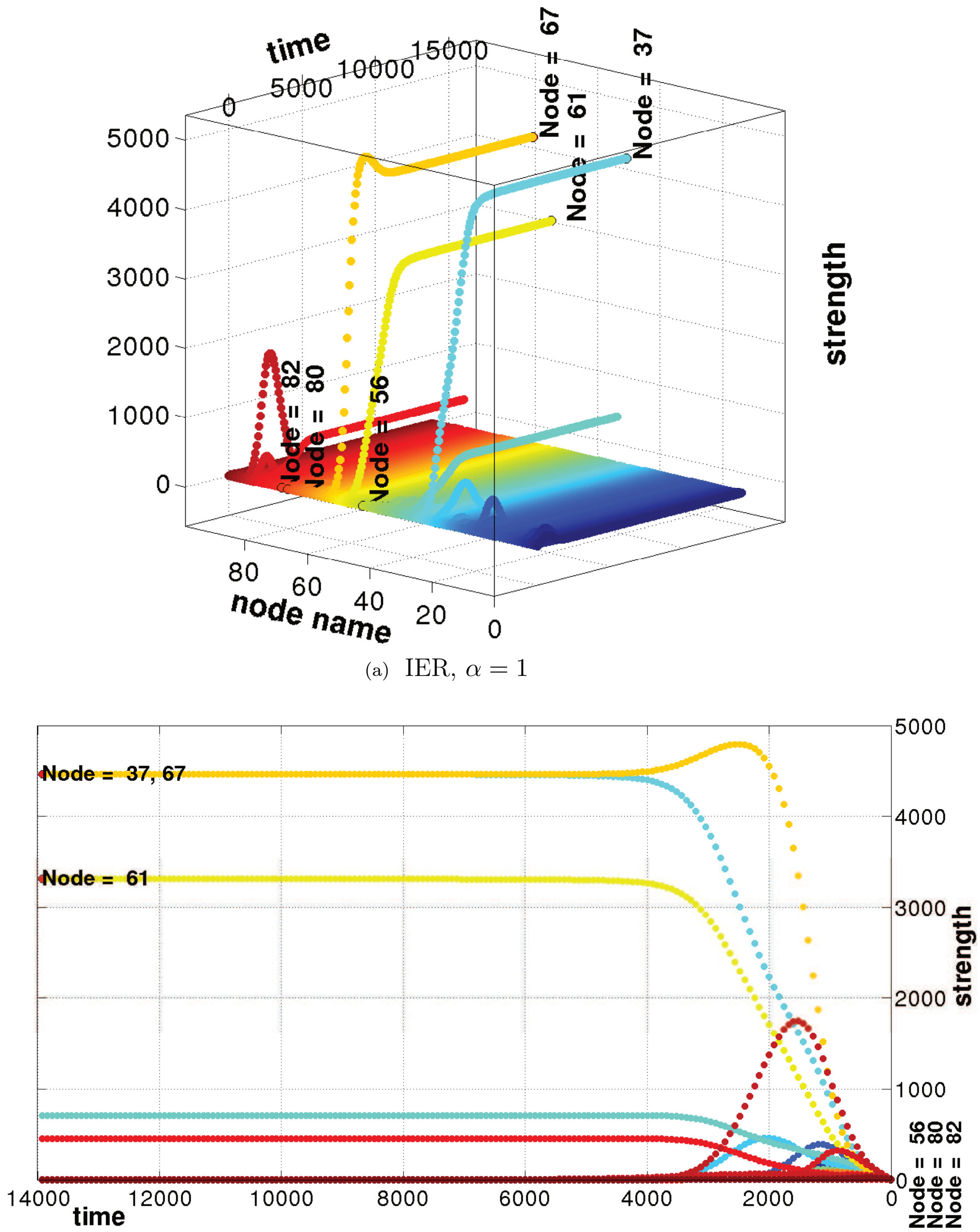
It means that there is a correlation between the initial node degree and the final features of the ER system. Indeed we can notice that

3. three nodes with highest strength and node size (width) at the final time step, have some of the highest initial degrees.

3.2 Strength distribution of an Erdos-Renyi with an initial nucleus

Going further with the analysis of the ER network with an initial structure, we can show the strength distribution with same parameters of the "original" Erdos-Renyi network (OER) in Figure 2.5(c): $\kappa = 0$, $\alpha = 1$, $N = 16384$ and average degree $\langle k \rangle = 10$.
We can remember that its distribution is a power law distribution with slope -1.05 and strength-range between 10^{-30} and 10^{5}.
In Figure 3.4(a), we observe very similar data for the IER strength distribution, with slope -1.05 and strength-range between 10^{-30} and 10^{5}.
Essentially this distribution does not change if we introduce an intial nucleus at the early times.

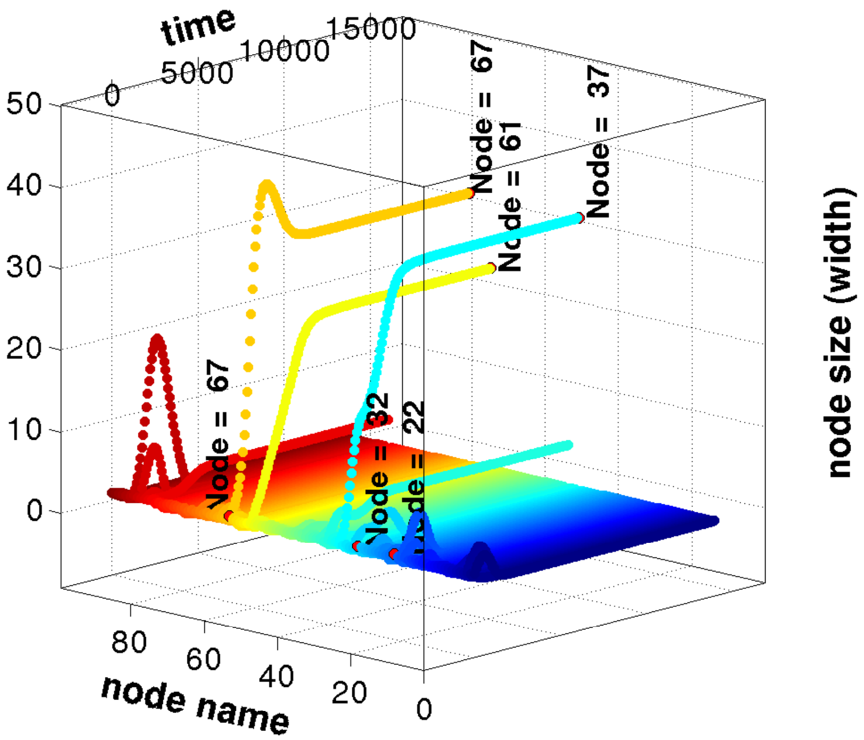
Another aspect that can be taken into account, is related to the "decomposition" of the above strength distribution for the different degrees.
It is shown in Figure 3.5, where each picture indicates the strenght distribution for each degree, with relative exponent n in the power law:



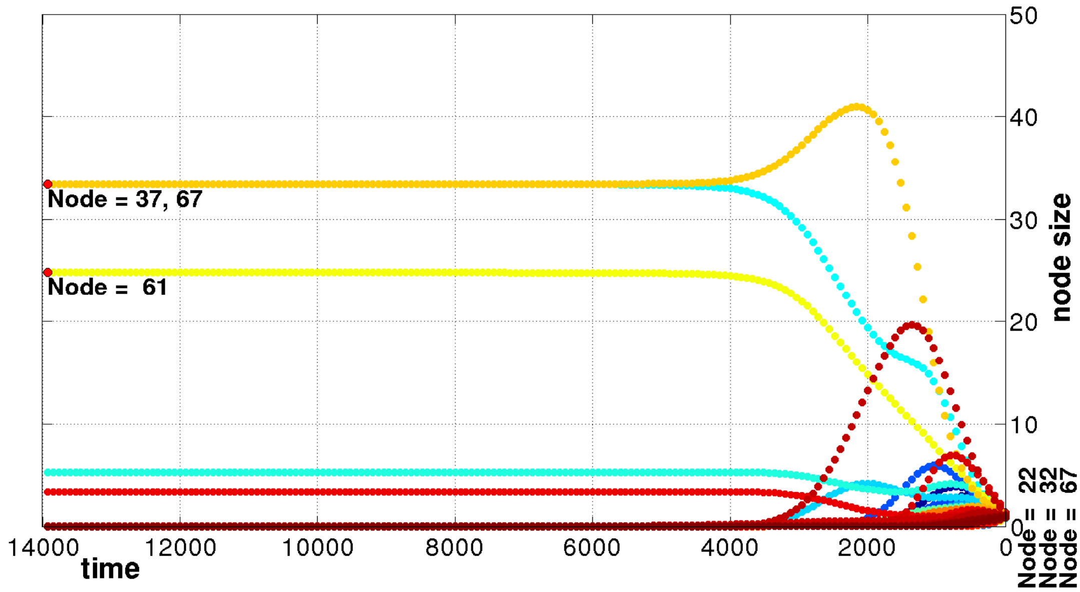
(a) IER, $\alpha = 1$

(b) IER, $\alpha = 1$

Figure 3.2: In Figure 3.2(a) we can observe the time evolution for the strenght of an ER network with initial nuclues ("initial" means at time $t = 0$) composed by 100 nodes. Each node has a different color, so it is possible to follow every behavior during the evolution.
The time range runs from $t = 0$ to $t = 14000$.
Approximately after time $t = 4000$, the system reaches a steady state and every strenght does not change anymore. At the first time step and at the last time step, highest strenght nodes are indicated, and we can notice that they are different. Figure 3.2(b) is a lateral 2D view: from this we can better understand the the time evolution of strenghts.



(a) IER, $\alpha = 1$

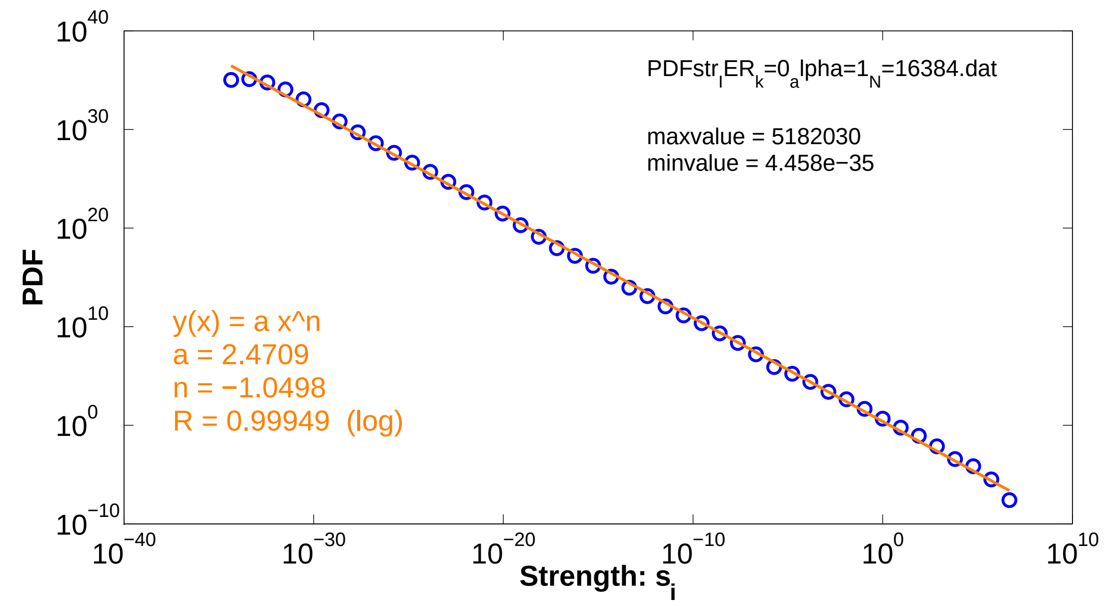


(b) IER, $\alpha = 1$

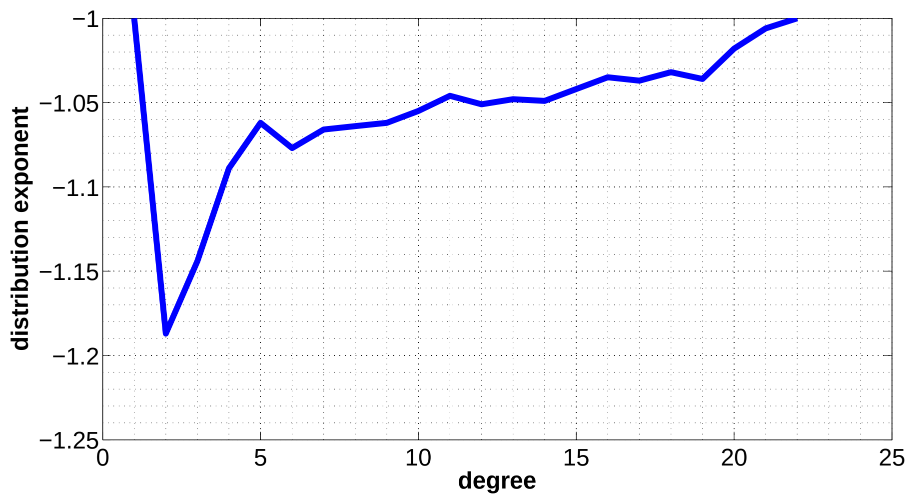
Figure 3.3: Figure 3.3(a) and Figure 3.3(a) show a similar analysis already seen in Figure 3.2, but for a different quantity of the system: instead to examine strenghts, we now show the time evolution for node sizes.
Here, nodes with highest size at $t = 0$ and at $t = 14000$ are different as well as in Figure 3.2 (except one of them).

$$y(x) = ax^n. \tag{3.1}$$

From a visual analysis we can infer different features: nodes with smallest degrees ($deg = 1, 2, 3$) and highest degrees ($deg = 21, 22$), have values of strength in a small range, respectively, between 10^{-35} and 10^{-25}, and between 10^{-25} and 10^{-15}.
Nodes with intermediate degrees (let say among $deg = 4$ and $deg = 20$) cover all the spectrum of strenghts as in Figure 3.4(a), but with an increasing value of the exponent n towards highest degrees. This behavior of exponent n is plotted in Figure 3.4(b).

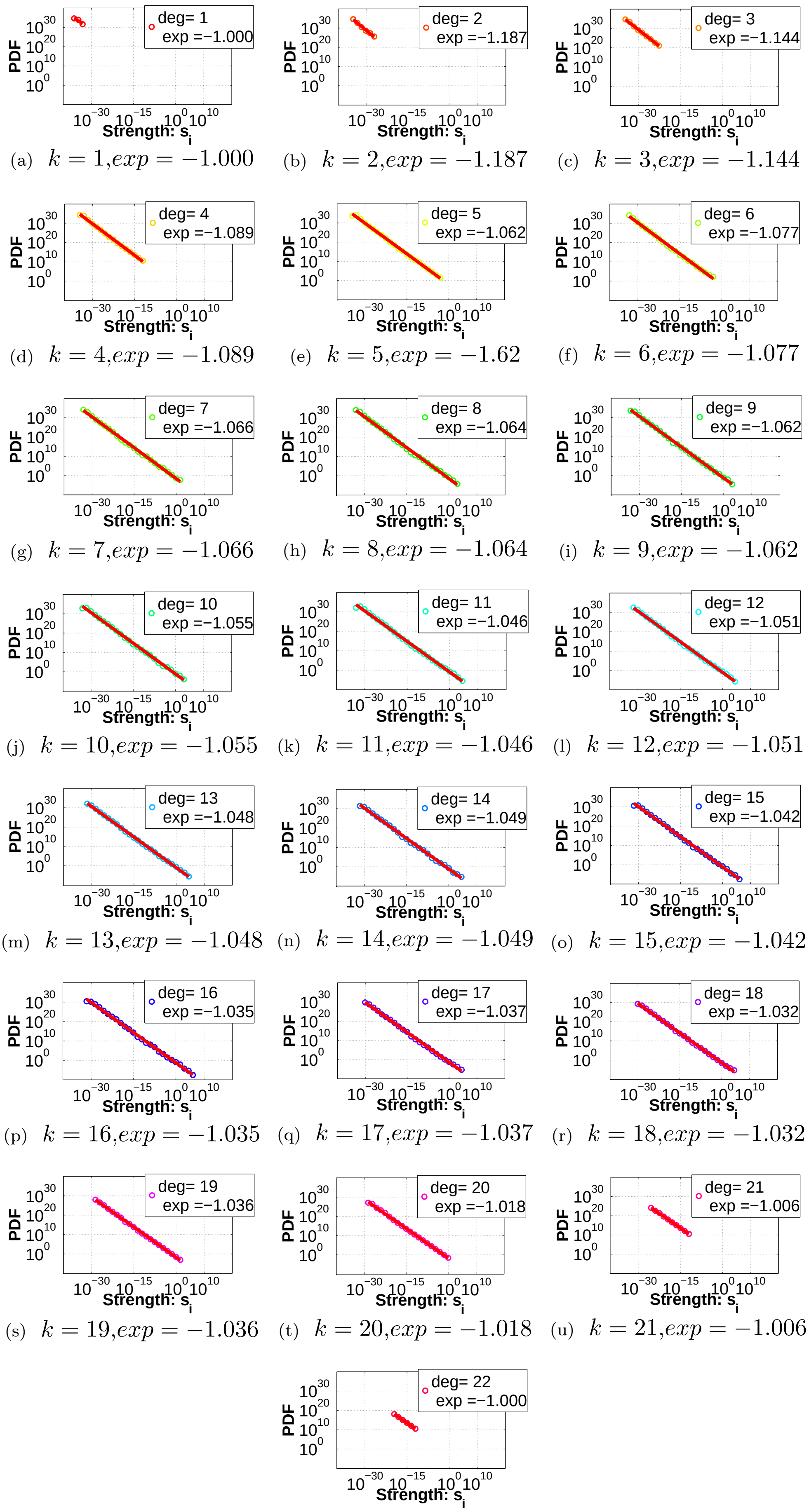


(a) IER, $\alpha = 1$



(b) IER, $\alpha = 1$

Figure 3.4: Figure 3.4(a) represents the strenght distribution for an Erdos-Renyi network with initial nucleus (IER). IER has been created with same values and parameters of the "original" Erdos-Renyi (OER): $\rightarrow$ $N = 16384$, average degree $\langle k \rangle = 10$, parameter for dissipation $\kappa = 0$, and $\alpha = 1$. Here the exponent for the power law distribution of IER is $n = -1.0498$, and for OER is $n = -1.0539$ as in Figure 2.5(c). Plot of Figure 3.4(b) indicates the "trend" of exponents n (of the power law distribution $y(x) = ax^n$) in Figure 3.5 with degrees k.



(a) $k = 1, exp = -1.000$ (b) $k = 2, exp = -1.187$ (c) $k = 3, exp = -1.144$

(d) $k = 4, exp = -1.089$ (e) $k = 5, exp = -1.62$ (f) $k = 6, exp = -1.077$

(g) $k = 7, exp = -1.066$ (h) $k = 8, exp = -1.064$ (i) $k = 9, exp = -1.062$

(j) $k = 10, exp = -1.055$ (k) $k = 11, exp = -1.046$ (l) $k = 12, exp = -1.051$

(m) $k = 13, exp = -1.048$ (n) $k = 14, exp = -1.049$ (o) $k = 15, exp = -1.042$

(p) $k = 16, exp = -1.035$ (q) $k = 17, exp = -1.037$ (r) $k = 18, exp = -1.032$

(s) $k = 19, exp = -1.036$ (t) $k = 20, exp = -1.018$ (u) $k = 21, exp = -1.006$

(v) $k = 22, exp = -1.000$

Figure 3.5: "Decomposition" of strenght distribution for each degree, for an Erdos-Renyi network with initial nucleus (IER). In every plot two quantities are indicated: degree k, and exponent n of the power law distribution $y(x) = ax^n$, which fits data.

Chapter 4

Conclusions

In this master thesis we studied the co-evolution of resources and edges, using equations 1.4 and 1.6 in our simulations, for the following kinds of netwroks: an Erdos-Renyi random network (“ER”), a 2-dimensional lattice (“REG”), a random k-Regular network (“RERA”), a fully connected network (“FULL”).

The reproduction of the Erdos-Renyi random network using the Montecarlo methods, which was in the task 1, turned out a good method, providing the same strength distribution as that one obtained by Aoki and Aoyagi in Table 1.2.
Please see the distribution of the “original” Erdos Renyi network (OER) in Figure 2.5, and the same distribution in Table 1.2 (for each value of α).

Goals of 1 and 2 have been achieved, with the computational reproduction of all kinds of networks and their degree distributions and strength distributions.
Authors of "Scale-Free Structures Emerging from Co-evolution of a Network and the Distribution of a Diffusive Resource on it" [5], showed that for the following choice of parameters, $\kappa = 0$ and $\alpha \geq 1$, the system acquires scale-free characteristics in the asymptotic state, as showed in Table 1.2. With same parameters ($\kappa = 0$ and $\alpha \geq 1$) we got power laws for the same strength distributions of Aoki and Aoyagi (ER and RERA), and for the other two structures of networks, the 2-dimensional lattice (REG) and the fully connected network (FULL). Please see OER, ORERA, OREG and OFULL in Figure 2.5, Figure 2.7, Figure 2.6, Figure 2.8.
“Exception to the rule”, is the strength distribution of OFULL for $\alpha = 1$, in Figure 2.8(c), which does not present a scale-free characteristics.

Task 3 was to study the evolution of networks applying the condition for deletion of links. As showed in Figure 2.6, Figure 2.7, Figure 2.8, the strength distributions of

LREG, LRERA, and LFULL lose the scale-free property for $\alpha = 1$ and $\alpha = 1.25$. LER maintains the scale invariance for $\alpha = 1$, but it loses it for $\alpha = 1.25$, as showed in Figure 2.5.

These new strength distributions, created with the links deletion condition, are "double power laws" for LER with $\alpha = 1.25$ (Figure 2.5(f)), and for LRERA with $\alpha = 1$ and $\alpha = 1.25$ (Figure 2.7(d) and Figure 2.7(f)).

LREG with $\alpha = 1$, has a strength distribution named "power law with an exponential cutoff"(Figure 2.6(d)).

The distribution of LREG with $\alpha = 1.25$, is not a well defined "power law with an exponential cutoff", but for some parts it can look like (Figure 2.6(f)).

Results of task 4 are contained in Chapter 3. Here, an Erdos Renyi (ER) network with an initial structure has been created, with same size and parameters of the ER network with a normal distribution for resources x_i and links weights w_{ij}. So with nodes $N = 16384$ and average degree $\langle k \rangle = 10$.

Strength distribution for the ER network with initial nucleus (a power law distribution with slope -1.05 and strength-range between 10^{-30} and 10^{5}) has very similar values of the "normal distributed" ER network.

It means that the introduction of a "nucleus" at early times (so with a different initial condition), does not affect the strength distribution of the network.

Moreover, as shown in Figure 3.5, nodes with smallest degrees ($deg = 1, 2, 3$) and highest degrees ($deg = 21, 22$), have values of strength in a small range, respectively, between 10^{-35} and 10^{-25}, and between 10^{-25} and 10^{-15}.

Nodes with intermediate degrees (among $deg = 4$ and $deg = 20$) cover all the spectrum of strenghts' values as in Figure 3.4(a).

Moreover in the "decomposition" of strenght distribution for each degree (in Chapter 3, Section 3.2), every exponent n of each power law, increases towards highest degrees, as plotted in Figure 3.4(b).

A last results of task 4, is a study of the time evolution of a small Erdos Renyi network with initial nucleus: $N = 100$ and average degree $\langle k \rangle = 10$.

Simulations up to time step $t = 14000$, reveal that the system reaches a steady state.

In the original paper, Aoki and Aoyagi studied the behavior of the co-evolving dynamics for a regular random graph showing that the system changes status all the time, up to time $t = 14000$.

Other information can be collected about the time evolution of our Erdos Renyi network with initial nucleus. Nodes with highest node size at the first time ($t = 0$), are different from nodes with highest values at the last time ($t = 14000$).

This also applies to another quantity of the system: strength of nodes.

Nodes with highest strength at the first time ($t = 0$), are different from nodes with highest values of strength at the last time ($t = 14000$).
But nodes with highest values of strength and node size at the last time ($t = 14000$), are the same.
It means that at the final time step, the same nodes simultaneously hold the highest values of strength and node size.

Appendix A

Erdos-Renyi with Montecarlo Method

In this appendix i will describe the generation of an Erdos-Renyi (ER) network, using the Metropolis *et al.* algorithm [8]. Following this procedure, the final network will have the same degree distribution of the Erdos-Renyi one, which is a Binomial distribution:

$$P(k) = \binom{N}{k} p^k (1-p)^{N-k}. \tag{A.1}$$

In this probability distribution of degrees, k represents the degree of a vertex, N the total number of nodes in the network, and p is the probability to have an edge.

In my simulation with `C++`, i applied the MC method to the Adjacency matrix A. We can provide a sort of definition of this matrix: elements A_{ij} indicate the presence, or not, of an edge between a pair of nodes i and j:

$$A_{ij} = \begin{cases} 1 & \text{if there is an edge joining i, j} \\ 0 & \text{otherwise.} \end{cases} \tag{A.2}$$

An example of a 5×5 matrix A is:

$$A = \begin{bmatrix} 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 \end{bmatrix}$$

Metropolis *et al.* algorithm is well described in a book of Raul Toral and Pere Colet, *Stochastic numerical methods: A master's course* [9], and for my simulation i focused my attention on the chapter 4, *Dynamical methods*, especially on the section 4.5, entitled *Metropolis et al. algorithm*.
I also found interesting a review paper of D.J.C. Mackay, *Introduction to Monte Carlo Methods* [10].
Metropolis sampler belongs to the category of the Markov chain Monte Carlo (MCMC) methods [10], and it samples from a target probability distribution $P(k)$, using a Markov chain. The Metropolis sampler creates a Markov chain that produces a sequence of states $k^{(t)}$ at iteration t. Samples from the chain, (after 'burnin'), reflect samples from the target distribution.
In our case we will see an application of the Metropolis sampler to a discrete distribution, where a Binomial random variable will take integer values $\widehat{\mathbf{x}} = k^{(t)}$, $k^{(t)} = 0, 1, 2, 3, ...$ with probability given in A.1.
A typical structure of the Metropolis sampler is the following:

1. Set $t = 1$
2. Generate a random initial state $k^{(t)}$
3. Repeat

    $t = t + 1$

    Generate a proposal k' from $g(k'|k^{(t-1)})$

    Calculate the acceptance probability
    $\alpha = min\big(1, \frac{P(k')}{P(k^{(t-1)})}\big)$

    Generate a random number u from the Uniform distribution $(0, 1)$

    If $u < \alpha$ accept the proposal and set $k^{(t)} = k'$, else set $k^{(t)} = k^{(t-1)}$
4. Until $t = T$ (T is the max number of iterations)

This was the general algorithm, but let me show the modified algorithm for our case:

1. Set $t = 1$
2. Generate a random initial state $k^{(t)}$
3. Repeat

    $t = t + 1$

    Pick random i and j for A_{ij}

Generate a proposal: $A_{ij} = 1 \rightarrow A_{ij} = 0$ (or $A_{ij} = 0 \rightarrow A_{ij} = 1$) corresponding to k' from $g(k'|k^{(t-1)})$

Calculate the acceptance probability
$\alpha = min\big(1, \frac{P(k')}{P(k^{(t-1)})}\big)$

Generate a random number u from the Uniform distribution $(0, 1)$

If $u < \alpha$ accept the proposal and set $A_{ij} = 0$ (or $A_{ij} = 1$) and $k^{(t)} = k'$, else set $A_{ij} = 1$ (or $A_{ij} = 0$) and $k^{(t)} = k^{(t-1)}$

4. Until $t = T$ (T is the max number of iterations)

To be more clear i would like to write explicitly the proposal:

$$g(k'|k^{(t-1)}) = \begin{cases} 1/2, & k' = k^{(t-1)} + 1 \\ & A_{ij} = 0 \rightarrow A_{ij} = 1 \\ 1/2, & k' = k^{(t-1)} - 1 \\ & A_{ij} = 1 \rightarrow A_{ij} = 0 \end{cases} \tag{A.3}$$

What does this proposal mean?
If a random number u is $u > 1/2$ then our proposal is to pass from $A_{ij} = 0$ to $A_{ij} = 1$, i.e. adding an edge between i and j, and it corresponds to make the proposal $k' = k^{(t-1)} + 1$. In the other case $u < 1/2$, i eliminate an edge passing from $A_{ij} = 1$ to $A_{ij} = 0$ and here $k' = k^{(t-1)} - 1$.
Now i will show the core of the algorithm with my code of `C++`:

```
// How to construct an ER random network with Montecarlo Method
//       P_k = coeffbin(N k) * (p)^k * (1-p)^(N-k)

// Some parameters:
double u_1;
double u_2;
int k;
int kinitial;
int k1;
int kmax = 8;
int time = 0;
int MCsteps = 190000;
double alfa;
int minunmezzo;
int maggunmezzo;

// Some used values for Monte Carlo Steps:
// MCsteps = 7500 for N = 900 (kmax=3)
// MCsteps = 18000 for N = 1600 (kmax=8)
// MCsteps = 190000 for N = 16384 (kmax=8)

```

```
// Creation of the Adjacency Matrix "A" for an ER random network:
if ( ERtype == "MC" )
        {
        totalnumberoflinks = 0;
        do {
                k = gsl_rng_uniform_int (r6,kmax);
           } while ( k < (kmax - 2) );
        kinitial = k;

        do {
                // Pick at random a pair of sites of the network:
                // i randomly select "i" and "j" of A[i][j]
                RandIndex1 = gsl_rng_uniform_int (r2,numberOfNodes);
                do {
                        RandIndex2 = gsl_rng_uniform_int (r3,numberOfNodes);
                   } while ( RandIndex1 == RandIndex2 );

                // Monte Carlo Method
                u_1 = gsl_rng_uniform (r4);
                if ((double) gsl_rng_uniform (r4) > (0.5) )
                   {
                         maggunmezzo++;
                         // Acceptance Ratio alpha
                         double num = ((double) ((numberOfNodes-k)*p) );
                         double den = ( (double) (k+1)*(1-p)  );
                         alfa =  ( ((double) num ) / ( (double) den ) );
                         u_2 = gsl_rng_uniform (r5);
                         if ((double) num > den )
                                {
                                 k = k1;
                                 A[RandIndex1][RandIndex2] = 1;
                                 A[RandIndex2][RandIndex1] = 1;
                                 totalnumberoflinks = totalnumberoflinks + 2;
                                }

                         else if ((double) gsl_rng_uniform (r5) < alfa )
                                {
                                 k = k1;
                                 A[RandIndex1][RandIndex2] = 1;
                                 A[RandIndex2][RandIndex1] = 1;
                                 totalnumberoflinks = totalnumberoflinks + 2;
                                }
                    }

                else
                   {
                         minunmezzo++;
                         // Acceptance Ratio alpha
                         double num = ((double) (k)*(1-p) );
                         double den = ( (double) (numberOfNodes-k+1)*(p)  );
                         alfa =  ( ( (double) num ) / ( (double) den ) );
                         if ((double) num > den )
                                {
                                 k = k1;
                                 A[RandIndex1][RandIndex2] = 0;
```

```
                                A[RandIndex2][RandIndex1] = 0;
                               }

                        else if ((double) gsl_rng_uniform (r5) < alfa )
                               {
                                k = k1;
                                A[RandIndex1][RandIndex2] = 0;
                                A[RandIndex2][RandIndex1] = 0;
                               }
                  }
                time = time + 1;
                } while ( time != MCsteps );
        }
```

In the last part of this section i will provide some mathematical details about the acceptance probability $\alpha = min\big(1, \frac{P(k')}{P(k^{(t-1)})}\big)$, expliciting the fraction $\frac{P(k')}{P(k^{(t-1)})}$. In the book of Raul Toral and Pere Colet [9], this ratio is called $q(k'|k^{(t-1)})$. As i wrote above in the proposal probability A.3 we have two cases and i will deal with them separately. In order to simplify the mathematical notation i remove the index $(t-1)$ in $k^{(t-1)}$ only here in these two 'derivations':

First case: $k' = k + 1$

$$
\begin{aligned}
q(k'|k) &= \frac{P(k')}{P(k)} \\
&= \frac{P(k+1)}{P(k)} \\
&= \frac{\binom{N}{k+1} p^{k+1}(1-p)^{N-k-1}}{\binom{N}{k} p^{k}(1-p)^{N-k}} \\
&= \frac{\frac{N!}{(k+1)!(N-k-1)!}}{\frac{N!}{k!(N-k)!}} \; \frac{p^{k+1}(1-p)^{N-k-1}}{p^{k}(1-p)^{N-1-k}} \\
&= \frac{k!(N-k)!}{(k+1)!(N-k-1)!}\left(\frac{p}{1-p}\right) \\
&= \left(\frac{N-k}{k+1}\right)\left(\frac{p}{1-p}\right)
\end{aligned}
\tag{A.4}
$$

Second case: $k' = k - 1$

$$
\begin{aligned}
q(k'|k) &= \frac{P(k')}{P(k)} \\
&= \frac{P(k-1)}{P(k)} \\
&= \frac{\binom{N}{k-1} p^{k-1} (1-p)^{N-k+1}}{\binom{N}{k} p^{k} (1-p)^{N-k}} \\
&= \frac{\frac{N!}{(k-1)!(N-k+1)!}}{\frac{N!}{k!(N-k)!}} \; \frac{p^{k-1}(1-p)^{N-k+1}}{p^{k}(1-p)^{N-k}} \\
&= \frac{k(N-k)!}{(N-k)!(N-k+1)} \left(\frac{1-p}{p}\right) \\
&= \left(\frac{k}{N-k+1}\right)\left(\frac{1-p}{p}\right)
\end{aligned}
\tag{A.5}
$$

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