A SHARP BILINEAR ESTIMATE FOR THE KLEIN–GORDON EQUATION IN $\mathbb{R}^{1+1}$

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Abstract. We prove a sharp bilinear estimate for the one dimensional Klein–Gordon equation. The proof involves an unlikely combination of five trigonometric identities. We also prove new estimates for the restriction of the Fourier transform to the hyperbola, where the pullback measure is not assumed to be compactly supported.

1. Introduction

Sharp constants and maximisers have been calculated for a number of space-time estimates for dispersive equations (see for example [1, 9, 13, 15, 22, 27]). Proving the existence of such maximisers has also received attention (see for example [4, 7, 8, 10, 11, 19, 24, 26]).

Regarding bilinear estimates for the Schrödinger equation, the sharp constant and maximisers were calculated in [23] for the estimate

$$
\left\| (-\Delta)^{\frac{2-d}{4}} (e^{it\Delta} f_1 e^{it\Delta} f_2) \right\|_{L^2_t L^2_x(\mathbb{R}^{d+1})} \leq C \|f_1\|_2 \|f_2\|_2,
$$

and, in [6], for the estimate

$$
\left\| e^{it\Delta} f_1 e^{it\Delta} f_2 \right\|^2_{L^2_t L^2_x(\mathbb{R}^{d+1})} \leq C \int_{\mathbb{R}^{2d}} |\hat{f}_1(\xi_1)|^2 |\hat{f}_2(\xi_2)|^2 |\xi_1 - \xi_2|^{-d-2} d\xi_1 d\xi_2.
$$

For the wave equation, the sharp constant and maximisers were calculated in [2] for the estimate

$$
\left\| e^{it\sqrt{-\Delta}} f_1 e^{it\sqrt{-\Delta}} f_2 \right\|^2_{L^2_t L^2_x(\mathbb{R}^{d+1})} \leq C \int_{\mathbb{R}^{2d}} |\hat{f}_1(\xi_1)|^2 |\hat{f}_2(\xi_2)|^2 \left|\frac{\xi_1}{|\xi_1||\xi_2|} \right|^\frac{d-3}{2} \left(1 - \frac{\xi_1 \cdot \xi_2}{|\xi_1||\xi_2|} \right)^\frac{d-3}{2} d\xi_1 d\xi_2.
$$

The estimates for the Schrödinger equation hold when the dimension $d = 1$, whereas no such estimate can hold for the wave equation in one dimension. This is connected to the fact that the parabola is curved, but the cone in $\mathbb{R}^{1+1}$ is not.

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Here we consider the Klein–Gordon equation, \(-\partial_t^2 u + \Delta u = u\) in space-time \(\mathbb{R}^{1+1}\).

The solution can be written as \(u = u_+ + u_-,\)

and

\[ u(0) = f_+ + f_- \quad \text{and} \quad \partial_t u(0) = i\sqrt{1 - \Delta} (f_+ - f_-). \]

Recall that \(e^{\pm it\sqrt{1 - \Delta}}\) is defined, initially on Schwartz functions, by

\[
e^{\pm it\sqrt{1 - \Delta}} f(x) = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} \hat{f}(\xi) e^{\pm it\sqrt{1 + \xi^2}} e^{i\xi x} d\xi,
\]

where \(\hat{f}\) is the Fourier transform defined by

\[
\hat{f}(\xi) = \int_{\mathbb{R}} f(x) e^{-ix\xi} dx.
\]

As in the case of the Schrödinger equation, the separation of the frequency supports is necessary in the following estimate, as otherwise the right hand side is infinite.

**Theorem 1.1.** Suppose that \(\text{supp}(\hat{f}_1) \cap \text{supp}(\hat{f}_2) = \emptyset.\) Then

\[
\|e^{it\sqrt{1 - \Delta}} f_1 e^{it\sqrt{1 - \Delta}} f_2\|^2_{L^2_t(\mathbb{R}^{1+1})} \leq \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} |\hat{f}_1(\xi_1)|^2 |\hat{f}_2(\xi_2)|^2 (1 + \xi_1^2)^{3/4} (1 + \xi_2^2)^{3/4} \frac{d\xi_1 d\xi_2}{|\xi_2 - \xi_1|}.
\]

The constant is sharp, but attained only if \(f_1 = 0\) or \(f_2 = 0.\)

The first estimates of this type were proven by Klainerman and Machedon [16, 17, 18] for the wave equation (see also [3], [14], [20], [21], [31], [32], [33] for \(L^p_{x,t}\) estimates).

### 2. Proof of Theorem 1.1

Amusingly, the proof relies on five trigonometric identities which we collect for the convenience of the reader:

- **(Pythagorean)** \(\sin^2 \theta = 1 - \cos^2 \theta\)
- **(Difference)** \(\sin(\phi - \theta) = \sin \phi \cos \theta - \cos \phi \sin \theta\)
- **(Double-angle)** \(\sin 2\theta = 2 \sin \theta \cos \theta\)
- **(Product-to-sum)** \(2 \sin \theta \sin \phi = \cos(\theta - \phi) - \cos(\theta + \phi)\)
- **(Power-reduction)** \(2 \sin^2 \theta = 1 - \cos 2\theta\)

We have that \(e^{it\sqrt{1 - \Delta}} f_1(x) e^{it\sqrt{1 - \Delta}} f_2(x)\) is equal to

\[
\frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} \hat{f}_1(\xi_1) \hat{f}_2(\xi_2) e^{i\xi(\xi_1 + \xi_2)} e^{-i(\sqrt{1 + \xi_1^2} + \sqrt{1 + \xi_2^2})} d\xi_1 d\xi_2
\]

\[= \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} F(\xi_1, \xi_2) e^{i\xi(\xi_1 + \xi_2)} e^{-i(\sqrt{1 + \xi_1^2} + \sqrt{1 + \xi_2^2})} d\xi_1 d\xi_2,
\]

where

\[ F(\xi_1, \xi_2) = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} \hat{f}_1(\xi_1) \hat{f}_2(\xi_2) e^{i\xi(\xi_1 + \xi_2)} e^{-i(\sqrt{1 + \xi_1^2} + \sqrt{1 + \xi_2^2})} d\xi_1 d\xi_2.
\]
where $F(\xi_1, \xi_2) = \frac{1}{2} (\hat{f}_1(\xi_1)\hat{f}_2(\xi_2) + \hat{f}_1(\xi_2)\hat{f}_2(\xi_1))$. As the integrand is now symmetric, we can write
\[
e^{it\sqrt{1-\Delta}} f_1(x) e^{it\sqrt{1-\Delta}} f_2(x) = \frac{2}{(2\pi)^2} \int_{\xi \geq \xi_1} F(\xi_1, \xi_2) e^{it(\xi_1 + \xi_2) + it\sqrt{1-\xi_1^2 + 1-\xi_2^2}} d\xi_1 d\xi_2,
\]
By the one-to-one change of variables
\[
\eta_1 = \xi_1 + \xi_2 \quad \text{and} \quad \eta_2 = \sqrt{1 + \xi_1^2 + 1 + \xi_2^2}
\]
we obtain
\[
(1) \quad e^{it\sqrt{1-\Delta}} f_1(x) e^{it\sqrt{1-\Delta}} f_2(x) = \frac{2}{(2\pi)^2} \int_{\xi \geq \xi_1} F(\xi_1, \xi_2) e^{it\eta_1 + it\eta_2} d\eta_1 d\eta_2 / |J(\xi_1, \xi_2)|,
\]
where the Jacobian $J$ is given by
\[
J(\xi_1, \xi_2) = \frac{\xi_2}{\sqrt{1 + \xi_2^2}} - \frac{\xi_1}{\sqrt{1 + \xi_1^2}}.
\]
Thus, by Plancherel’s theorem and reversing the change of variables,
\[
\|e^{it\sqrt{1-\Delta}} f_1 e^{it\sqrt{1-\Delta}} f_2\|_{L^2_{t,x}(\mathbb{R}^{1+1})}^2 = \frac{4}{(2\pi)^2} \int_{\xi \geq \xi_1} |F(\xi_1, \xi_2)|^2 \frac{d\eta_1 d\eta_2}{|J(\xi_1, \xi_2)|^2} = \frac{4}{(2\pi)^2} \int_{\xi \geq \xi_1} |F(\xi_1, \xi_2)|^2 \frac{d\xi_1 d\xi_2}{|J(\xi_1, \xi_2)|}.
\]
We will require the sharp estimate
\[
(3) \quad \frac{\xi_2 - \xi_1}{(1 + \xi_2^2)^{3/4}(1 + \xi_1^2)^{3/4}} \leq \frac{\xi_2}{\sqrt{1 + \xi_2^2}} - \frac{\xi_1}{\sqrt{1 + \xi_1^2}}, \quad \xi_2 \geq \xi_1.
\]
An unsharp version of this estimate, for large $\xi_1, \xi_2$, was obtained by Segal [25]. In order to see this, we note that by (Product-to-sum),
\[
2 \sin \theta \sin \phi \leq 1 - \cos(\theta + \phi),
\]
so that by (Power-reduction),
\[
\sqrt{\sin \theta \sin \phi} \leq \sin \left(\frac{\phi + \theta}{2}\right).
\]
Now if $0 \leq \theta \leq \phi \leq \pi$, then both sides are nonnegative. Using the fact that $\cos(\phi - \theta) \leq 1$ and multiplying both sides by $2 \sin \left(\frac{\phi - \theta}{2}\right)$, we obtain
\[
2 \sin \left(\frac{\phi - \theta}{2}\right) \cos \left(\frac{\phi - \theta}{2}\right) \sqrt{\sin \theta \sin \phi} \leq 2 \sin \left(\frac{\phi - \theta}{2}\right) \sin \left(\frac{\phi + \theta}{2}\right).
\]
Using (Double-angle) to change the left hand side and (Product-to-sum) to change the right, this yields the trigonometric inequality
\[
(4) \quad \sin(\phi - \theta) \sqrt{\sin \theta \sin \phi} \leq \cos \theta - \cos \phi, \quad 0 \leq \theta \leq \phi \leq \pi.
\]
Now using (Difference), this is equivalent to
\[
(\sin \theta)^{1/2} (\sin \phi)^{3/2} \cos \theta - (\sin \theta)^{1/2} (\sin \phi)^{1/2} \cos \phi \leq \cos \theta - \cos \phi,
\]
and finally by (Pythagorean), this is equivalent to

\[(1 - \cos^2 \theta)^{3/4}(1 - \cos^2 \phi)^{3/4} \cos \theta - (1 - \cos^2 \theta)^{3/4}(1 - \cos^2 \phi)^{3/4} \cos \phi \leq \cos \theta - \cos \phi.\]

Finally, by making the substitution

\[\cos \phi = \frac{\xi_1}{\sqrt{1 + \xi_1^2}} \quad \text{and} \quad \cos \theta = \frac{\xi_2}{\sqrt{1 + \xi_2^2}},\]

and noting that

\[1 - \cos^2 \phi = \frac{1}{1 + \xi_1^2} \quad \text{and} \quad 1 - \cos^2 \theta = \frac{1}{1 + \xi_2^2},\]

we obtain (3).

Inserting (3) into (2), we see that

\[
\|e^{it\sqrt{-\Delta}} f_1 e^{it\sqrt{-\Delta}} f_2\|_2^2 \leq \frac{4}{(2\pi)^2} \int_{\xi_2 \geq \xi_1} |F(\xi_1, \xi_2)|^2 (1 + \xi_1^2)^{3/4}(1 + \xi_2^2)^{3/4} \frac{d\xi_1 d\xi_2}{|\xi_2 - \xi_1|}
\]

\[
= \frac{2}{(2\pi)^2} \int_{\mathbb{R}^2} |F(\xi_1, \xi_2)|^2 (1 + \xi_1^2)^{3/4}(1 + \xi_2^2)^{3/4} \frac{d\xi_1 d\xi_2}{|\xi_2 - \xi_1|},
\]

where we have again used the fact that the integrand is symmetric. Given that \(f_1\) and \(f_2\) have disjoint frequency supports,

\[
|F(\xi_1, \xi_2)|^2 = \frac{1}{2} \left( |\hat{f}_1(\xi_1)|^2 |\hat{f}_2(\xi_2)|^2 + |\hat{f}_1(\xi_2)|^2 |\hat{f}_2(\xi_1)|^2 \right),
\]

so that \(\|e^{it\sqrt{-\Delta}} f_1 e^{it\sqrt{-\Delta}} f_2\|_2^2\) is less than or equal to

\[
\frac{1}{2(2\pi)^2} \int_{\mathbb{R}^2} \left( |\hat{f}_1(\xi_1)|^2 |\hat{f}_2(\xi_2)|^2 + |\hat{f}_1(\xi_2)|^2 |\hat{f}_2(\xi_1)|^2 \right) (1 + \xi_1^2)^{3/4}(1 + \xi_2^2)^{3/4} \frac{d\xi_1 d\xi_2}{|\xi_2 - \xi_1|},
\]

which yields the desired inequality.

To see that this is sharp, we first show that (3) is sharp. Note that

\[
\frac{\xi_2}{\sqrt{1 + \xi_2^2}} - \frac{\xi_1}{\sqrt{1 + \xi_1^2}} = \int_{\xi_1}^{\xi_2} \frac{d\omega}{(1 + \omega^2)^{3/2}} \leq \frac{\xi_2 - \xi_1}{(1 + \xi_1^2)^{3/2}},
\]

Thus, if we could improve on (3), and it were true that

\[
\frac{\xi_2 - \xi_1}{(1 + \xi_2^2)^{3/4}(1 + \xi_1^2)^{3/4}} \leq c \left( \frac{\xi_2}{\sqrt{1 + \xi_2^2}} - \frac{\xi_1}{\sqrt{1 + \xi_1^2}} \right), \quad \xi_2 \geq \xi_1,
\]

with \(c < 1\), then by combining with (6), we would have

\[
\frac{1}{(1 + \xi_2^2)^{3/4}} \leq c \frac{1}{(1 + \xi_1^2)^{3/4}}, \quad \xi_2 \geq \xi_1,
\]

which is manifestly false. Thus (3), and hence our bilinear estimate, is sharp.

On the other hand, to see that there are no nontrivial maximisers, it will suffice to see that equality in (3) is only attained in trivial circumstances. Now equality in (3) is attained if and only if the equality in our trigonometric inequality (4) is
attained. This happens if and only if \( \cos(\frac{\phi - \theta}{2}) = 1 \), which happens if and only if \( \phi = \theta \). By (5), this happens if and only if

\[
\frac{\xi_1}{\sqrt{1 + \xi_1^2}} = \frac{\xi_2}{\sqrt{1 + \xi_2^2}},
\]

which happens if and only if \( \xi_1 = \xi_2 \). Thus, there are no nontrivial maximisers, and we are done.

3. Restriction of the Fourier transform

We require the following estimate which we will again see is equivalent to a trigonometric inequality. In the previous section we proved this estimate with \( \alpha = 1 \) and \( C_\alpha = 1 \).

Lemma 3.1. There is a constant \( C_\alpha \) such that

\[
\frac{|\xi_2 - \xi_1|^\alpha}{(1 + \xi_1^{1/2 + \alpha/4})(1 + \xi_2^{1/2 + \alpha/4})} \leq C_\alpha \left| \frac{\xi_2}{(1 + \xi_2^{1/2})} - \frac{\xi_1}{(1 + \xi_1^{1/2})} \right|
\]

if and only if \( 1 \leq \alpha \leq 2 \).

Proof. By the substitution (5) and (Pythagorean), this is equivalent to

\[
\left( (\sin \theta)^{1/\alpha - 1/2} (\sin \phi)^{1/\alpha + 1/2} \cos \theta - (\sin \theta)^{1/\alpha + 1/2} (\sin \phi)^{1/\alpha - 1/2} \cos \phi \right)^\alpha \leq C_\alpha (\cos \theta - \cos \phi),
\]

where \( 0 \leq \theta \leq \phi \leq \pi \). By (Difference), this is equivalent to

\[
\left( (\sin \theta \sin \phi)^{1/\alpha - 1/2} \sin(\phi - \theta) \right)^\alpha \leq C_\alpha (\cos \theta - \cos \phi).
\]

By (Double-angle) and (Product-to-sum), this is the same as

\[
(\sin \theta \sin \phi)^{1 - \frac{\alpha}{2}} (2 \sin \left( \frac{\phi - \theta}{2} \right) \cos \left( \frac{\phi - \theta}{2} \right))^{\alpha} \leq C_\alpha 2 \sin \left( \frac{\phi - \theta}{2} \right) \sin \left( \frac{\phi + \theta}{2} \right).
\]

When \( 1 \leq \alpha \leq 2 \), combining (Product-to-sum) and (Power-reduction) as before, it suffices to show that

\[
\left( \sin \left( \frac{\phi + \theta}{2} \right) \right)^{1 - \alpha} (2 \sin \left( \frac{\phi - \theta}{2} \right))^{\alpha - 1} \left( \cos \left( \frac{\phi - \theta}{2} \right) \right)^\alpha \leq C_\alpha.
\]

As \( \sin \left( \frac{\phi - \theta}{2} \right) \leq \sin \left( \frac{\phi + \theta}{2} \right) \) and \( \cos \theta \leq 1 \), we have proven the inequality.

To see that the inequality of the statement is not true when \( \alpha \notin [1, 2] \), we note that it is equivalent to (7) which, by an application of (Double-angle), is equivalent to

\[
(\sin \theta \sin \phi)^{1 - \alpha/2} (\sin(\phi - \theta))^{\alpha - 1} \cos \left( \frac{\phi - \theta}{2} \right) \leq C_\alpha \sin \left( \frac{\phi + \theta}{2} \right).
\]

Using the fact that \( \sin \theta \approx \theta \) when \( 0 \leq \theta \leq \pi/2 \) and \( \cos \theta \approx 1 \) when \( 0 \leq \theta \leq \pi/4 \), for \( 0 \leq \phi, \theta \leq \pi/2 \), this is in turn equivalent to

\[
(\theta \phi)^{1 - \alpha/2} (\phi - \theta)^{\alpha - 1} \leq C_\alpha (\phi + \theta).
\]

Taking \( \theta \) close to zero, we see that such an inequality is not possible when \( \alpha > 2 \), and when \( \phi \) and \( \theta \) are close we see that such an inequality is not possible when \( \alpha < 1 \). □
The first estimate of the following kind was proven by Stein, who meaningfully restricted the Fourier transform to the sphere. Fefferman, Stein and Zygmund [12, 34] restricted to the circle in the sharp range. The same estimates holds for a compact piece of the parabola. For certain Lebesgue exponents, the estimate for the parabola is scale invariant, and so the whole parabola can be considered.

The following estimate is well-known when the measure is assumed to be compactly supported (see [29, pp. 412]). The following global version was proven by Segal [25] in the case \( q = 2 \) and \( p = 6/5 \), and extended by Strichartz [30] to the range \( q = 2 \) and \( 1 < p \leq 6/5 \). In fact the result of Segal followed directly (by duality) from the earlier work of Sjölin [28] who treated the whole line \( 3q = p' \) with \( 1 \leq p < 4/3 \) (the authors thank J. Wright for bringing this to their attention). We remark that the power of the weight in the definition of the measure is not predicted by scaling (considering \( \sqrt{1 - \Delta} \approx \sqrt{-\Delta} \)). However, we will prove that it is sharp in the sense that a weight with less decay would not suffice. It is the precise price one has to pay for the fact that the hyperbola has almost no curvature near to infinity.

**Theorem 3.2.** Let \( 3q \leq p' \leq 4q \) and \( \frac{2}{p} + \frac{2}{q} < 3 \). Then

\[
\left( \int_{\mathbb{R}} |\hat{F}(\xi, \sqrt{1 + \xi^2})|^q \frac{d\xi}{\sqrt{1 + \xi^2}} \right)^{1/q} \leq C\|F\|_{L^p(\mathbb{R}^2)}.
\]

**Proof.** By duality, the desired estimate is equivalent to

\[
\|e^{it\sqrt{1-\Delta}}f\|_{L^q_{x,t}} \leq C\|(1 + |\cdot|^2)^{3p'/4} \hat{f}\|_p, \quad 1 < \frac{2}{p} + \frac{2}{q},
\]

where \( 3p' \leq q \leq 4p' \). We follow the well-known argument which can be traced back to the work of Carleson–Sjölin [5] for Bochner–Riesz means. By (1), we have

\[
e^{it\sqrt{1-\Delta}}f(x) e^{it\sqrt{1-\Delta}}f(x) = \frac{2}{(2\pi)^2} \int_{\xi_2 \geq \xi_1} \hat{f}(\xi_1) \hat{f}(\xi_2) e^{i\xi_1 \eta_1 + it\eta_2} \frac{d\eta_1 d\eta_2}{|J(\xi_1, \xi_2)|},
\]

where the Jacobian \( J \) is given by

\[
J(\xi_1, \xi_2) = \frac{\xi_2}{\sqrt{1 + \xi_2^2}} - \frac{\xi_1}{\sqrt{1 + \xi_1^2}}.
\]

This time we use the Hausdorff-Young inequality, instead of using Plancherel’s theorem, so that

\[
\|e^{it\sqrt{1-\Delta}}f\|_{L^q_{x,t}}^2 = \|e^{it\sqrt{1-\Delta}}f e^{it\sqrt{1-\Delta}}f\|_{L^{q/2}_{x,t}}
\leq C \left( \int_{\xi_2 \geq \xi_1} |\hat{f}(\xi_1) \hat{f}(\xi_2)|^{(q/2)'} \frac{d\eta_1 d\eta_2}{|J(\xi_1, \xi_2)|^{(q/2)'}} \right)^{(q/2)'}
\leq C \left( \int_{\mathbb{R}^2} |\hat{f}(\xi_1) \hat{f}(\xi_2)|^{(q/2)'} \frac{d\xi_1 d\xi_2}{|J(\xi_1, \xi_2)|^{(q/2)' - 1}} \right)^{(q/2)'}.
\]

Using Lemma 3.1, this is bounded by

\[
C_\alpha \left( \int_{\mathbb{R}^2} |\hat{f}(\xi_1) \hat{f}(\xi_2)|^{(q/2)'} \left( \frac{1 + \xi_1^2}{\xi_2 - \xi_1^a} \right)^{(1/2 + a/4)(1 + \xi_2^2)^{1/2 + a/4}} \frac{d\xi_1 d\xi_2}{|\xi_2 - \xi_1|^a} \right)^{(q/2)'}.
\]
for all $1 \leq \alpha \leq 2$. The proof is completed by Hölder’s inequality followed by an application of the Hardy–Littlewood–Sobolev inequality (see for example [30]). For this we require that $q > 2(1 + \alpha)$ and $\frac{1}{p'} = \frac{2 + \alpha}{q}$, which yields the result. □

We remark that a larger range of $p$ and $q$ can be obtained by interpolating between this estimate and that due to Strichartz mentioned previously.

The fact that $3q \leq p'$ is a necessary condition is well-known (see for example [30] or [32] or the references therein). When $3q = p'$ our condition $2p + 2q \leq 3$ becomes $q \geq 4/3$ which is also well-known to be sharp, however we have no reason to believe that the condition is sharp when $3q < p'$. Similarly we have no reason to believe that the condition $p' \leq 4q$ is necessary, however we cannot extend the range via these arguments as Lemma 3.1 is false when $\alpha > 2$.

To see that the power of the weight is sharp, we consider the dual inequality

$$
\| e^{it\sqrt{1-\Delta}} f \|_{L^q_{x,t}} \leq C \|(1 + |\cdot|)^{s/2} \hat{f} \|_p.
$$

No such inequality can hold when $\sqrt{1-\Delta}$ is replaced by $\sqrt{-\Delta}$, because then the waves do not disperse, making the left hand side infinite. For the Klein–Gordon equation, these waves keep together for only a finite amount of time, and it is such a wave which proves that $s \geq 1/p'$ is necessary.

Indeed, consider the datum $\hat{f}(\xi) = \psi(R^{-1} \xi)$, where $\psi$ is smooth and supported in the unit annulus. Then, by scaling,

$$
e^{it\sqrt{1-\Delta}} f(x) = \frac{R}{2\pi} \int_{\mathbb{R}} \psi(\xi) e^{i(Rx\xi + Rt\sqrt{\xi^2 + R^{-2}})} d\xi.
$$

Now as $\sqrt{\xi^2 + R^{-2}} = \xi + \frac{1}{2\pi} R^{-2} + O(R^{-4})$, we see that if $|x + t| \leq \frac{1}{8} R^{-1}$ and $|t| \leq \frac{1}{8} R$, then

$$
|Rx\xi + Rt\sqrt{\xi^2 + R^{-2}}| = |R(x + t)\xi - Rt\xi| \leq |R(x + t)\xi + \frac{1}{2\pi} R^{-1} + O(R^{-4})|
$$

so that the phase is not large enough for the exponential function to oscillate. This yields

$$
\| e^{it\sqrt{1-\Delta}} f \|_{L^q_{x,t}} \geq c R^{1/q} R^{-1/q} = c R.
$$

On the other hand, it is easy to see that

$$
\|(1 + |\cdot|^2)^{s/2} \hat{f} \|_p \leq C R^{1/p + s},
$$

so that a necessary condition for (8) to hold is that $s \geq 1/p'$.

This is a natural continuation of the results obtained in [2] and [23]. The authors would like to thank Neal Bez and Yoshio Tsutsumi for the numerous conversations which have influenced this work.
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