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A constraint algorithm for singular Lagrangians subjected to nonholonomic constraints

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We construct a constraint algorithm for singular Lagrangian systems subjected to nonholonomic constraints which generalizes that of Dirac for constrained Hamiltonian systems. © 1997 American Institute of Physics. [S0022-2488(97)00806-2]

I. INTRODUCTION

The natural geometrical setting for mechanics are tangent and cotangent bundles.¹ In fact, a Lagrangian function $L=L(q^A, \dot{q}^A)$ is a function defined on the tangent bundle TQ of the configuration manifold Q . TQ is the space of velocities, and it is connected with the phase space of generalized momenta T^*Q via the Legendre transformation. The geometries of TQ and T^*Q permit us to derive in an intrinsic way the motion equations (Euler–Lagrange and Hamilton equations). The procedure works well if L is regular, which is the case for natural Lagrangians, say L is of the form $L=T-V$, where T is a kinetic energy derived from some Riemannian metric on Q and V is a potential energy. However, if L is singular, the motion equations have no solutions in general. Motivated by the problem of quantization of singular systems, Dirac² developed a constraint algorithm giving a final constraint submanifold where a solution (up to some gauge ambiguity) exists. The so-called Dirac–Bergmann algorithm was later globalized by Gotay and Nester.^{3,4}

On the other hand, we have recently developed a geometrical setting for nonholonomic Lagrangian systems.^{5–10} In some cases, the system does not admit solutions on the given constraint submanifold. Thus, we have developed a constraint algorithm which is very similar to that of Dirac.

The purpose of this paper is to construct a constraint algorithm for singular Lagrangian systems subjected to nonholonomic constraints, in such a way that both algorithms are combined.

II. SINGULAR LAGRANGIANS SUBJECTED TO NONHOLONOMIC CONSTRAINTS

Let $L:TQ\rightarrow\mathbb{R}$ be a Lagrangian function defined on the space of velocities TQ of an n -dimensional configuration manifold Q . We denote by $\tau_Q:TQ\rightarrow Q$ the canonical projection. The fibred coordinates in TQ are denoted by (q^A, v^A) , $1\leq A\leq n$. L is said to be regular if the Hessian matrix

$$\left(\frac{\partial^2 L}{\partial v^A \partial v^B} \right)$$

is regular. Otherwise, it is called singular or degenerate. The energy E_L is defined to be $E_L = CL - L$, where $C = v^A (\partial/\partial v^A)$ is the Liouville vector field on TQ . We denote by α_L the

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Poincaré–Cartan one-form defined by $\alpha_L = J^*(dL)$ and by $\omega_L = -d\alpha_L$ the Poincaré–Cartan two-form, where $J = dq^A \otimes \partial/\partial v^A$ is the canonical almost tangent structure on TQ . (For a definition of C and J we refer to Ref. 1.) As we know, L is regular if and only if ω_L is symplectic. In this case, the motion equation

$$i_X \omega_L = dE_L \quad (1)$$

has a unique solution ξ_L on TQ . ξ_L is a SODE (second-order differential equation) which is called the Euler–Lagrange vector field. The SODE character is geometrically expressed by the algebraic condition $J\xi_L = C$. The solutions of ξ_L (that is, the projections of its integral curves) are just the solutions of the Euler–Lagrange equations.

If L is singular, ω_L is presymplectic and Eq. (1) has in general no solution on TQ . However, we can develop the Gotay and Nester algorithm³ (a geometrization of the Dirac–Bergmann algorithm²) and obtain a sequence of submanifolds

$$\cdots \rightarrow P_k \rightarrow \cdots \rightarrow P_2 \rightarrow P_1 = TQ.$$

If the algorithm stabilizes at some integer k , say $P_k = P_{k+1} = P_f$, we obtain a solution X on the final constraint submanifold P_f , that is, there exists $X \in \mathfrak{X}(P_f)$ such that

$$(i_X \omega_L = dE_L)|_{P_f}.$$

Now, we suppose that L is subjected to a system of m nonholonomic constraints $\{\phi_i; 1 \leq i \leq m\}$ (with $m < n$), which are affine in the velocities; that is, $\phi_i: TQ \rightarrow \mathbb{R}$ is a function which can be locally expressed as follows:

$$\phi_i = (\mu_i)_A(q)v^A + h_i(q), \quad (2)$$

where $(\mu_i)_A$ and h_i are functions on Q . Let us recall that an affine function $\phi = \mu_A(q)v^A + h$ on TQ may be globally defined by $\phi = \hat{\mu} + h^V$, where $\mu = \mu_A(q)dq^A$ is a one-form on Q and $h^V = h \circ \tau_Q$. Here $\hat{\mu}: TQ \rightarrow \mathbb{R}$ denotes the function defined by $\hat{\mu}(X_q) = \langle \mu(q), X_q \rangle$, $\forall X_q \in T_q Q$. Thus, there are m one-forms $\{\mu_i\}$ and m functions $\{h_i\}$ defined on Q such that

$$\phi_i = \hat{\mu}_i + h_i^V, \quad 1 \leq i \leq m,$$

with $\mu_i = (\mu_i)_A dq^A$.

The dynamics of the constrained system is determined by a vector field which is tangent to the submanifold defined by the vanishing of the constraints. In geometrical terms, we have to modify the motion equation (1) and obtain the following system of equations:

$$i_X \omega_L = dE_L + \lambda^i \mu_i^V, \quad d\phi_i(X) = 0, \quad (3)$$

where $\mu_i^V = \tau_Q^* \mu_i$. The functions λ^i are Lagrange multipliers.

The case when L is regular has been studied in many recent papers (see Refs. 5, 9, and 11–23, and the references therein.) Even in that case, Eqs. (3) have no solutions in general (see Ref. 5). The purpose of this paper is to develop an algorithm which generalizes that by Gotay and Nester.^{3,4}

We put $\bar{P}_1 = P_1 = TQ$ and define the subset $\bar{P}_2$ of P_1 as follows:

$$\bar{P}_2 = \{x \in TQ / \langle dE_L + \lambda^i \mu_i^V, \ker \omega_L \rangle(x) = 0, \phi_i(x) = 0, \text{ for some } \lambda^i \in \mathbb{R}\}.$$

This means the following. If $\dim \ker \omega_L = r$, we take a local basis $\langle W_1, W_2, \dots, W_r \rangle$ of $\ker \omega_L$ defined on a neighborhood U of a point $x \in TQ$. Hence, the condition

$$\langle dE_L + \lambda^i \mu_i^V, \ker \omega_L \rangle(x) = 0$$

is equivalent to

$$W_a(E_L) + \lambda^i \mu_i^V(W_a) = 0, \quad 1 \leq i \leq m, \quad 1 \leq a \leq r. \quad (4)$$

If the rank of the matrix $C_1 = (\mu_i^V(W_a))$ is constant and

$$\text{rank}(\mu_i^V(W_a)) = \text{rank}(\mu_i^V(W_a), W_a(E_L)),$$

then Eqs. (4) have at least a solution for some values of the Lagrange multipliers λ_i . It is possible that (4) gives directly an inconsistency (type 0 = 1), and then we say that Eqs. (4) are inconsistent. In order to avoid this problem, we impose, as in the Dirac–Bergmann–Gotay–Nester algorithm,^{2,3} the condition that these equations do not involve an inconsistency. If the rank of the matrix C_1 is equal to r_1 with $r_1 \leq \min(r, m)$, the number of Lagrange multipliers λ^i determined by Eqs. (4) is $m - r_1$. It should be noticed that the determinants of the submatrices obtained by enlarging a regular submatrix of C_1 of order r_1 yield the new constraints which define the submanifold $\bar{P}_2$.

If Eqs. (4) do not involve an inconsistency, there exists at least a vector field X along $\bar{P}_2$ which verifies (4), but in general it is not tangent to $\bar{P}_2$. Therefore, we take the collection $\bar{P}_3$ of the points in $\bar{P}_2$ where there exists a solution which is tangent to $\bar{P}_2$. To do this, for every $x \in \bar{P}_2$, we define

$$T_x \bar{P}_2^\perp = \{v \in T_x(TQ) / \omega_L(x)(u, v) = 0, \forall u \in T_x \bar{P}_2\}.$$

Notice that $\ker \omega_L(x) \subset T_x \bar{P}_2^\perp$, for all $x \in \bar{P}_2$. Now, the obstruction to find solutions which are tangent to $\bar{P}_2$ is just $T\bar{P}_2^\perp$. Thus, we define

$$\bar{P}_3 = \{x \in \bar{P}_2 / \langle dE_L + \lambda^i \mu_i^V, T\bar{P}_2^\perp \rangle(x) = 0, \text{ for some } \lambda^i \in \mathbb{R}\}.$$

Again, if we choose a local basis for $T\bar{P}_2^\perp$ (for instance, enlarging the above basis $\{W_1, \dots, W_r\}$ for $\ker \omega_L$ to $\{W_1, \dots, W_r, V_1, \dots, V_s\}$), we obtain a system of equations

$$W_a(E_L) + \lambda^i \mu_i^V(W_a) = 0, \quad V_{\bar{a}}(E_L) + \lambda^i \mu_i^V(V_{\bar{a}}) = 0,$$

$1 \leq a \leq r$, $1 \leq \bar{a} \leq s$, and we can get the local equations defining $\bar{P}_3$.

There exists a solution X along $\bar{P}_3$ which is tangent to $\bar{P}_2$. However, X does not need to be tangent to $\bar{P}_3$. We proceed as above and obtain a sequence of constraint submanifolds

$$\dots \rightarrow \bar{P}_k \rightarrow \dots \rightarrow \bar{P}_3 \rightarrow \bar{P}_2 \rightarrow P_1 = TQ,$$

where, for all $k > 2$, we have

$$\bar{P}_{k+1} = \{x \in \bar{P}_k / \langle dE_L + \lambda^i \mu_i^V, T\bar{P}_k^\perp \rangle(x) = 0, \text{ for some } \lambda^i \in \mathbb{R}\},$$

with

$$T_x \bar{P}_k^\perp = \{v \in T_x TQ / \omega(x)(u, v) = 0, \forall u \in T_x \bar{P}_k\}.$$

If this algorithm stabilizes, that is, there exists an integer k such that $\bar{P}_k = \bar{P}_{k+1}$, and $\dim \bar{P}_k > 0$, we obtain a final constraint submanifold $\bar{P}_k$ where a completely consistent solution of the dynamics exists. We denote this submanifold by $\bar{P}_f$ and it will be called the final constraint

submanifold. Then there exists a vector field ξ on $\bar{P}_f$ which is a solution of Eq. (3). (Of course, $\xi + Z$, $Z \in \ker \omega_L \cap T\bar{P}_f$ is a solution, too.) In this process, some Lagrange multipliers may remain undetermined and the others ones will rest completely fixed on $\bar{P}_f$.

Example: Let $L: T\mathbb{R}^2 \rightarrow \mathbb{R}$ be given by

$$L(q^1, q^2, v^1, v^2) = \frac{1}{2}(v^1)^2 + \frac{1}{2}(q^1)^2 q^2.$$

We have

$$E_L = \frac{1}{2}(v^1)^2 - \frac{1}{2}(q^1)^2 q^2, \quad \omega_L = dq^1 \wedge dv^1,$$

from which we deduce that

$$\ker \omega_L = \left\langle \frac{\partial}{\partial q^2}, \frac{\partial}{\partial v^2} \right\rangle.$$

If we study the dynamics for the free problem, we obtain the following constraints submanifolds:

$$P_2 = \{(q^1, q^2, v^1, v^2) \in \mathbb{R}^4 / q^1 = 0\} \quad \text{and} \quad P_3 = \{(q^1, q^2, v^1, v^2) \in \mathbb{R}^4 / q^1 = 0, v^1 = 0\}.$$

In that case, the dynamics is fully undetermined on the final constraint submanifold P_3 .

Now, we suppose that the Lagrangian system is subjected to the constraint $\phi = v^1$. Thus, the motion equations are

$$i_X \omega_L = dE_L + \lambda dq^1, \quad dv^1(X) = 0.$$

In such a case, if we apply our constraint algorithm, we obtain that the dynamics stabilizes on the submanifold $\bar{P}_2 = P_3$ given by

$$\bar{P}_2 = \{(q^1, q^2, v^1, v^2) \in \mathbb{R}^4 / q^1 = 0, v^1 = 0\}.$$

Taking coordinates (q^2, v^2) on $\bar{P}_2$ we obtain that the dynamics is determined by the vector field

$$f \frac{\partial}{\partial q^2} + g \frac{\partial}{\partial v^2},$$

where $f, g \in C^\infty(\bar{P}_2)$.

Next, consider the following constraint: $\phi = v^2$. Then, the motion equations are

$$i_X \omega_L = dE_L + \lambda dq^2, \quad dv^2(X) = 0.$$

We will compute the points x such that

$$\langle dE_L + \lambda dq^2, \ker \omega_L \rangle(x) = 0.$$

We obtain that all $x \in TQ$ satisfy this equation for the following particular value of the Lagrange multiplier:

$$\lambda = \frac{1}{2}(q^1)^2.$$

Moreover, the final constraint submanifold is

$$\bar{P}_2 = \{(q^1, q^2, v^1, v^2) \in \mathbb{R}^4 / v^2 = 0\},$$

and the dynamics is determined by the vector fields

$$X = v^1 \frac{\partial}{\partial q^1} + q^1 q^2 \frac{\partial}{\partial v^1} + f \frac{\partial}{\partial q^2},$$

where $f \in C^\infty(\bar{P}_2)$ and coordinates (q^1, q^2, v^1) on $\bar{P}_2$ are taken.

Remark 1: Notice that the example shows that the behavior of the algorithms for the free singular Lagrangian system and the constrained Lagrangian system may dramatically differ.

III. THE HAMILTONIAN FORMALISM

Let T^*Q be the cotangent bundle with canonical projection $\pi_Q: T^*Q \rightarrow Q$. Denote by $\text{Leg}: TQ \rightarrow T^*Q$ the Legendre mapping defined by a Lagrangian L . As we know, Leg is locally written by $\text{Leg}(q^A, V^A) = (q^A, p_A = \partial L / \partial V^A)$. If L is singular, Leg is not a diffeomorphism. However, we suppose that L is almost regular, i.e., $M_1 = \text{Leg}(TQ)$ is a submanifold of T^*Q and, Leg is a submersion onto M_1 with connected fibers. The restriction of Leg to its image will be denoted by $\text{Leg}_1: TQ \rightarrow M_1$ and it is a fibration. The submanifold M_1 will be called the primary constraint submanifold. Let $\lambda_Q = p_A dq^A$ be the Liouville one-form and $\omega_Q = -d\lambda_Q$ the canonical symplectic form on T^*Q . Here, (q^A, p_A) are fibered coordinates in T^*Q . Since the Lagrangian is almost regular, the energy E_L is constant along the fibers of Leg . Therefore, E_L projects onto a function h_1 on M_1 : $h_1(\text{Leg}(x)) = E_L(x)$, $\forall x \in TQ$. This construction is just the globalization of the one by Dirac.^{2,3}

If we denote by ω_1 the restriction of ω_Q to M_1 , then (M_1, ω_1) is a presymplectic manifold. In order to obtain the Hamiltonian formalism for the singular Lagrangian system subjected to the nonholonomic constraints, we assume that the constraints $\{\phi_i; 1 \leq i \leq m\}$ are Leg projectable, that is,

$$Z(\phi_i) = 0, \quad \forall Z \in \ker T \text{Leg}, \quad \forall i \in \{1, \dots, m\}.$$

In such a case, we obtain constraints $\bar{\phi}_i$ ($1 \leq i \leq m$) on the Hamiltonian side which are defined by $\bar{\phi}_i(\text{Leg}(x)) = \phi_i(x)$, $\forall x \in TQ$. Moreover, if we consider the one-forms $\bar{\mu}_i = \pi_Q^* \mu_i$, we obtain that $\text{Leg}^* \bar{\mu}_i = \mu_i^V$. We write the following system of equations on the submanifold M_1 :

$$i_{\bar{X}} \omega_1 = dh_1 + \bar{\lambda}^i \bar{\mu}_i, \quad d\bar{\phi}_i(\bar{X}) = 0. \quad (5)$$

As in the above section, we can develop a constraint algorithm for this system of equations and obtain the following sequence of submanifolds:

$$\bar{M}_2 = \{\bar{x} \in M_1 / \langle dh_1 + \bar{\lambda}^i \bar{\mu}_i, \ker \omega_1 \rangle(\bar{x}) = 0, \quad \bar{\phi}_i(\bar{x}) = 0, \text{ for some } \bar{\lambda}^i \in \mathbb{R}\},$$

and, given $\bar{M}_k$, we obtain $\bar{M}_{k+1}$ as follows:

$$\bar{M}_{k+1} = \{\bar{x} \in \bar{M}_k / \langle dh_1 + \bar{\lambda}^i \bar{\mu}_i, T\bar{M}_k^\perp \rangle(\bar{x}) = 0, \text{ for some } \bar{\lambda}^i \in \mathbb{R}\},$$

where

$$T_{\bar{x}} \bar{M}_k^\perp = \{\bar{v} \in T_{\bar{x}} \bar{M}_1 / \omega_1(\bar{x})(\bar{u}, \bar{v}) = 0, \quad \forall \bar{u} \in T_{\bar{x}} \bar{M}_k\}.$$

(We put $\bar{M}_1 = M_1$.)

A direct computation shows that $\text{Leg}_1: TQ \rightarrow M_1$ satisfies $\text{Leg}_1(\bar{P}_2) = \bar{M}_2$, and the restriction $\text{Leg}_2: \bar{P}_2 \rightarrow \bar{M}_2$ is again a fibration. Proceeding further we obtain a sequence of fibrations $\text{Leg}_k: \bar{P}_k \rightarrow \bar{M}_k$ which relates the Lagrangian and the Hamiltonian sides. Consequently, the behav-

ior of the algorithms is the same in both sides, i.e., if the algorithm stabilizes on some k in the Lagrangian side, the same holds for the Hamiltonian side and conversely. We then obtain the following commutative diagram:

Thus, we have proved that the Lagrangian and Hamiltonian formulations are equivalent. That is, given a vector field $\xi \in \mathfrak{X}(\bar{P}_f)$ which is a solution of (3) and Leg_f projectable, then its projection $\bar{\xi} = T \text{Leg}_f(\xi)$ is a solution of (5). Conversely, if $\bar{\xi} \in \mathfrak{X}(\bar{M}_f)$ is a solution of (5), then each projectable vector field on $\bar{P}_f$ onto $\bar{\xi}$ is a solution of (3).

Example (continued): Let L be the Lagrangian function defined in the example and suppose that it is subjected to the constraint $\phi = v^1$.

Since the Legendre mapping is given by

$$\text{Leg}(q^1, q^2, v^1, v^2) = (q^1, q^2, v^1, 0),$$

we obtain that

$$M_1 = \text{Leg}(TQ) = \{(q^1, q^2, p_1, p_2) \in T^*Q / p_2 = 0\}$$

and

$$\omega_1 = dq^1 \wedge dp_1, \quad h_1 = \frac{1}{2} p_1^2 - \frac{1}{2} (q^1)^2 q^2,$$

taking coordinates (q^1, q^2, p_1) on M_1 .

Since the constraint ϕ projects onto the constraint $\bar{\phi} = p_1$, we have that the motion equations are

$$i_{\bar{X}} \omega_1 = dh_1 + \bar{\lambda} dq^1, \quad \bar{X}(p_1) = 0. \quad (6)$$

Applying the algorithm, we arrive to the final constraint submanifold:

$$\bar{M}_2 = \{(q^1, q^2, p_1, p_2) \in T^*Q / q^1 = 0, p_1 = 0, p_2 = 0\}.$$

The dynamics is undetermined, that is, each vector field $g(\partial/\partial q_2)$ with $g \in C^\infty(M_2)$ is a solution of the dynamics on the Hamiltonian side.

IV. THE SECOND-ORDER DIFFERENTIAL EQUATION PROBLEM

In Sec. II, we found a final constraint submanifold $\bar{P}_f$ where there exists at least a completely consistent solution ξ of the dynamics. However, in general, these solutions do not verify the SODE condition, that is, $(J\xi = C)_{/\bar{P}_f}$. (It should be noticed that the Euler–Lagrange equations are, in fact, of second order.) To solve this problem we will proceed as in the case of free singular Lagrangians (see Ref. 4).

Assume that ξ is Leg_f projectable onto a vector field $Z \in \mathfrak{X}(\bar{M}_f)$. We remark that since Leg_f is a fibration, we can always choose a Leg_f -projectable solution ξ . In fact, we pick a solution Z of Eq. (5) and then lift Z to $\bar{P}_f$ by means of a connection in the fibration $\text{Leg}_f: \bar{P}_f \rightarrow \bar{M}_f$, for instance. Suppose that ξ is locally given by

$$\xi = \mathcal{A}^A \frac{\partial}{\partial q^A} + \mathcal{B}^A \frac{\partial}{\partial v^A}.$$

Since ξ verifies Eq. (3), we deduce that $\xi^* = J\xi - C \in \ker \omega_L \cap \text{Im } J = \ker T \text{Leg}$. Since ξ is projectable we get that $\mathcal{A}^A$ is constant along the fibers of the fibration $\text{Leg}_f: \bar{P}_f \rightarrow \bar{M}_f$. Take an arbitrary point $(q_0^A, v_0^A) \in \bar{P}_f$ and put $\text{Leg}(q_0^A, v_0^A) = z_0$. The integral curve of ξ^* with that initial datum is $\sigma(t) = (q_0^A, \mathcal{A}^A - e^{-t}(\mathcal{A}^A - v_0^A))$. A direct calculation shows that

$$\lim_{t \rightarrow +\infty} \sigma(t) = (q_0^A, \mathcal{A}^A). \quad (7)$$

Therefore the limit point is in the fiber over z_0 . Thus, we have obtained a global section $s: \bar{M}_f \rightarrow \bar{P}_f$ of Leg_f and a submanifold $S = s(\bar{M}_f)$ of $\bar{P}_f$. A direct computation from (7) shows that $(\xi^*)_{/S} = 0$. Hence, if we denote by $\tilde{\xi}$ the vector field $Ts(Z) \in \mathfrak{X}(S)$, we get

$$(i_{\tilde{\xi}} \omega_L = dE_L + \lambda^i \mu_i^V)_{/S}, \quad (d\phi_i(\tilde{\xi}) = 0)_{/S}, \quad (J\tilde{\xi} = C)_{/S}. \quad (8)$$

Example (continued): Following with the Lagrangian L and the constraint $\phi = v^1$, we now study the second-order differential equation problem on the Lagrangian side.

If we choose a projectable solution of the motion equations of $\bar{P}_f$, namely

$$\xi = f(q^2) \frac{\partial}{\partial q^2},$$

we obtain

$$S = \{(0, q^2, 0, f(q^2)) \in TQ / q^2 \in \mathbb{R}\},$$

and

$$\tilde{\xi} = f(q^2) \frac{\partial}{\partial q^2} + f(q^2) \frac{\partial f}{\partial q^2} \frac{\partial}{\partial v^2}.$$

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