Closed-Form Solution to the Position Analysis of Watt–Baranov Trusses Using the Bilateration Method

The exact position analysis of a planar mechanism reduces to compute the roots of its characteristic polynomial. Obtaining this polynomial almost invariably involves, as a first step, obtaining a system of equations derived from the independent kinematic loops of the mechanism. The use of kinematic loops to this end has seldom been questioned despite deriving the characteristic polynomial from them requires complex variable eliminations and, in most cases, trigonometric substitutions. As an alternative, the bilateration method has recently been used to obtain the characteristic polynomials of the three-loop Baranov trusses without relying on variable eliminations nor trigonometric substitutions and using no other tools than elementary algebra. This paper shows how this technique can be applied to members of a family of Baranov trusses resulting from the circular concatenation of the Watt mechanism irrespective of the resulting number of kinematic loops. To our knowledge, this is the first time that the characteristic polynomial of a Baranov truss with more than five loops has been obtained, and hence, its position analysis solved in closed form. [DOI: 10.1115/1.4004031]

Keywords: Baranov trusses, Assur kinematic chains, position analysis, bilateration, distance-based formulations

1 Introduction

The position analysis of planar linkages has been dominated by resultant elimination and tangent-half-angle substitution techniques applied to sets of kinematic loop equations. This analysis is thus reduced to finding the roots of a polynomial in one variable, the characteristic polynomial of the linkage. When this polynomial is obtained, it is said that the problem is solved in closed form. This approach is usually preferred to numerical approaches because the degree of the polynomial specifies the greatest possible number of assembly configurations of the linkage and modern software of personal computers provides guaranteed and fast computation of all real roots of a polynomial equation and hence of all assembly configurations of the analyzed linkage.

A nonoverconstrained linkage with zero-mobility from which an Assur group can be obtained by removing any of its links is defined as an Assur kinematic chain, basic truss [1,2], or Baranov1 truss when no slider joints are considered [3]. Hence, a Baranov truss, named after the Russian kinematician Baranov [4] who first stated it in 1952 [5], corresponds to multiple Assur groups. The relevance of the Baranov trusses derives from the fact that, if the position analysis of a Baranov truss is solved, the same process can be applied to solve the position analysis of all its corresponding Assur groups. Curiously enough, despite this importance, it is commonly accepted that the Baranov trusses with more than nine links have not been properly catalogued yet while all Assur groups with up to 12 links have been identified (see Table 1) [3]. It is worth mentioning here that Yang and Yao found that the number of Baranov trusses with 11 links is 239 using an algorithm that certainly requires further attention [6].

While the standard closed-form position analysis leads to complex systems of nonlinear equations derived from independent kinematic loop equations, the bilateration method avoids the computation of loop equations as usually understood. It has recently been shown to be a powerful technique by obtaining the characteristic polynomial of the three 3-loop Baranov trusses without relying on variable eliminations nor half-angle tangent substitutions [7].

At the end of the 19th century, it was known that there were only two six-link single degrees of freedom planar hinged linkages. At a suggestion of Burmester [8], these two linkages were called the Watt linkage and the Stephenson linkage. Several Stephenson linkages can be concatenated leading to what the Baranov trusses, which will be called Stepheson–Baranov and Watt–Baranov trusses, respectively (Fig. 1).

The position analysis of the Stephson–Baranov truss of four loops has been solved in closed form at least in Refs. [11–14], and more recently by Wohlhart in Ref. [15], thus reaching what the

Table 1 Number of Baranov trusses as a function of the number of links (alternatively, number of loops), and number of different Assur groups resulting from eliminating one link from the Baranov trusses in each class [3,6]

<table>
<thead>
<tr>
<th>Links</th>
<th>Loops</th>
<th>Baranov trusses</th>
<th>Resulting Assur groups</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>5</td>
<td>2</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>7</td>
<td>3</td>
<td>1</td>
<td>10</td>
</tr>
<tr>
<td>9</td>
<td>4</td>
<td>1</td>
<td>173</td>
</tr>
<tr>
<td>11</td>
<td>5</td>
<td>239</td>
<td>5442</td>
</tr>
<tr>
<td>13</td>
<td>6</td>
<td>unknown</td>
<td>251638</td>
</tr>
</tbody>
</table>

1 Some authors misspell it as Barranov.

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author considers to be the limit of Sylvester’s elimination method. The position analysis of the Watt–Baranov truss of four loops was solved in closed form by Han et al. in Ref. [16] and more recently by Borrás and Gregorio [17]. Elimination methods seem to reach their limit with the analysis of Baranov trusses with four or five loops, depending on their topology. Actually, the closed-form position analysis of a Baranov truss with more than five loops has not been reported to the best of our knowledge, and only the position analysis of a Baranov truss with an arbitrary number of kinematic loops. To this end, it is shown how the bilateration method can be applied to the analysis of Watt–Baranov trusses, with up to six loops, using the bilateration method.

This paper is organized as follows. In Sec. 2, the basic formula required to apply the bilateration method is briefly reviewed. Then, in Sec. 3, it is shown how the bilateration method can be applied to obtain the characteristic polynomial of a Watt–Baranov truss with an arbitrary number of kinematic loops. To this end, it is first shown how to derive a single scalar radical equation, which is satisfied if, an only if, the truss can be assembled and, then, how the characteristic polynomial is derived by simply clearing radicals. This last step is actually the only costly step in the whole process. Two examples are analyzed in Sec. 4, including a six-loop Watt–Baranov truss—whose characteristic polynomial is of degree 126— with 76 assembly modes.

2 Bilateration

The bilateration problem consists of finding the feasible locations of a point, say \( P_k \), given its distances to two other points, say \( P_i \) and \( P_j \), whose locations are known. Then, according to Fig. 2, the result, in matrix form, can be expressed as

\[
P_k = Z_{i,j,k} \cdot P_j
\]

where

\[
Z_{i,j,k} = \frac{1}{D(i,j)} \begin{vmatrix} D(i,j,k) & \pm \sqrt{D(i,j,k)} \\ \pm \sqrt{D(i,j,k)} & D(i,j,k) \end{vmatrix}
\]

is called a bilateration matrix, and

\[
D(i_1, \ldots, i_n; j_1, \ldots, j_n) = 2 \left( \frac{-1}{2} \right)^n \begin{vmatrix} 0 & 1 & \ldots & 1 \\ 1 & s_{i_1,j_1} & \ldots & s_{i_1,j_n} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & s_{i_n,j_1} & \ldots & s_{i_n,j_n} \end{vmatrix}
\]

(3)

with \( s_{ij} = d_{ij}^2 = \| p_i - p_j \|^2 \), where \( P_i = p_i \) and \( P_j = p_j \). This determinant is known as the Cayley–Menger bideterminant of the point sequences \( P_{i_1}, \ldots, P_{i_n} \), and \( P_{j_1}, \ldots, P_{j_n} \) and its geometric interpretation plays a fundamental role in the so-called distance geometry, the analytical study of Euclidean geometry in terms of invariants [19]. When the two point sequences are the same, it is convenient to abbreviate \( D(i_1, \ldots, i_n; i_1, \ldots, i_n) \) by \( D(i_1, \ldots, i_n) \), which is simply called the Cayley–Menger determinant of the involved points.

Now, it is important to observe that this kind of matrices constitute an Abelian group under product and addition and if \( v = Zw \), where \( Z \) is a bilateration matrix, then \( \|v\|^2 = \det(Z)\|w\|^2 \). The interested reader is addressed to Ref. [7] for a more detailed treatment of bilateration matrices and some basic geometric operations that can be performed with them.

3 Position Analysis of the General N-Link Watt–Baranov Truss

Figure 3 shows the general \( n \)-link Watt–Baranov truss, a structure with \( k = (n-1)/2 \) loops and \( v = 3/2(n-1) \) revolute joints.
The $k$-ary link is defined by $P_1P_2P_3\ldots P_{n-1}P_n$, and the $k$ ternary links by the triangles $P_1P_2P_3$, $P_1P_2P_5$, $P_1P_3P_5$, $P_2P_3P_5$, and $P_{n-3}P_{n-1}P_n$. The position analysis problem for this structure consists in, given the dimensions of all links, calculating all relative possible transformations between them all. To solve this problem, instead of directly computing the relative Cartesian poses of all links through loop-closure equations, we will compute the set of values of $s_{1,3}$ compatible with all binary and ternary links side lengths. Thus, this procedure is entirely posed in terms of distances.

On the one hand, according to Fig. 3, $p_{1,4}$, $p_{1,7}$, ..., $p_{1,x-5}$, $p_{1,x-2}$ can be expressed as a function of $p_{1,3}$ by computing $3k-2$ bilaterations as follows:

\[
p_{1,4} = Z_{1,3,4}p_{1,3} \tag{4}
\]

\[
p_{1,7} = Z_{1,4,7}p_{1,4} = Z_{1,4,7}Z_{1,3,4}p_{1,3} \tag{5}
\]

\[
p_{1,10} = Z_{1,7,10}p_{1,7} = Z_{1,7,10}Z_{1,4,7}Z_{1,3,4}p_{1,3} \tag{6}
\]

\[\vdots\]

\[
p_{1,x-5} = Z_{1,x-8,5}Z_{1,x-11,8}Z_{1,4,7}Z_{1,3,4}p_{1,3} \tag{7}
\]

\[
p_{1,x-2} = Z_{1,x-5,2}Z_{1,x-8,5}Z_{1,4,7}Z_{1,3,4}p_{1,3} \tag{8}
\]

On the other hand, for the ternary link $P_1P_2P_3$, we have

\[
p_{4,5} = Z_{4,3,5}p_{1,3} \tag{9}
\]

\[
p_{4,1} + p_{1,5} = Z_{4,3,5}(p_{1,4} + p_{1,3}) \tag{10}
\]

\[
p_{4,1} + p_{1,5} = Z_{4,3,5}(p_{1,3} + p_{1,4}) \tag{11}
\]

Likewise, for the ternary links $P_1P_2P_3$, $P_1P_5P_6$, and $P_{n-2}P_{n-3}P_{n-4}$, we obtain

\[
p_{1,6} = p_{1,7} + Z_{1,5,6}(p_{1,5} - p_{1,7}) \tag{12}
\]

\[
p_{1,8} = p_{1,7} + Z_{1,6,8}(p_{1,6} - p_{1,7}) \tag{13}
\]

\[\vdots\]

\[
p_{1,x-3} = p_{1,x-4} + Z_{1,x-2,4}(p_{1,x-4} - p_{1,x-3}) \tag{14}
\]

\[
p_{1,x-1} = p_{1,x-2} + Z_{1,x-2,3}(p_{1,x-3} - p_{1,x-2}) \tag{15}
\]

Now, substituting Eqs. (4)–(8) in Eqs. (9)–(13) and then replacing the resulting expression for $p_{1,3}$ in that for $p_{1,6}$, and the resulting expression for $p_{1,4}$ after this substitution in that for $p_{1,8}$, and so on till an expression is obtained for $p_{1,x-1}$, we get

\[
p_{1,x-1} = Q_n p_{1,3} \tag{16}
\]

Moreover, for the ternary link $P_1P_2P_3$, we have

\[
p_{1,2} = Z_{1,2,3}p_{1,3} \tag{17}
\]

Finally, using Eqs. (14) and (15), we get

\[
p_{v-1,v} = p_{v-1,v} + p_{1,v} = (-Q_n + Z_{1,2,3}Z_{1,3,2})p_{1,3} \tag{18}
\]

Therefore,

\[
\det(-Q_n + Z_{1,2,3}Z_{1,3,2}) = \frac{s_{v-1,v}}{s_{1,3}} \tag{19}
\]

The left hand side of the above equation is a function of the $k-1$ unknown squared distances $s_{1,3}$ and $s_{5,7}$, $s_{8,10}$, ..., $s_{2k-5}$, $s_{2k-2}$.

Since using the same procedure to obtain Eq. (16) allows us to obtain

\[
p_{5,7} = -p_{1,5} + p_{1,7} = D_{n_s} p_{1,3} \tag{20}
\]

\[
p_{8,10} = -p_{1,8} + p_{1,10} = D_{n_s} p_{1,3} \tag{21}
\]

The substitution of Eqs. (22)–(25) into Eq. (17) yields a scalar equation in a single variable, $s_{1,3}$. The roots of this equation, in the range in which the signed areas of the triangles $P_1P_2P_3$ and $P_1P_3P_4$ are real, that is, the range

\[
\left[\max\{(d_{1,2} - d_{3,4})^2, (d_{1,4} - d_{3,4})^2\}, \min\{(d_{1,2} + d_{3,4})^2, (d_{1,4} + d_{3,4})^2\}\right]
\]

determine the assembly modes of the general $n$-link Watt–Baranov truss. These roots can be readily obtained using, for example, an interval Newton method for the $2^n$ possible combinations for the signs of the signed areas of the triangles $P_1P_2P_3$, $P_1P_3P_4$, and $P_1P_2P_3$, $P_1P_4P_5$, ..., $P_1P_{2k-5}P_{2k-4}$, $P_1P_{2k-4}P_{2k-3}$.

In order to obtain the characteristic polynomial, it just remains to clear all square roots in the obtained scalar equation by isolating one at a time and squaring the result till no square root remains. Using a computer algebra system, it can be seen that this clearing process leads to

\[
D_n = 0 \tag{26}
\]

where $D_n$ is a polynomial in $s_{1,3}$ of degree $2k+1 - 2$. The extraneous roots at $s_{1,3} = 0$, ..., $s_{2k-4} = 0$ were introduced when clearing denominators, so they can be dropped. For each of the real roots of polynomial $D_n$, we can determine the Cartesian position of the $v-k$ revolute pair centers of the ternary links, with respect to the $k$-ary link, using Eqs. (9)–(13) and (15), and the
4 Examples

4.1 Five-Loop Watt–Baranov Truss. Consider an 11-link Watt–Baranov truss. Since in this case $k = 5$ and $v = 15$, Eq. (17) reduces to

$$\det(-Q_{11} + Z_{1,2,15} Z_{1,3,2}) = \frac{s_{14,15}}{s_{1,3}}$$

(27)

where

$$Q_{11} = Z_{1,10,11} Z_{1,7,10} Z_{1,4,7} Z_{1,3,4} + Z_{13,12,14} Z_{13,11,12}$$

and

$$Q_{11} = (Z_{1,7,10} Z_{1,4,7} Z_{1,3,4} + Z_{10,9,11} Z_{10,8,9} (Z_{1,4,7} Z_{1,3,4} Z_{7,5,6})$$

and

$$s_{1,3} = 30.6486 \quad s_{1,3} = 39.0249 \quad s_{1,3} = 47.1860 \quad s_{1,3} = 48.6406$$

$$s_{1,3} = 69.9863 \quad s_{1,3} = 77.3161 \quad s_{1,3} = 90.1506 \quad s_{1,3} = 130.0000$$

$$s_{1,3} = 132.2178 \quad s_{1,3} = 134.2206 \quad s_{1,3} = 134.9836 \quad s_{1,3} = 140.6611$$

$$s_{1,3} = 142.9286 \quad s_{1,3} = 143.7773 \quad s_{1,3} = 148.1286 \quad s_{1,3} = 151.6614$$

Fig. 4 The assembly modes of the analyzed 11-link Watt–Baranov truss
and Eqs. (22)–(25) reduce to

\[ s_{5,7} = \det(D_{11})_{5,13} \]

(28)

\[ s_{8,10} = \det(D_{12})_{5,13} \]

(29)

\[ s_{11,13} = \det(D_{13})_{5,13} \]

(30)

where

\[ D_{11} = -Z_{1,3,4} - Z_{4,3,5} (I - Z_{1,3,4}) + Z_{4,4,7} Z_{1,3,4} \]

\[ D_{12} = -Z_{4,4,7} Z_{1,3,4} - Z_{7,6,8} Z_{7,6,2} + Z_{4,4,3,5} (I - Z_{1,3,4}) - Z_{4,4,7} Z_{1,3,4} + Z_{7,10,7} Z_{4,4,7} Z_{1,3,4} + Z_{7,10,13} Z_{4,4,7} Z_{1,3,4} \]

Finally, the square roots in Eq. (36) can be eliminated by properly twicing squaring it. This operation yields

\[ -\Phi_{1} A_{1,2,3} A_{2,4,3} + 2\Phi_{2} \Phi_{4} A_{1,2,3} A_{2,4,3} + 4\Phi_{2} \Phi_{4} A_{1,2,3} A_{2,4,3} \]

\[ -\Phi_{2} A_{1,2,3} - \Phi_{4} A_{1,2,3} = \Phi_{1} = (2+\Phi_{2} - 2\Phi_{2} \Phi_{4} \Phi_{4} \Phi_{4} + 2\Phi_{2} \Phi_{4}) \]

\[ A_{1,2,3} A_{2,4,3} + 2\Phi_{2} A_{1,2,3} A_{2,4,3} + 2\Phi_{4} A_{1,2,3} A_{2,4,3} = 0 \]

(37)

which, when fully expanded, leads to

\[ s_{16}^{15} P_{1}(s_{1,3}, 0)^{9} P_{2}(s_{1,3}, 0)^{4} P_{3}(s_{1,3}, 0)^{2} \Delta_{11} = 0 \]

(38)

where \( \Delta_{11} \) is a polynomial in \( s_{1,3} \) of degree 62. The extraneous roots at \( s_{4,5} = 0 \), \( s_{8,10} = 0 \) and \( s_{11,13} = 0 \) were introduced when clearing denominators to obtain Eq. (35), so they can be dropped.

Finally, let us suppose that \( s_{1,2} = 40, s_{1,4} = 13, s_{3,5} = 26, s_{1,10} = 34, s_{1,13} = 17, s_{1,15} = 13, s_{2,5} = 50, s_{2,15} = 17, s_{3,8} = 81, s_{3,9} = 90, s_{4,9} = 90, s_{4,10} = 49, s_{4,11} = 52, s_{5,6} = 126, s_{5,7} = 40, s_{6,8} = 9, s_{7,8} = 37, s_{7,10} = 20, s_{7,13} = 45, s_{8,9} = 136, s_{10,9} = 53, s_{11,9} = 50, s_{11,10} = 50, s_{11,13} = 121, s_{12,11} = 50, s_{12,14} = 9, s_{13,14} = 65, and s_{14,15} = 29. Then, proceeding as explained above, we obtain the characteristic polynomial

\[ s_{1,3}^{62} - 4091.5078 s_{1,3}^{61} + 8.3074 \times 10^{6} s_{1,3}^{60} - 1.1186 \times 10^{10} s_{1,3}^{59} + 1.1260 \times 10^{13} \]

(39)

where \( \Psi \) is a linear equation in \( A_{13,11,12} \), i.e.,

\[ a_{1} \Psi_{1} + a_{2} \Psi_{2} + \cdots + a_{n} \Psi_{n} = 0 \]

with \( \Psi_{i}, i = 1, \ldots, 2^{3} \), polynomials in \( s_{1,3}, s_{5,7}, s_{8,10}, \) and \( s_{11,13} \).

Thus, this polynomial has 16 real roots. The values of these roots as well as the corresponding assembly modes, for the case in which
$P_1 = (12, 10)^T$, $P_2 = (10, 13)^T$, $P_3 = (13, 15)^T$, $P_{10} = (17, 13)^T$, and $P_{13} = (16, 9)^T$, appear in Fig. 4.

The coefficients of the above polynomial have to be computed in rational arithmetic. Otherwise, numerical problems make impracticable the correct computation of its roots. Although these coefficients are given here in floating point arithmetic for space limitation reasons, they could be of interest for comparison with other possible methods.

### 4.2 Six-Loop Watt–Baranov Truss

Let us consider a 13-link Watt–Baranov truss where $s_{1,1} = 58$, $s_{1,2} = 18$, $s_{1,3} = 40$, $s_{1,4} = 53$, $s_{1,5} = 50$, $s_{1,6} = 20$, $s_{1,7} = 41$, $s_{1,8} = 52$, $s_{1,9} = 13$, $s_{1,10} = 64$, $s_{1,11} = 18$, $s_{1,12} = 34$, $s_{1,13} = 10$, $s_{1,14} = 41$, $s_{1,15} = 68$, $s_{1,16} = 50$, $s_{1,17} = 50$, $s_{1,18} = 74$, $s_{1,19} = 10$, $s_{1,20} = 68$, $s_{1,21} = 50$, $s_{1,22} = 52$, $s_{2,1} = 65$, $s_{2,2} = 10$, $s_{2,3} = 68$, $s_{2,4} = 9$, $s_{2,5} = 9$, $s_{2,6} = 29$, $s_{2,7} = 61$, $s_{2,8} = 65$, $s_{2,9} = 65$, $s_{2,10} = 10$, $s_{2,11} = 13$, $s_{2,12} = 13$, $s_{2,13} = 44$, $s_{2,14} = 10$, $s_{2,15} = 113$, $s_{2,16} = 40$, $s_{2,17} = 13$.

Fig. 5 The assembly modes of the analyzed 13-link Watt–Baranov truss (Part 1)
Fig. 6 The assembly modes of the analyzed 13-link Watt–Baranov truss (Part 2)
Fig. 7 The assembly modes of the analyzed 13-link Watt–Baranov truss (Part 3)
This polynomial, which was computed using exact rational arithmetic and is presented here only for comparison purposes of eventual future works, has 76 real roots. The values of these roots as well as the corresponding configurations, for the case in which \( P_1 = (12, 8), P_2 = (9, 11), P_3 = (10, 14), P_4 = (14, 15), P_5 = (17, 13), \) and \( P_6 = (16, 10), \) appear in Figs. 5–7.

5 Conclusion

Given a Watt–Baranov truss, it has been shown how a scalar radical equation—which is satisfied if, and only if, it is assemblable—can be straightforwardly derived using bilaterations, independent of the number of its kinematic loops. Clearing radicals from this equation leads to the characteristic polynomial of the corresponding Watt–Baranov truss. Although conceptually simple, this clearing operation is computationally costly as it yields an exponential number of terms with the number of involved bilaterations. The whole process has been carried out for Watt–Baranov trusses with up to six loops and two examples have been presented. Obtaining the characteristic polynomial of a Watt–Baranov truss with more than six loops becomes a huge task. This suggests the convenience of working with the compact expression including radicals whenever possible, depending on the application.

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References


