STOCHASTIC RESONANCE IN EXTENDED SYSTEMS: THE ROLE OF THE COUPLING MECHANISM

Horacio S. Wio\(^{(a)}\)

Instituto de Fisica de Cantabria
Universidad de Cantabria-CSIC

\(^{(a)}\) Electronic address: wio@ifca.unican.es
URL: http://www.ifca.unican.es/~wio/

XI LAWNP, Buzios, Brazil, Oct. 05-09, 2009
COLLABORATORS:

S. Bouzat 1 *
F. Castelpoggi 3 *
R. Deza 2
B. von Haeften 4 *
G. Izús 2 *
M. Kuperman 1 **
S. Mangioni 2 **
J.A. Revelli 6
A. Sánchez 2 **
C. Tessone 5 *

1) Centro Atómico Bariloche & Instituto Balseiro, Argentina.
2) Universidad Nacional de Mar del Plata, Argentina.
3) CitiBank, Buenos Aires, Argentina.
4) Universidad de Vigo, Spain
5) ETH, Zurich, Swiss
6) Inst. Fisica de Cantabria, Spain
Organization of the Talk:

• Introduction: Stochastic Resonance in 0-d and spatially extended systems;
• Far from Equilibrium Potentials: Brief review;
• Reaction-Diffusion Systems: Example of a scalar system;
• Stochastic Resonance in Extended Systems:
  (a) Non-local Interactions;
  (b) Bounded KPZ system;
  (c) KPZ plus non-local interaction;
  ...
• Final Comments …
STOCHASTIC RESONANCE IN 0-D SYSTEMS:

**Figure 4.** A paddlefish uses stochastic resonance to locate zooplankton.

**Figure 11.** A scheme of the experimental setting, with the paddlefish between the electrodes. Using the antenna-like rostrum, the fish detects electrical signals emitted by its prey, the plankton *Daphnia*. In the experiment, the probability of the fish capturing a prey located at a distance $d$ above the rostrum is observed for different strengths of the external noise produced by the electrodes. After [21].
TWO STATE THEORY:


ASSUMPTIONS:

1. $P_{\pm}(t)$: probability of finding the system in $\pm C$

2. Non-stationary Master Equation for $P_{\pm}(t)$
   
   \[
   \frac{d}{dt}P_+(t) = W_-P_-(t) - W_+P_+(t)
   \]
   
   (“adiabatic approx.”)

3. Perturb. up to 1st order in $B$,

   $W_{\pm} \approx W_0 - \left(\frac{B}{D}\right) \sin[\omega_0 t]$

Kramers-like approximation:

The knowledge of $P_{\pm}(t)$ allows to obtain the correl. function

\[
C(t, t') = \langle x(t)x(t') \| x_i \rangle
\]

Its Fourier transform give us the Power Spectral Density
TWO STATE THEORY:

$S(\omega) -$ The PSD results:

$$S(\omega) \approx \frac{4\alpha_o \langle x^2 \rangle}{\alpha_o^2 + \omega^2} + \frac{\pi \langle x^2 \rangle}{\alpha_o^2 + \omega_o^2} \delta(\omega - \omega_o)$$
TWO STATE THEORY:

$S(\omega)$ - The PSD results:

$$S(\omega) \approx \frac{4\alpha_o \langle x^2 \rangle}{\alpha_o^2 + \omega^2} + \frac{\pi \langle x^2 \rangle}{\alpha_o^2 + \omega_o^2} \delta(\omega - \omega_o)$$

The Signal-to-Noise Ratio (SNR) results:

$$SNR = \left( \frac{B \Delta V_0}{D} \right)^2 \exp \left\{ 2 \frac{\Delta V_0}{D} \right\}$$
FAR FROM EQUILIBRIUM POTENTIAL:

**Dynamical Systems:**

**Gradient (or variational):**

\[
\dot{x}_j = -\frac{\partial}{\partial x_j} V(x_1, \ldots, x_n),
\]

\[
\frac{dV}{dt} = \sum \frac{\partial V}{\partial x_j} \frac{dx_j}{dt} = -\sum \left(\frac{\partial V}{\partial x_j}\right)^2 \leq 0.
\]

**Relaxational or non-gradient:**

\[
\dot{x}_j = -\sum (T)_{ji} \frac{\partial V}{\partial x_i},
\]

\[
\frac{dV}{dt} = -\sum (T)_{ji} \frac{\partial V}{\partial x_j} \frac{\partial V}{\partial x_i} \leq 0.
\]

**Non-relaxational & non-gradient:**

\[
\dot{x}_j = -\sum (IK)_{ji} \frac{\partial V}{\partial x_i},
\]

\[
\mathbf{K} = \mathbf{S} + \mathbf{F}
\]

\[
\mathbf{S} = \frac{1}{2} (\mathbf{K} + \mathbf{K}^T) \quad \mathbf{S} = \mathbf{S}^T
\]

\[
\mathbf{F} = \frac{1}{2} (\mathbf{K} - \mathbf{K}^T) \quad \mathbf{F} = -\mathbf{F}^T.
\]

\[
\frac{dV}{dt} = -\sum (\mathbf{S})_{ji} \frac{\partial V}{\partial x_j} \frac{\partial V}{\partial x_i} - \sum (\mathbf{F})_{ji} \frac{\partial V}{\partial x_j} \frac{\partial V}{\partial x_i} \leq 0,
\]
FAR FROM EQUILIBRIUM POTENTIAL:

Including stochastic terms:

\[
\frac{dx_j}{dt} = F_j(x) + \sum_l g_{jl} \xi_l(t),
\]

Associated Fokker-Planck equation:

\[
\frac{\partial}{\partial t} P(x, t) = \sum_j \frac{\partial}{\partial x_j} \left( -[F_j(x) P(x, t)] + \gamma \sum_l \frac{\partial}{\partial x_l} [G_{jl} P(x, t)] \right)
\]

If:

\[
F_j(x) = -\sum_l (\$)_{jl} \frac{\partial}{\partial x_l} V(x),
\]

and

\[
G_{lj} = \sum_k g_{lk} \times g^{T}_{kj}
\]

\[
P_{st}(x) \sim e^{-V(x)/\gamma},
\]

\[
V(x) = -\lim_{\gamma \to 0} \gamma \ln P_{st}(x)
\]

\[
P_{st}(\{x\}) = Z[\{x\}] \exp \left\{ -\frac{V(\{x\})}{\gamma} + O(\gamma) \right\}
\]

\[
V[\{x\}] \quad \text{solution of Hamilton-Jacobi –like equation, independent of}
\]

(that is a solution of a 1st order pdf)
NEP Example: Scalar System

Balast resistor – Schlögl model:

\[
\frac{\partial}{\partial t} \phi = D \frac{\partial^2}{\partial y^2} \phi - \phi + \phi_n \theta(\phi - \phi_c).
\]

\[
\left. \frac{\partial \phi(y,t)}{\partial y} \right|_{y=\pm y_L} = \mp k \phi(\pm y_L, t),
\]
NEP Example: Scalar System

\[ F[\phi, k, y_L] = \int_{-y_L}^{y_L} \left\{ -\int_{0}^{\phi(y,t)} \left[ -\phi' + \phi_h \theta(\phi' - \phi_c) \right] d\phi' + \frac{D}{2} \left( \frac{\partial}{\partial y} \phi(y,t) \right)^2 \right\} dy + \frac{k}{2} \phi(y,t)^2 \]
FAR FROM EQUILIBRIUM POTENTIAL
SR in Extended Systems:
SR in Extended Systems:
**SR in Extended Systems:**

Transitions rates (~ Kramer theory)

\[ W_i = \tau_0^{-1} \exp \left\{ -\frac{\Delta F_i[\phi, t]}{\gamma} \right\} \]

1st order in the “perturbation”

\[ \Delta F_i[\phi, t] = \Delta F_i[\phi] + \delta \phi \left( \frac{\partial \Delta F_i[\phi]}{\partial \phi^*} \right)_{\phi^*=\phi^*} \cos(\omega t) \]

\[ W_i \approx \frac{1}{2} \left( \mu_i + \alpha_i \frac{\delta \phi^*}{\gamma} \cos(\omega t) \right) \]

\[ \mu_i \approx \exp \left\{ -\frac{\Delta F_i[\phi]}{\gamma} \right\}, \quad \alpha_i \approx \pm \mu_i \left( \frac{d \Delta F_i}{d \phi^*} \right)_{\phi^*} \]

Signal-to-Noise Ratio:

\[ SNR = \frac{\pi}{4 \mu_1 \mu_2} \frac{(\alpha_2 \mu_1 + \alpha_1 \mu_2)^2}{\mu_1 + \mu_2} = \frac{\pi}{4 \gamma^2} \frac{\mu_1 \mu_2}{\mu_1 + \mu_2} \Phi, \]
Stochastic-Resonance in a Scalar System: Non-Local Kernel

\[
\begin{align*}
\frac{\partial u(x, t)}{\partial t} &= D_u \frac{\partial^2 u(x, t)}{\partial x^2} + f(u(x, t)) - v(x, t), \\
\frac{\partial v(x, t)}{\partial t} &= D_v \frac{\partial^2 v(x, t)}{\partial x^2} + \beta u(x, t) - \gamma v(x, t), \\
\frac{\partial u(x, t)}{\partial t} &= D_u \frac{\partial^2 u(x, t)}{\partial x^2} + f(u) - \beta \int_{-L}^{L} G(x, x') u(x') \, dx'.
\end{align*}
\]

\[\mathcal{F}[u] = \int_{-L}^{L} dx \left\{ \frac{D_u}{2} \left( \frac{\partial u}{\partial x} \right)^2 - \int_{-L}^{L} f(w) \, dw + \frac{\beta}{2} \int_{-L}^{L} dx' G(x, x') u(x') \, u(x) \right\}.\]

\[V(u(x)) = -\frac{a}{2} u(x)^2 + \frac{b}{4} u(x)^4\]
Stochastic-Resonance in a Scalar System: Non-Local Kernel

Parameters:
D=1,  L= 2 π
- β = 0  l = 0.05
- β = 0.01  l = 0.2
- β = 0.01  l = 0.3
Bounded KPZ:

Variational formulation for the KPZ and related kinetic equations,

Case of KPZ-like equation

\[
\frac{\partial}{\partial t} h(z, t) = \nu \frac{\partial^2}{\partial z^2} h + \frac{\lambda}{2} \left( \frac{\partial}{\partial z} h \right)^2 - \frac{\delta V[h]}{\delta h} + \xi(z, t)
\]
Bounded KPZ:


Case of KPZ-like equation

\[ \frac{\partial}{\partial t} h(z, t) = \nu \frac{\partial^2}{\partial z^2} h + \frac{\lambda}{2} \left( \frac{\partial}{\partial z} h \right)^2 - \frac{\delta V[h]}{\delta h} + \xi(z, t) \]

with the nonequilibrium potential

\[ \Phi[h] = \int dx \left[ \frac{\nu}{2} (\partial_x h)^2 + V[h] \right] - \frac{\lambda}{2} \int dx \int^h d\psi (\partial_x \psi)^2 \]

fulfilling

\[ \frac{\partial}{\partial t} h(z, t) = -\frac{\delta \Phi[h]}{\delta h} + \xi(z, t) \]

\[ \frac{\partial}{\partial t} \Phi[h] = - \left( \frac{\delta \Phi[h]}{\delta h} \right)^2 \]
Bounded KPZ:

Variational formulation for the KPZ and related kinetic equations,

Case of KPZ-like equation

\[
\frac{\partial}{\partial t} h(z, t) = \nu \frac{\partial^2}{\partial z^2} h + \frac{\lambda}{2} \left( \frac{\partial}{\partial z} h \right)^2 - \frac{\delta V[h]}{\delta h} + \xi(z, t)
\]

with the nonequilibrium potential

\[
\Phi[h] = \int dx \left[ \frac{\nu}{2} (\partial_x h)^2 + V[h] \right] - \frac{\lambda}{2} \int dx \int^h d\psi (\partial_x \psi)^2
\]

fulfilling

\[
\frac{\partial}{\partial t} h(z, t) = -\frac{\delta \Phi[h]}{\delta h} + \xi(z, t)
\]

\[
\frac{\partial}{\partial t} \Phi[h] = -\left( \frac{\delta \Phi[h]}{\delta h} \right)^2
\]

The approximate form (expansion around a reference state)

\[
\Phi[h] \approx \int dx \left[ \left( \frac{\nu}{2} - \frac{\lambda}{2} h \right)(\partial_x h)^2 + V[h] \right]
\]

allows to exploit all previous results in order to obtain the SNR
Bounded KPZ:


\[ V(h(x)) = -\frac{a}{2} h(x)^2 + \frac{b}{4} h(x)^4 \]

Parameters:

\[ \lambda = 0; 0.25; 0.5 \quad (\nu = 2; a = b = 30) \]
Bounded KPZ: Non-Local Kernel

Parameters:
\[ \lambda = 0.1 \quad (\nu = 2; \ a = b = 30) \]
\[ \beta = 0.01; \quad l = 0.05; 0.25; 0.5, 1. \]
**Final Comments:**

- The knowledge of the NEP is extremely useful to analyze and understand the system’s dynamics (even when the NEP is not known in full detail);
- The knowledge of the NEP, when accessible, allows us to:
  - clear understand the role played by each of the different systems parameters;
  - analyze different forms to enhance and/or control the phenomenon of SR;
  - understand the physical origin of the different trends;
- Some future research lines:
  1. to analyze very general situations with selective or state dependent coupling;
  2. to analyze the effect of other boundary conditions;
  3. to analyze the effect of other kind of noises: colored, non-Gaussian, $f^{-\nu}$;
  4. to analyze the effect of other forms of coupling;
  5. ……
RELEVANT PAPERS

THANKS!