Cosmological perturbations in Loop Quantum Cosmology: Some numerical results

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Introduction

- Quantum Field Theory in Curved Spacetime provides a good description of formation of primordial structure during inflation.

- Loop Quantum Cosmology gives a rigorous quantization of FLRW spacetimes in which the singularities are avoided. Besides, its effective dynamics favours inflation.

The hybrid approach combines Fock and loop representations to quantize inhomogeneous models. [Martín-Benito, Garay & Mena Marugán]

We have quantized a (scalarly) perturbed FLRW model with

- a (massive) scalar field as matter content,
- compact spatial sections.

(Classical) Cosmological Perturbations

3+1 decomposition

Let $(\mathcal{M}, g)$ be a globally hyperbolic spacetime.

Let t be a global time function.

t foliates the spacetime in spacelike hypersurfaces Σ_t.

We define

$h_{ab} \rightsquigarrow$ 3-metric induced on Σ_t by $g_{\alpha\beta}$

$N^a \rightsquigarrow$ shift vector

$N \rightsquigarrow$ lapse function

$(a, b, \ldots = 1, 2, 3 \rightsquigarrow$ spatial indices$)$

The line element can be written as

$$ds^2 = -(N^2 - N_a N^a)dt^2 + 2N_a dt dx^a + h_{ab} dx^a dx^b.$$

Parametrization

FLRW (with a scalar field) + inhomogeneities

$$\Phi(t,x) = \frac{1}{l_0^{3/2}\sigma}\left[\varphi(t) + \delta\varphi(t,x)\right],$$
$$h_{ab}(t,x) = \sigma^2 e^{2\alpha(t)}\left[{}^0h_{ab}(x) + \epsilon_{ab}(t,x)\right],$$
$$N(t,x) = \sigma\left[N_0(t) + \delta N_0(t,x)\right],$$
$$N_a(t,x) = \delta N_a(t,x).$$

[Halliwell & Hawking 85]

$$\sigma^2 = \frac{4\pi G}{3l_0^2}$$
$$l_0^3 = \int_\Sigma d^3x\sqrt{{}^0h}$$

${}^0h_{ab} \rightsquigarrow$ fiducial metric on Σ.

The inhomogeneities can be Fourier expanded, e.g.

$$\delta\varphi(t,x) = \sum_n f_n(t)\tilde{Q}^n(x)$$

$\tilde{Q}_n$ are real eigenfunctions of the Laplace-Beltrami operator,

$$^0\Delta\tilde{Q}_n = -\omega_n^2\tilde{Q}_n$$

Constraints

We truncate the action at quadratic order in the coefficients of the inhomogeneities expansions.

After a Legendre transform, we obtain a Hamiltonian which is a linear combination of constraints:

- The corrected Hamiltonian constraint $C_0 + \sum C^n_{|2}$ (which appears with the homogeneous lapse).
- Linear constraints $C^n_{|1}$ and $C^n_{\_1}$ (with the perturbations of the lapse and the shift, resp.).

We fix the linear constraints classically.

Gauge fixing

Consider e.g. the longitudinal gauge, in which

$$h_{ab} \propto {}^{0}h_{ab} \text{ and } N_a = 0.$$

After the reduction, one is left with the homogeneous α and φ, the field-like $\delta\varphi$, and their former momenta, which do no longer have canonical (Dirac) brackets.

Nonetheless, we can find a new set of canonical variables.

We take advantage of this change to introduce a field description adapted to the quantization.

Reparametrization of the system

We scale the field and change its momentum in the following way:

$$\bar{f}_n = e^{\alpha} f_n, \quad \bar{\pi}_{\bar{f}_n} = e^{-\alpha}(1 + F_n)\pi_{f_n} + G_n f_n$$

$O(\omega_n^{-2})$ background functions

while the homogeneous variables

get 2nd-order corrections ($\rightsquigarrow$ backreaction).

In these variables, $C^n_{|2}$ adopts a Klein-Gordon-like form with background-dependent mass and $O(\omega_n^{-2})$ corrections.

Dynamical equations:

$$\ddot{\bar{f}}_n + r_n \dot{\bar{f}}_n + (\omega_n^2 + s + s_n)\bar{f}_n = 0$$
$$\dot{\bar{f}}_n = (1 + p_n)\pi_{\bar{f}_n} + q_n \bar{f}_n$$
$$p_n, q_n, r_n, s_n = O(\omega_n^{-2})$$

Choice of the field description

Classically, the parametrization of the inhomogeneities is irrelevant.
But different parametrizations lead to inequivalent quantum theories.

In a classical background, the scaling and the momentum redefinition are necessary if we require

- a vacuum invariant under the spatial isometries and
- unitarily implementable field dynamics.

Moreover, these criteria select

a class of unitarily equivalent Fock representations for the field.

[Cortez, Mena Marugán, Olmedo & Velhinho]

A representative of the class of preferred Fock representations can be constructed from the annihilation-like variables

$$a_{\bar{f}_n} = \frac{1}{\sqrt{2\omega_n}}(\omega_n \bar{f}_n + i\pi_{\bar{f}_n}).$$

Quantization

Hybrid Quantization

We adopt

- a polymer representation of the homogeneous gravitational d.o.f.,
  - a Schrödinger representation for the homogeneous field,
- a standard Fock quantization for its field-like perturbation.

In this approximation,

only the background incorporates the effects of quantum geometry

but an infinite number of d.o.f. can be treated.

The kinematical Hilbert space of the theory

is constructed as the product

$$\mathcal{H}^{\text{tot}}_{\text{kin}} = \mathcal{H}^{\text{LQC}}_{\text{kin}} \otimes \mathcal{H}^{\varphi}_{\text{kin}} \otimes \mathcal{F}.$$

$\mathcal{H}^{\text{LQC}}_{\text{kin}}$: $L^2(\mathbb{R}_{\text{B}}, d\mu_{\text{B}})$

$\mathcal{H}^{\varphi}_{\text{kin}}$: $L^2(\mathbb{R}, d\varphi)$

$\mathcal{F}$: Fock space

Homogeneous sector: Ashtekar variables

Flat case: $\Sigma = T^3$

In a homogeneous and isotropic universe,

the Ashtekar-Barbero connection and the densitized triad

can be parametrized by two variables, c and p, satisfying

$$\{c, p\} = \frac{8\pi G\gamma}{3}, \quad |p| = l_0^2\sigma^2 e^{2\bar{\alpha}}, \quad pc = -\gamma l_0^3 \sigma^2 \bar{\pi}_{\bar{\alpha}}.$$

In terms of these variables,

the classical Hamiltonian constraint of the homogeneous system is

$$C_0 = \frac{1}{|p|^{3/2}} \left(-\frac{6}{\gamma^2} c^2 p^2 + 8\pi G(\pi_\phi^2 + m^2 |p|^3 \phi^2) \right),$$

Holonomy-flux algebra

However, the fundamental variables for quantization are not the connection and the triad, but

- Holonomies of the connection along straight edges of length $l_0\bar{\mu}(p)$, parametrized by the functions $N_{\bar{\mu}} = e^{i\bar{\mu}c/2}$.
  The improved dynamics scheme has been adopted: $l_0\bar{\mu} = l_0\sqrt{\Delta/p}$, where Δ is an imput from Loop Quantum Gravity: the minimum non-zero eigenvalue of the area operator.
- Fluxes of the densitized triad (proportional to p).

Fundamental algebra: $\{N_{\bar{\mu}}, p\} = \frac{4\pi i G\gamma\bar{\mu}}{3} N_{\bar{\mu}}.$

Representation

Mimicking the representation employed in LQG, the holonomy-flux algebra is represented in $\mathcal{H}_{\text{kin}}^{\text{LQC}} = L^2(\mathbb{R}_{\text{Bohr}}, d\mu_{\text{Bohr}})$.

The momentum representation is more frequently employed:

- Orthonormal basis: $\{|v\rangle \mid v \in \mathbb{R}\}$, $\langle v|v'\rangle = \delta_{vv'}$.
- Fundamental operators: $\hat{N}_{\bar{\mu}}|v\rangle = |v+1\rangle, \quad \hat{p}|v\rangle = p(v)|v\rangle$.

As this representation is not continuous, there is no operator for c.

The Hamiltonian constraint must be regularized.

This is done by following the programme of Loop Quantum Gravity

Regularization

The term cp can be expressed in terms of a holonomy around a closed squared loop in the limit of vanishing area of the loop.

Instead of a vanishing area, we take a loop with the minimum one, Δ.

Thus, we obtain $(cp)^2 \to \hat{\Omega}_0^2$, where

$$\hat{\Omega}_0 = \frac{|\hat{p}|^{3/4}}{4i\sqrt{\Delta}}\left[\widehat{\mathrm{sgn}(p)}\left(\hat{N}_{2\bar{\mu}} - \hat{N}_{-2\bar{\mu}}\right) + \left(\hat{N}_{2\bar{\mu}} - \hat{N}_{-2\bar{\mu}}\right)\widehat{\mathrm{sgn}(p)}\right]|\hat{p}|^{3/4}.$$

In addition, inverse powers of p are regularized expressing them in terms of Poisson brackets of the fundamental operators. Then, the brackets are promoted to commutators. The result is

$$\widehat{\left[\frac{1}{|p|^{1/2}}\right]} = \frac{3}{4\pi\gamma G\hbar\sqrt{\Delta}}\widehat{\mathrm{sgn}(p)}\sqrt{|\hat{p}|}\left(\hat{N}_{-\bar{\mu}}\sqrt{|\hat{p}|}\hat{N}_{\bar{\mu}} - \hat{N}_{\bar{\mu}}\sqrt{|\hat{p}|}\hat{N}_{-\bar{\mu}}\right).$$

Second-order constraint

The 2nd-order Hamiltonian has the structure

$$C^n_{|2} \propto \frac{1}{2} e^{-\alpha} \left(E^n_{\pi\pi} \bar{\pi}^2_{\bar{f}_n} + 2 E^n_{f\pi} f_n \bar{\pi}_{\bar{f}_n} + E^n_{ff} \bar{f}^2_n \right),$$

where the E-coefficients are functions of the homogeneous variables.

The prescription we follow to quantize it is:

- Normal ordering for annihilation and creation operators.
- Symmetrizations: $\phi\pi_\phi \rightsquigarrow \frac{1}{2}(\hat{\phi}\hat{\pi}_\phi + \hat{\pi}_\phi\hat{\phi})$, $AV^k \rightsquigarrow \hat{V}^{k/2}\hat{A}\hat{V}^{k/2}$
  - $(cp)^{2k} \rightsquigarrow \hat{\Omega}_0^{2k}$
  - $(cp)^{2k+1} \rightsquigarrow |\hat{\Omega}_0|^k \hat{\Lambda}_0 |\hat{\Omega}_0|^k$

$$\hat{\Lambda}_0 = \frac{1}{8i\sqrt{\Delta}} \hat{V}^{1/2} \left[\widehat{\mathrm{sgn}(v)} \left(\hat{N}_{4\bar{\mu}} - \hat{N}_{-4\bar{\mu}} \right) + \left(\hat{N}_{4\bar{\mu}} - \hat{N}_{-4\bar{\mu}} \right) \widehat{\mathrm{sgn}(v)} \right] \hat{V}^{1/2}.$$

In this way, the superselection sectors are preserved.

Effective Dynamics

Derivation of the effective dynamics

Now we have a quantum model, but it is very intrinate

As a first approach we studied its effective dynamics

in the massless case.

In simple models, the peaks of certain semiclassical states

follow simple trajectories

which obey the effective constraint obtained by

$$\hat{p} \to p$$
$$\hat{N}_{\bar{\mu}} \to N_{\bar{\mu}}$$

There are two types of corrections:

- Regularization of $\widehat{|p|^{-1/2}} \to$ inverse-triad corrections
- Regularization of $cp \to$ holonomy corrections

This algorithm has proven useful in more involved systems

(of course, one should check its validity!)

Implementation

We introduce holonomy corrections in our model by making

$$(pc)^2 \to \Omega^2 = \left(p \frac{\sin(\bar{\mu} c)}{\bar{\mu}} \right)^2$$

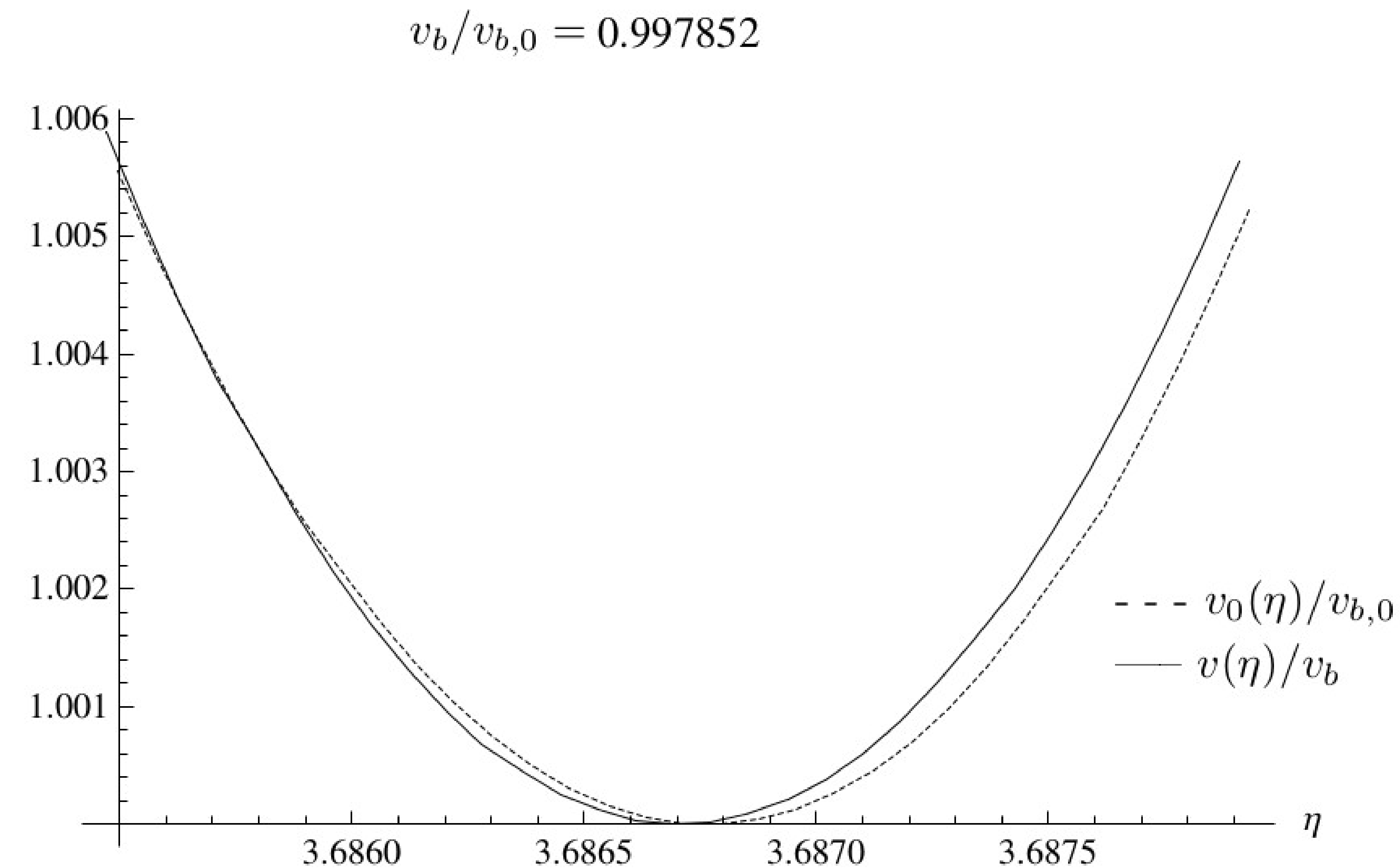
$$pc \to \Lambda = p \frac{\sin(2\bar{\mu} c)}{2\bar{\mu}}$$

This implementation of the effective dynamics, with two steps, has qualitative as well as quantitative consequences, e.g. the bounce in the volume does not coincide with the max. density.

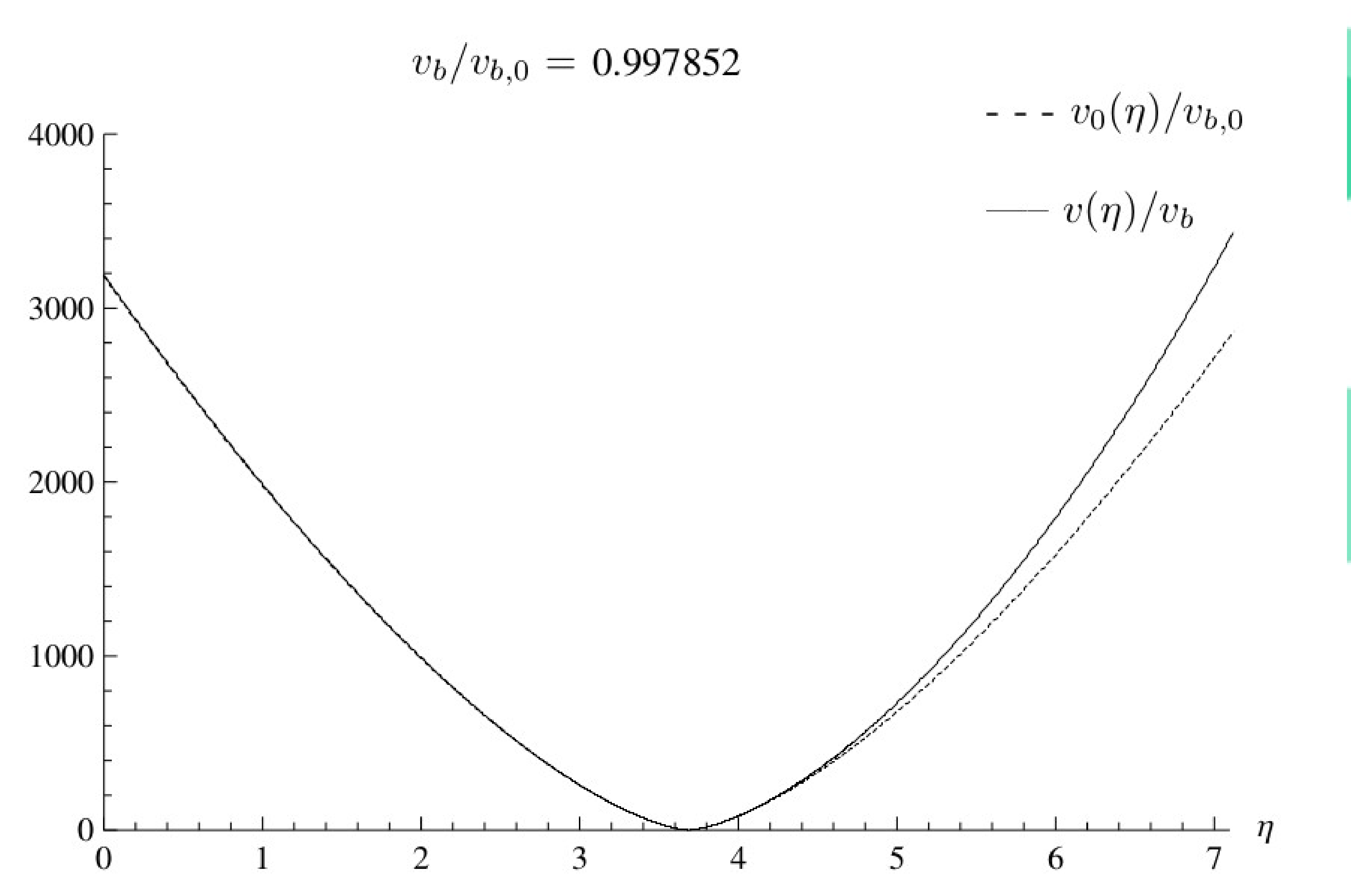
Nonetheless, we would expect other effects of the backreaction to appear in other prescriptions as well.

E.g., the energy transfer between background and perturbations.

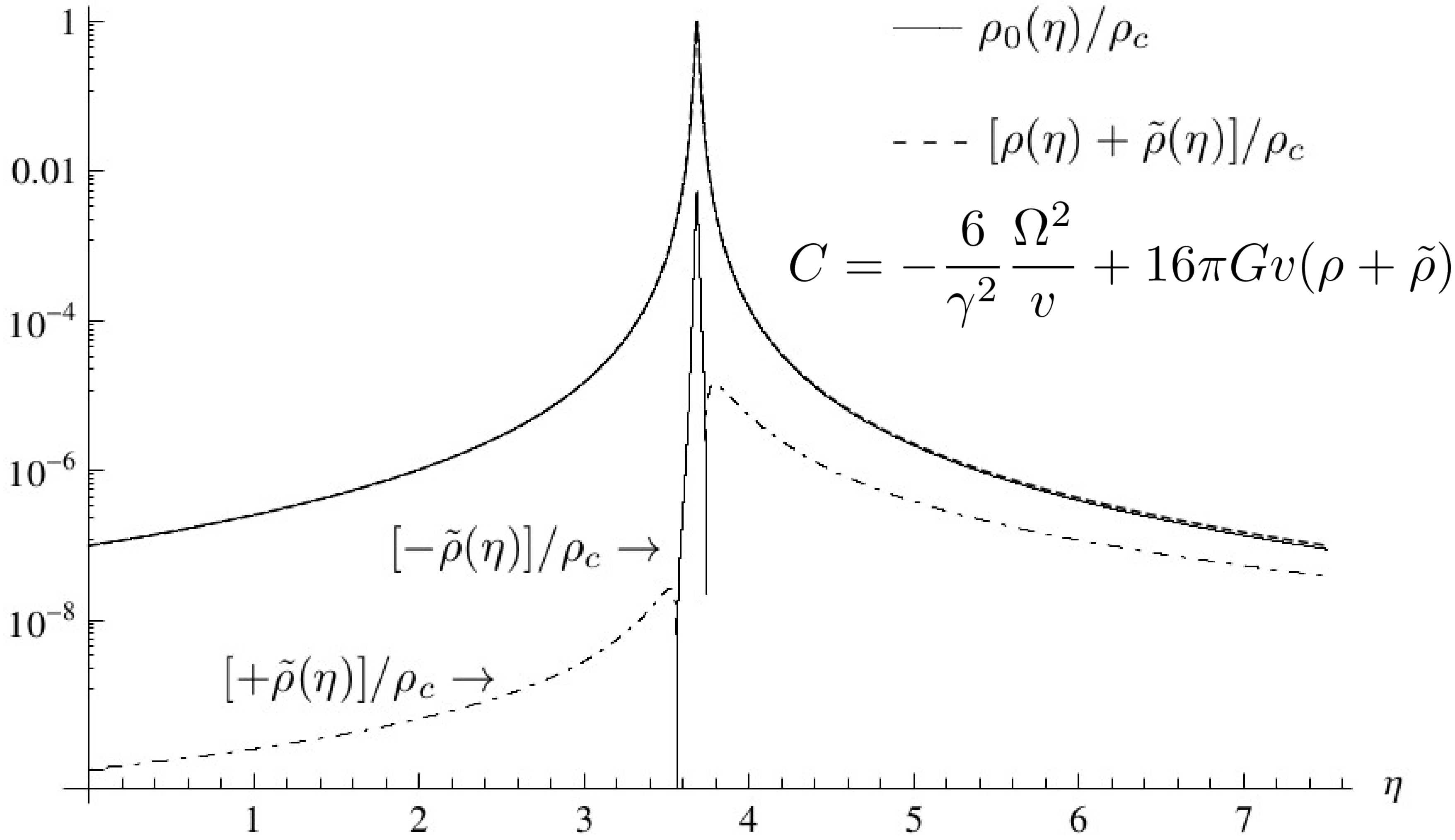
Effects of the backreaction (I)



Effects of the backreaction (II)



Effects of the backreaction (III)



Perturbation amplification (I)

We studied the evolution of the system

setting initial conditions well before the bounce (at η_0)

and evolving them until long after the bounce (at η_{f}).

Initial conditions for the inhomogeneities:

- Gaussian distribution for the amplitude
- Homogeneous distribution for the phase α

(idea: mimicking a vacuum state)

Statistically, the perturbations are amplified through the bounce.

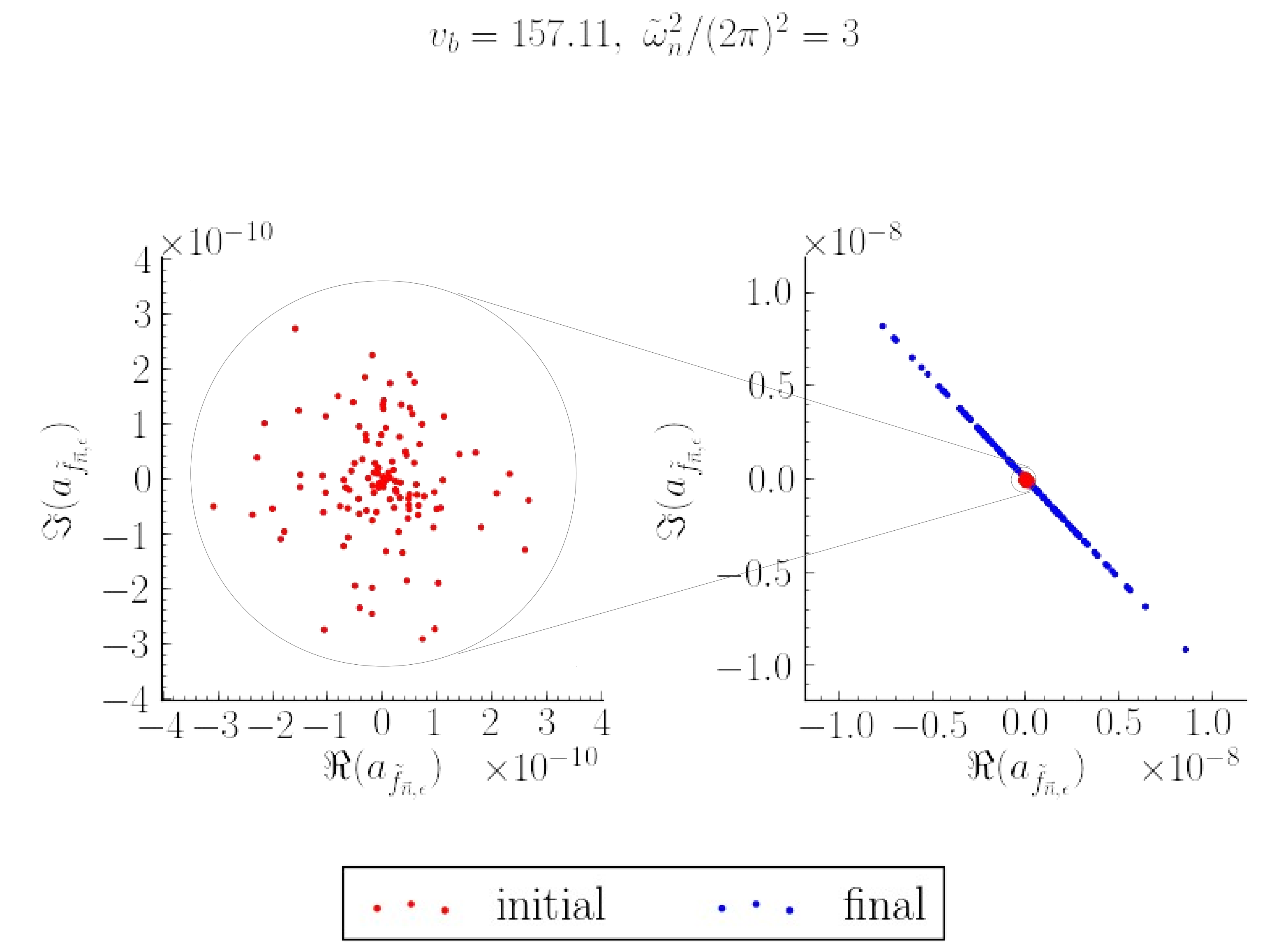
The average amplification is modulated by the frequency.

Besides, there is an effect of alignment of the phases.

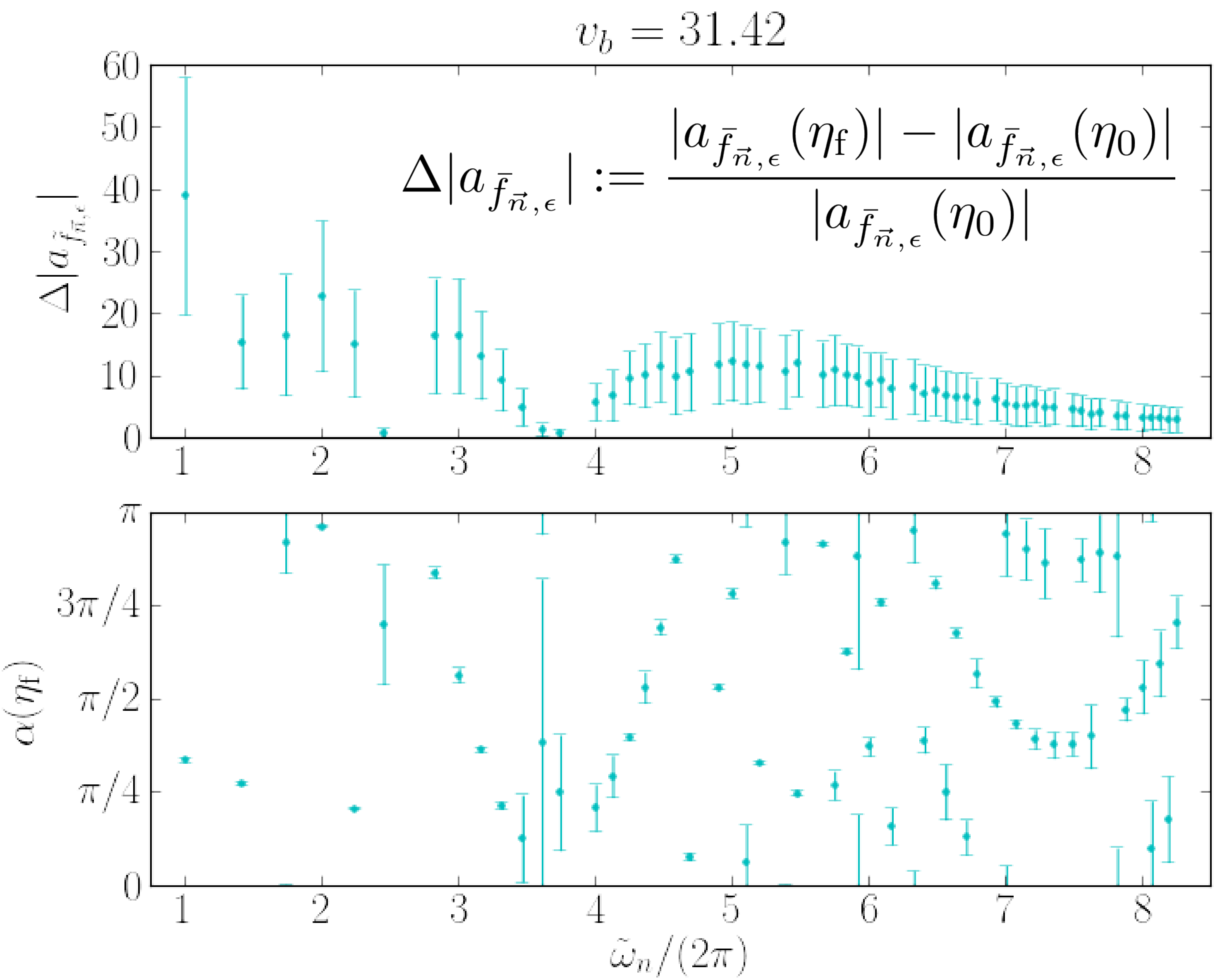
The following figures were obtained neglecting the backreaction

and choosing $\eta_{\mathrm{f}} - \eta_{\mathrm{bounce}} = \eta_{\mathrm{bounce}} - \eta_0$

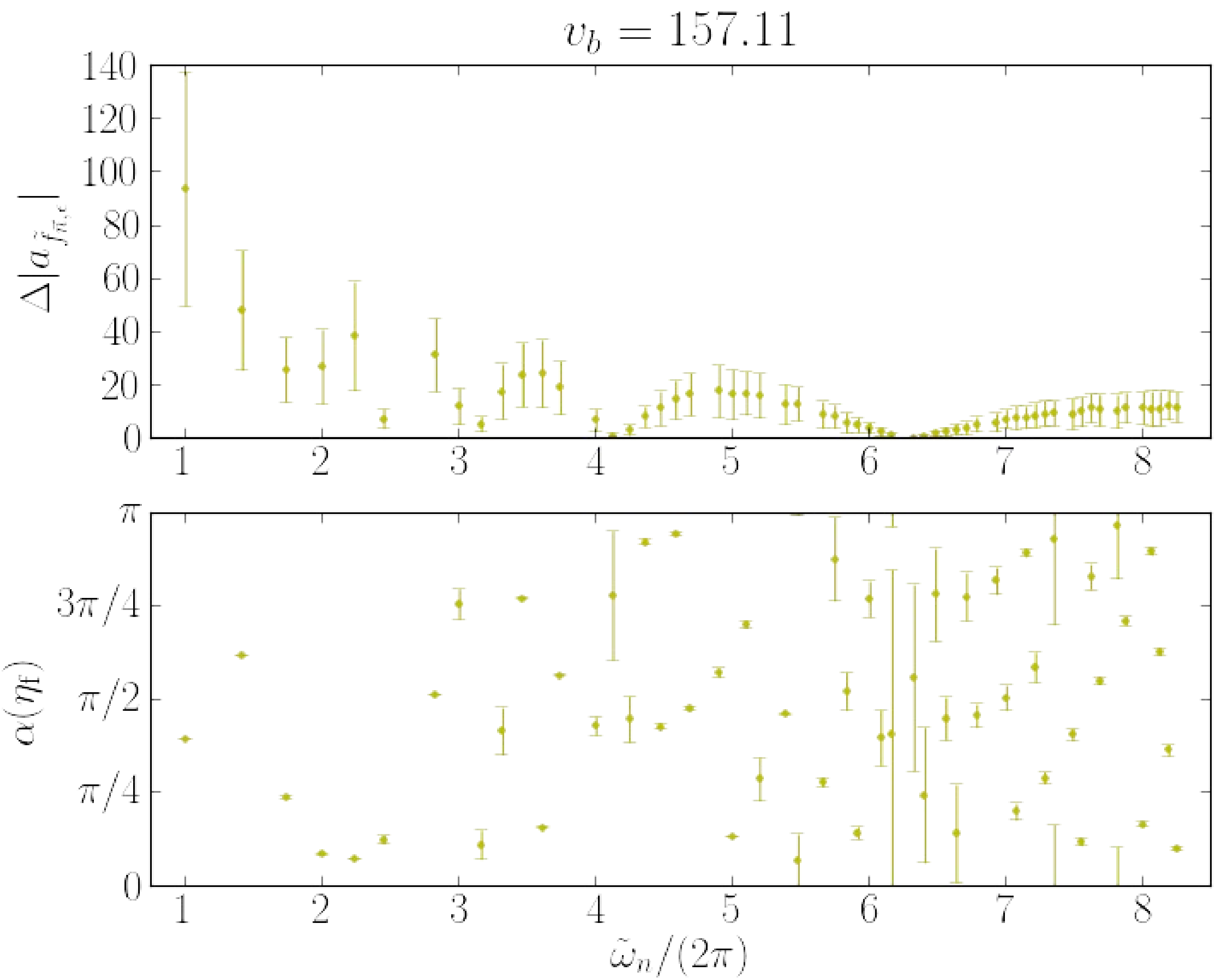
Perturbation amplification (II)



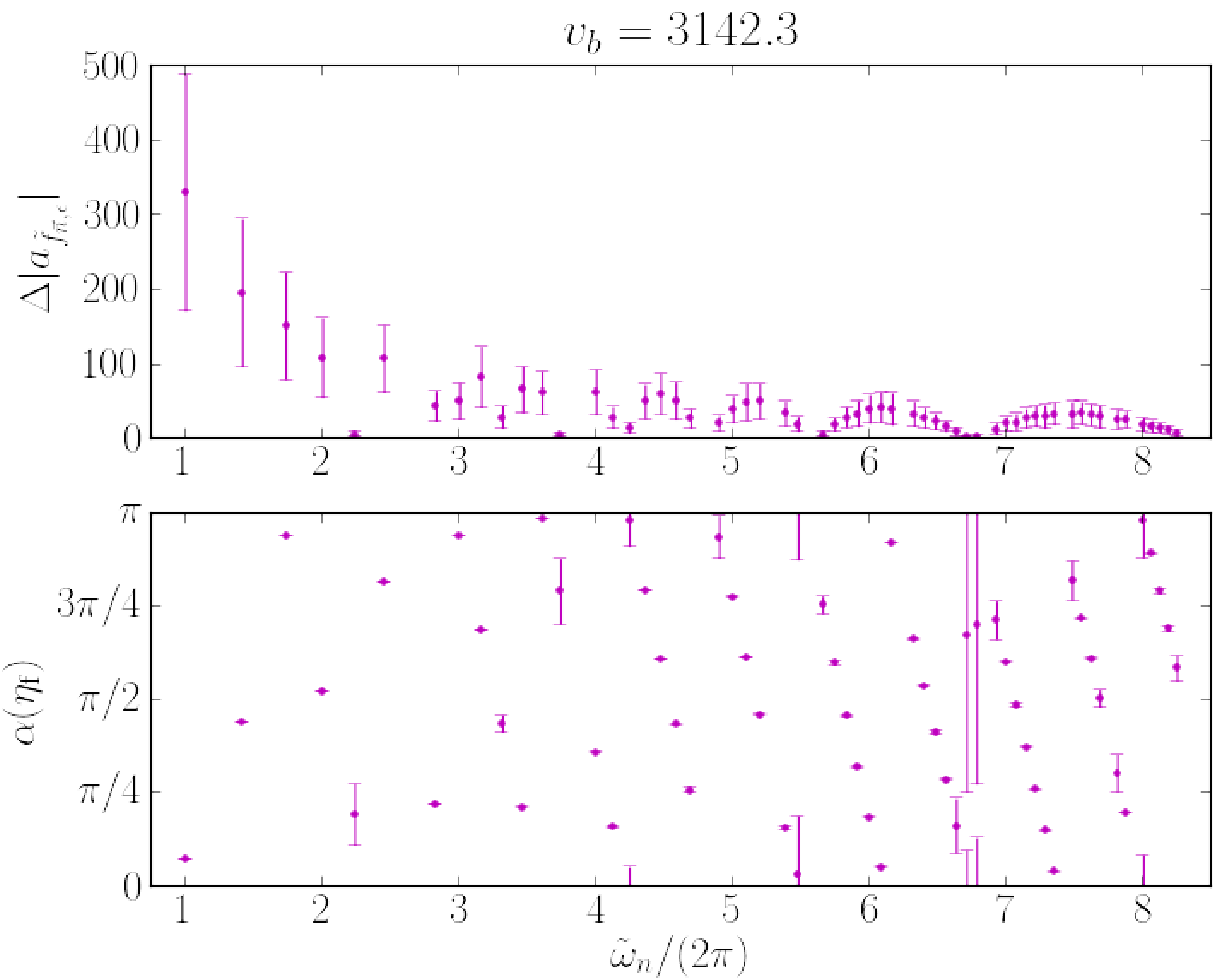
Modulation of the amplification (I)



Modulation of the amplification (II)



Modulation of the amplification (III)



Conclusions

Conclusions

We have studied the plausible effective dynamics
of the hybrid quantization of the perturbed FLRW model.

Results:

- The perturbations are boosted in the bounce.
  The average amplification oscillates with the frequency.
  The ultraviolet modes are not amplified significantly.
- There is a parallel effect of alignment of the phases.
- There is an energy transfer between the background
  and the perturbations.

Thank you!