Logarithmic bred vectors. A new ensemble method with adjustable spread and calibration time

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[1] The breeding method is a conceptually simple and computationally cheap ensemble generation technique. Bred vectors (BV) are dynamically obtained from the nonlinear model and correspond to the spatial structures with fast-growing fluctuations at each time. These vectors have a characteristic localized spatial structure, with only a small number of significant values corresponding to the leading fluctuation areas. The temporal and spatial growth of the BV interacts making the spatial structure a key factor of the dynamics. In this paper we introduce a new breeding technique, Logarithmic Bred Vectors (LBV), which allows growing vectors with tunable spatial structure, more (or less) localized. This yields ensembles with different spatiotemporal dynamics (different spread, etc.). This is done by introducing a new parameter (the geometrical mean) which controls the spatial correlation of the bred vectors, from uncorrelated vectors similar to random perturbations, to fully correlated and localized vectors similar to standard bred vectors. The new method increases the diversity of the ensemble and allows the spread to grow faster preserving the model performance in terms of the root mean square error. Consequently, the ensembles can be calibrated for a desired lead time (for instance a shorter forecast range). The concepts are illustrated using a chain of diffusively coupled Lorenz systems. Preliminary evidence of the possible application in operative weather prediction models is also presented.


1. Introduction

[2] One of the main problems in numerical weather forecasting is the quantification of the different sources of uncertainty influencing regional and global circulation models [Kalnay, 2002]. Ensemble Prediction Systems (EPS) are the main operative tool to deal with this problem. These techniques work by estimating the evolution of errors using a finite sample, or ensemble, of deterministic forecast integrations. Among the main sources of uncertainty, errors in the initial conditions due to nonlinearity (chaos) has been the most studied in meteorology [Lorenz, 1965]. In this case, the individual members of the ensemble are obtained by perturbing the initial condition (the analysis) according to the estimated analysis errors. However, the phase-space of the operational forecast models is very high-dimensional, and the size of the ensemble is strongly limited by the computational cost required to integrate each of the members. Therefore it is important to sample the fastest growing directions, so they evolve to significantly different spatiotemporal patterns, capturing different sources of uncertainty.

The main difficulty of the EPS approach lies in the choice of equiprobable spatial perturbations representative of the fastest growing directions in the phase space.

[3] One of the most popular perturbation techniques is the breeding method, which computes a limited set of fast-growing vectors in phase space (Bred Vectors, BV) that are used to generate the initial perturbations for the ensemble [Toth and Kalnay, 1993, 1997]. BV are generated dynamically from the nonlinear model in a two step growing/rescaling process. Thus an advantage of this method is that it does not require additional information such as the tangent linear counterpart used in the singular vector approach [see, e.g., Molteni et al., 1996]. However, as a shortcoming, the spatiotemporal effects of nonlinearities have not been fully understood in this framework due to the lack of an appropriate theory. While the finite-time and finite-resolution effects on predictability of extended systems have been extensively studied using a generalization of standard indicators [Boffetta et al., 2002], the use of finite perturbations as ensembles of prediction systems has been scarcely explored. Both are two distinct aspects of the predictability problem which are essential in the short and intermediate forecasting range. In a recent paper, López et al. [2004] characterize the spatiotemporal growth of finite, as opposed to infinitesimal, perturbations working in the framework of kinetic rough surfaces theory [Barabási and Stanley, 1995]. This theory characterizes the dynamics by considering not only the spatially averaged exponential
growth in time (the Lyapunov exponents) but also the
interplay with the changing spatial structure, characterized
in terms of the spatial correlation, which becomes a key
component in spatiotemporal systems. The growth of finite
perturbations is described by precise power laws using an
analogy with the growth of perturbations in self-affine
surfaces [Pikovsky and Politi, 1998].

[4] In this paper we apply this characterization to the
generation of BVs, which are also finite perturbations
growing in time, but rescaled at fixed time intervals. We
show that the standard rescaling procedure (using a norm),
confers a characteristic localized spatial structure to the
resulting vectors, which tend to the leading local Lyapunov
vector (in an homogeneous system). Thus the variety of
the resulting ensemble is limited, since all the bred vectors
(started from different random initial perturbations) tend
to similar spatial patterns. We present a modified technique
(logarithmic bred vectors, LBV) which allows breeding
fluctuations around the control trajectory with a desired
correlation structure, more or less localized, controlled by a
new parameter, the geometric mean of the vector (the norm
of the logarithmic vector). Moreover, we show that this
parameter also controls the diversity of the different bred
vectors and, hence, the growth of the ensemble spread. Thus
using the proposed methodology the spread can be tuned
more or less dispersive preserving the ensemble perfor-
mance. We also analyze the implications of the proposed
method in the calibration process needed to adjust the
practical operative range of the resulting EPS systems.
Calibration is needed to estimate the operative range of
the EPS method.

[5] The concepts are illustrated using a chain of diffusively
coupled Lorenz systems. This toy model displays
many of the characteristics of atmospheric numerical weather
prediction models and has been previously used in similar
studies [Lorenzo et al., 2003]. However, we also give some
preliminary evidence showing that some of the results also
stand in the operative ECMWF medium-range numerical
weather EPS system. Thus this technique could result in an
important advance of operative ensemble forecast methods,
although further research in atmospheric models is needed.

[6] This paper is organized as follows. In section 2 we
describe the simple model used in this paper. Section 3
describes the standard bred vector method used to perturb
the initial conditions for generating ensembles. Section 4
analyzes the dynamical growth of finite fluctuations, and
presents the main results characterizing the spatiotemporal
evolution of BV. Logarithmic bred vectors are introduced in
section 5. Section 6 analyzes the ensembles resulting from
this method. On the one hand, section 6.1 analyzes the
spread of the resulting ensembles; on the other hand,
section 6.2 illustrates their calibration using Talagrand
diagrams. Some results concerning atmospheric circulation
models are presented in section 7. Finally, the main con-
clusions are given in section 8.

2. Description of the Models

[7] Coupled chaotic “toy” systems are simple models
which display many of the characteristics of numerical
atmospheric circulation models, such as sensitive depen-
dence to the initial conditions, characteristic spatial struc-
tures, etc. [see Lorenzo et al., 2003 for further details].
These “toy” models are useful to illustrate new ideas and
methods which may be later extended to more complex
systems with a multitude of coexisting scales. We consider a
one-dimensional chain of diffusively coupled Lorenz mod-
els in chaotic regime ($\sigma = 10$, $R = 28$ and $b = 8/3$):

$$\begin{align*}
\dot{x}_i &= \sigma(y_i - x_i) \\
\dot{y}_i &= R x_i - y_i - x_i z_i + D (y_{i-1} - 2y_i + y_{i+1}) \\
\dot{z}_i &= x_i y_i - b z_i
\end{align*}$$

where the variable $y_i$ of each system is coupled with the two
immediate neighbors $y_{i-1}$ and $y_{i+1}$. $l = 2$, \ldots, $L - 1$ (circular
conditions are used for the boundary cells). A diffusion
coefficient $D$ regulates the spatial coupling. This model has
dissipative and forced terms in order to emulate atmospheric
circulation models. Other kind of problems more focused on
pure turbulence phenomena are modeled by coupled maps
with conservative properties that produce scale invariant
patterns [Bohr et al., 1992]. Figure 1 shows the evolution of
the $x$ field of a chain of $L = 50$ Lorenz systems integrated
with an Euler method with a time step of $dt = 0.002$. Note
that from now on we will focus the study on the $x$ field,
although the same results are obtained with the $y$ field.
Two different coupling strengths are shown in this figure: $D = 0$
and $D = 0.5$, corresponding to uncoupled systems (that is,
with a low-dimensional attractor) and spatial chaos (a high-
dimensional attractor), respectively. The dynamics of the
system results in spatial patterns with a characteristic scale.
This characteristic length can be computed as the length
where the spatial correlation function $c(s) = \langle y_j(t) y_{j+s}(t) \rangle_{l,t}$
saturates; $\langle \rangle_{l,t}$ denotes an average in $l$ and $t$, and $\langle \rangle$
denotes the standardized variables, with zero mean and unit variance.
Figure 2 shows the spatial correlation for the chain of
Lorenz systems shown in Figure 1 which saturates for
$l = 1$ and $l = 6$ for $D = 0$ and $D = 0.5$, respectively. Note that in
the uncoupled case there is no spatial dynamics, whereas in
the coupled case the spatial dynamics confer a characteristic
spatial correlation to the system.
Toth and Kalnay

Spatial correlation function for the chain of

\[ D = 0.5 \]

which gives characteristic spatial scales of \( l = 1 \) and \( l = 6 \), respectively.

It is important to remark that this simple homogeneous system has a single characteristic spatial scale (the correlation length), whereas global atmospheric circulation models exhibit different characteristic scales in an inhomogeneous space. An extension of the Lyapunov analysis can be made using finite size Lyapunov exponents [Boffetta et al., 1998] in systems with low dimensional chaos. In extended systems the study is more difficult, since different effects in time and space are mixed together obscuring the analysis. However, a recent paper shows that the same basic features of spatiotemporal systems are present both in coupled chains as in weather models [Primo et al., 2007]. Thus these toy models are a simple testbed to introduce new methods and algorithms.

3. Bred Vectors

During the operative cycle of numerical weather forecasting, assimilation procedures provide the best estimation of the true state of the atmosphere by combining information from model and observations: the so-called control analysis. However, different sources of uncertainty in the initial conditions (such as errors in the measurements, lack of complete data coverage, etc.), as well as in the assimilation process, produce differences between the analysis and the true state of the atmosphere: the so-called “analysis error”. The main goal of EPS is sampling these errors by perturbing the analysis to obtain a set of equiprobable initial conditions; then, multiple runs of the model yield an ensemble of forecasts which provides a particular posterior prediction based on the prior initial uncertainty.

Several techniques have been proposed for generating the initial perturbations, sampling by different procedures the fastest growing directions in the high-dimensional phase space of the system. The breeding method, proposed by Toth and Kalnay [1993], is a two step growing/rescaling process where an initial random perturbation is dynamically grown and periodically rescaled to avoid divergence, keeping the spatial structure. The perturbation is obtained as the difference between the control and a perturbed trajectory both computed using the original nonlinear system. The resulting rescaled difference has certain skill representing the instabilities that generate the “errors of the day” [Corazza et al., 2003] and can, therefore, be used to estimate the shape or pattern of the forecast error.

3.1. The Breeding Cycle

For the model described in section 2, a standard BV can be obtained from an initial condition (the control analysis), \( x_0(0) \), \( l = 1, \ldots, 50 \), through the following breeding cycle:

1. Start with an arbitrary perturbation (e.g., random normal) \( \delta x(0) \) with a prescribed fixed arbitrary norm, for instance the Euclidean norm \( \| \delta x(0) \| = \mu_0 \), which is kept constant during the breeding cycle. Add the perturbation to the analysis to obtain the perturbed initial condition \( x_0(t) = x_0(0) + \delta x(0) \).
2. Use the nonlinear system to integrate an arbitrary time step \( \Delta t \) (the breeding rescaling time) both the control and the perturbed trajectories, from the initial conditions \( x_0(k-1) \Delta t \) and \( x_0(k-1) \Delta t \), respectively.
3. Obtain the resulting fluctuation by subtracting the perturbed integration from the unperturbed one: \( \delta x(k \Delta t) = x_0(k \Delta t) - x_0(k \Delta t) \).
4. Rescale the fluctuation down to the initial norm \( \mu_0 \) obtaining the rescaled perturbation (bred vector) at time \( k \Delta t \):

\[
\text{bv}(l, k \Delta t) = \frac{\mu_0}{\mu(k \Delta t)} \delta x(l)(k \Delta t),
\]

where \( \mu(t) = \| \delta x(t) \| \).
5. Redefine the perturbed trajectory \( x_0(k \Delta t) = x_0(k \Delta t) + \text{bv}(l, k \Delta t) \), and set \( k = k + 1 \) for the next iteration step.

The rescaling equation (2) avoids the exponential divergence of BV and, thus, the breeding process can be interpreted as a dynamical redistribution of the constant norm favoring fast-growing perturbation areas, i.e., those regions where the perturbations have grown the most in the recent past. Thus in principle, these vectors are the best candidates for introducing fast-growing uncertainty in the initial condition.

An ensemble of forecasts is obtained by breeding a set of BV from different initial perturbations following the above procedure for each of them, and using the vectors resulting at each time to perturb the initial conditions.

Figures 3a–3c show the temporal sequence of bred vectors, \( \text{bv}(l, t) \), for three different breeding cycles obtained applying the above procedure to a chain of \( L = 50 \) diffusively coupled Lorenz systems with diffusion parameter \( D = 0.5 \), starting from the same initial condition but with different initial random perturbations and different rescaling norms \( \mu_0 = 0.5 \) (a), 0.1 (b), and \( 10^{-4} \) (c). The above figures show the characteristic localized spatial structure of BV which evolve dynamically according to the larger error growth areas. Note that the choice of the rescaling norm only affects the amplitude of the resulting bred vectors, but not their spatial pattern. Note that in all these cases the amplitude of the vectors is small enough so that nonlinear effects do not occur.

3.2. Properties of BV

The breeding process does not depend neither on the choice of the norm [Corazza et al., 2003] nor on the rescaling time \( \Delta t \), as long as it is kept small enough [Kalnay et al., 2002]. In this paper we consider the standard Euclidean norm and normalize every integration step \( \Delta t = \)
$dt = 0.002$ (see section 2). In practice, the rescaling amplitude for BV is usually chosen at the same level as the analysis errors, so the resulting BV can be directly used to perturb the initial conditions. However, an arbitrary rescaling amplitude can be used during the breeding cycle, renormalizing the resulting BV to the desired value before using them to perturb the initial condition in the operative cycle. In this paper we use a constant value, equivalent to 5% of the amplitude of the system variables, as the norm of the analysis errors.

[22] Regarding the independence of the BV, although the three breeding cycles above have been started from three different initial perturbations, after a short transient the bred vectors are very similar to each other, indicating a “low dimensional” breeding space. This is the result of the leading finite-time Lyapunov exponent which dominates the growth of perturbations and, thus, all bred vectors are dominated by this component of the dynamics. Note that real numerical atmospheric models are not homogeneous spatial systems, like those studied in this paper, and different spatial regions coexist within the same space. This situation makes difficult the understanding of the properties associated with BV. For instance, there seems to be enough evidence to claim that bred vectors remain globally distinct, rather than converging to a single vector. However, it has been recently found that the local similarity of BV is higher than it could be expected in the light of the global result [see, e.g., Annan, 2004]. The lack of a theory explaining the growth of finite fluctuations (a generalization of Lyapunov theory) obstructs the understanding of these problems. Thus in practice, several ad-hoc techniques have been introduced to overcome different problems arising in practical applications of the breeding technique. For instance, to generate more diverse ensembles of BV, different alternatives have been introduced, such as adding small random perturbations during the breeding process, or growing locally orthogonal bred vectors [Annan, 2004], but no spatiotemporal nonlinear theoretical framework has been used so far to understand the limitations of the standard breeding technique.

Figure 3. Evolution of bred vectors in a chain of $L = 50$ diffusively coupled Lorenz systems with diffusion parameter $D = 0.5$. Panels (a)–(c) show three different standard bred vectors obtained with constant norms $\mu_0 = 0.5, 0.1, \text{ and } 10^{-4}$, respectively. Panels (d)–(f) show three different logarithmic bred vectors with $\rho_0 = 0.5, 0.1 \text{ and } 10^{-2}$, respectively.
In this paper we use a recent characterization of the spatiotemporal dynamics of finite perturbations, or fluctuations, as a dynamical “interface” developing (growing) in a spatial support. This characterization is based on a well-established theoretical framework of kinetic rough surfaces [see, e.g., Barabási and Stanley, 1995]. In the next section we briefly describe this theory.

4. Growth of Noninfinitesimal Perturbations

As shown in the above section, bred vectors are generated as finite fluctuations growing in time, but rescaled at fixed time intervals to avoid exponential divergence. The growth of errors, or finite fluctuations, depends on the interplay between the following two key components of the dynamics [López et al., 2004]:

- The exponential growth of errors in time, which follows a non-normal distribution even in simple low-dimensional models. For instance in the work of Boffetta et al. [2002] it can be seen how for any finite time fluctuation the lognormal distribution is a good approximation, although this result is not valid for extreme multiplicative events. This problem makes the analysis more difficult, since standard Gaussian statistical techniques are not appropriate; for instance, in the Lorenz model, the average growth rate is two times shorter than the most probable one.

- The spatial correlation which also grows/decays in time interacting with the exponential growth of errors. For instance, an initially spatially uncorrelated random perturbation added to the system becomes correlated in time as a result of the system dynamics. This fact interacts with the exponential growth of errors reinforcing or weakening it in local regions of the space.

A first attempt to characterize the growth of finite perturbations considering jointly the above two factors was proposed by [López et al., 2004]. They show that the spatiotemporal dynamics of finite fluctuations cannot be analyzed using the original variables but using the logarithmic transformed counterparts, which are approximately Gaussian. To illustrate this point, we have computed the fluctuation values $\delta x(t)$ for a 50-member ensemble of initial perturbations for different time values $t = 0, 1, 3, 5, 15$. There were considered 50 different control trajectories. In Figure 4 we show the distribution of the error fluctuations for $D = 0.5$. The perturbations are normally random in the initial step (see panel(a)). However, as time goes by, the spatial correlation grows and the distribution of errors becomes quickly localized in space (panels b-d). The histograms seem to be just a single bar because most of the perturbations are really small compared to a few big values. The inset of these figures show the distributions of the corresponding logarithmic fluctuations $P(\log(\delta x(t)))$. Insets (b)–(d) show that even if the histograms of the perturbation distribution become correlated in time as a result of the system dynamics (panel d). Therefore it follows that finite fluctuations can be characterized in terms of lognormal distributions.

Figures 5a–5c shows the logarithm of the bred vectors shown in Figure 3. The resulting patterns are no longer localized and can be analyzed using Gaussian techniques. For instance, Figure 6a shows the spatial correlation of these patterns which results in a characteristic spatial correlation length. Then, the prescribed fixed norm does not affect the spatial correlation. Note that the spatial correlation of the original bred vectors (Figure 3) cannot be directly computed, due to the lognormal character of the spatial patterns which leads a fluctuating ill-posed correlation function.

The above results suggest that the spatiotemporal analysis of bred vectors requires the study of the logarithmic patterns, since statistics like the spatial correlation, the mean fluctuation and the variance are well defined for these patterns.

5. Logarithmic Bred Vectors (LBV)

According to the above results, we propose a new breeding method working in the logarithmic space. Instead of rescaling the fluctuations in the original space using (2),
we rescale the variables in the logarithmic space (note that this is equivalent to rescaling the variables in the original space, but using the geometric mean instead of a norm). Thus we introduce a new breeding method replacing (2) by

$$lbv(l, k, \Delta t) = \frac{r_0}{\rho(k, \Delta t)} \delta x_l(k, \Delta t),$$

where

$$\rho(t) = \left( \prod_{l=1}^{L} |x_l(t)| \right)^{1/L}$$

is the geometric mean of the vector. We refer to this method as “logarithmic breeding”, and we denote the resulting vectors as Logarithmic Bred Vectors (LBV). For instance, Figures 3d–3f show the evolution of three different logarithmic breeding cycles generated with different rescaling values, $r_0 = 0.5, 0.1$ and $10^{-4}$, for the same control trajectory. Comparing these figures with those corresponding to the standard method, shown in panels (a)–(c), it can be easily shown that the spatial patterns of the resulting LBV depend on the value $r_0$, contrary to the case of standard BV. Thus the spatial pattern of the bred vectors can be tuned using the parameter $r_0$ to obtain very localized bred vectors (corresponding to the leading exponential growth area, similar to the standard BV), or more uniform bred vectors (resulting from a mixture of different exponential growth areas, i.e., different Lyapunov exponents). Moreover, the LBV tend to be more independent as the parameter $r_0$ is increased or, in other words, as the vectors are less localized and take into account the effect of more fast growing modes. Note that logarithmic bred vectors with small $r_0$ value are similar to standard bred vectors, as shown in panel (f). Note that, in this case, the size of the fluctuations keep small enough to avoid nonlinear saturation effects due to the finite size of the system. These nonlinear effects take place for larger values of the rescaling parameter $r_0$. For instance, for $r_0 = 0.1$, the fluctuations take maximum local values close to the system amplitude, which is the maximum value attained by the system variables. As we shall see later, this effect modifies the dynamics of the breeding cycle freezing the spatial correlation to an arbitrary value, thus allowing to choose the

**Figure 5.** As in Figure 3 but with the evolution of the interfaces (logarithms of the bred vectors).
characteristic spatial scale of fluctuations (from localized delta-like ones, to uniformly spaced on the system). Simply stated, when the leading fluctuations reach the amplitude of the system, they cannot grow any longer and alternative fluctuations grow at a different location revealing an alternative region with fast growth of errors. It is important to remark here that although the fluctuations can locally reach a large amplitude, they are real fluctuations generated by the system and, hence, are compatible with the system dynamics; in other words, the perturbed orbit is within the “attractor” of the system, as well as the control orbit. Therefore these fluctuations do not create any distortion on the system, making it unstable or divergent. That could only happen if the perturbations are not real fluctuations of the nonlinear system, for instance random perturbations or finite fluctuations generated at a different time. This surprising effect of large fluctuations being dynamically compatible with the system dynamics is a property of nonlinear systems and require appropriate techniques for their understanding.

[31] One may argue that the large fluctuations generated with the logarithmic breeding technique are not appropriate for representing the “error of the day” structure, since they are affected by nonlinear saturation effects which do not seem to be connected with fast growing areas. However, as we shall see later, this nonlinear effect acts by filtering the leading fast growing vector allowing several independent fast growing areas to coexist simultaneously in the same bred vector. Thus in spite of their large amplitude, logarithmic bred vectors are associated with fast growing perturbation areas. In this case, a rescaling of the amplitude (norm) of the bred vectors is required to adapt their amplitude to the analysis errors before using them to produce an ensemble.

[32] As shown in Figures 3d–3f, logarithmic bred vectors grow following different spatiotemporal patterns depending on the initial amplitude value. This behavior can be theoretically described by using an analogy with the theory of kinetic rough interfaces [see Barabási and Stanley, 1995], for an introduction to this field]. Rough interfaces (curves or surfaces) appear in many practical problems of Physics, Geology and Biology, and their spatiotemporal dynamics is statistically characterized according to precise fractal power laws. The first application of this theory to the study of fluctuations was done by Pikovsky and Politi [1998], showing that the growth of logarithmic infinitesimal errors in spatiotemporal systems is described by precise power laws and exponents that also appear in a wide class of systems, known as the Kardar-Parisi-Zhang (KPZ) universality class [see Barabási and Stanley, 1995, for more details]. This allowed them to characterize Lyapunov exponents in this framework.

[33] An extension of this study to finite perturbations has been done by López et al. [2004], considering the logarithm of absolute values of fluctuations $g(t) = \log(|x(t)|)$ as a kinetic roughening process defined in space $l$ and time $t$. Within this framework, the spatiotemporal dynamics of $g(l, t)$ is characterized by three parameters: the spatial mean, the width, and the correlation length. In particular, bred vectors generated in the breeding cycle (3) are finite perturbations, so applying the above methodology to $h(l, t) = \log(|lbv(l, t)|)$, we can describe the dynamics of the LVB as follows:

\[ h(l) = \frac{1}{E} \sum_{j=1}^{L} h(l, t) = \ln(|\text{lbv}(l, t)|)^2 = \ln \rho_0. \]  

Therefore the mean value of the transformed bred vectors keeps constant in time due to the rescaling procedure used for logarithmic bred vectors.

[35] If we consider the growth of $g(l, t)$ instead, we find that the mean value grows in time as $\ln \rho(t)$. Moreover, since the geometrical mean evolves as $\rho(t) \sim \rho_0 e^{\lambda t}$ [see López et al., 2004], then the mean of the curve evolves as $\ln \rho(t) \sim \ln \rho_0 + \lambda t$, where $\lambda$ is the leading Lyapunov exponent. Thus the breeding technique described in (3) can be seen as a practical form of removing the dominant exponential behavior of the leading Lyapunov vector from the spatiotemporal dynamics of fluctuations.

[36] The width (or roughness) of the curve is defined by the standard deviation of fluctuations around the mean:

\[ w(t) = \left( \frac{\langle h(t) \rangle - \langle h(t) \rangle^2}{\langle h(t) \rangle^2} \right)^{1/2}. \]  

According to López et al. [2004], the width grows as a power law of the form $w(t) \sim t^\beta$, with $\beta = 1/3$, and it saturates after a time $t_s \sim L^2$ to a system-size dependent value $w_s \sim L^\alpha$, being $z = \alpha/\beta = 3/2$ (the dynamic exponent), $L$ the size of the system, and $\alpha = 1/2$ (the roughness exponent in one dimension). These exponents correspond to the KPZ universality class. In particular, bred vectors also evolve following these power laws. Figure 7 shows the scaling behavior of the width for (a) standard BV and (b) LBV with different $\mu_0$ and $\rho_0$ values, respectively. To
avoid fluctuations dependent on the control orbit, the curves shown correspond to averaged values obtained from fifty realizations of the control orbit. The log-log representation in panel (b) shows the trend $\log w(t)^2 = 2\beta \log t$, in agreement with the theory. Note that all standard BV exhibit the same saturation regime, whereas LBV saturates at different times, depending on the value $\rho_0$. This phenomenon is known as premature saturation due to nonlinearity [see Primo et al., 2006, for more details].

[37] The correlation length $\xi(t)$ is defined as the characteristic distance over which the points are correlated in space. The correlation length grows as $\xi(t) \sim t^{1/z}$ until saturation due to the finite size $L$ of the system (the case of unconstrained fluctuations) or to the finite amplitude of the attractor (the case of renormalized perturbations using LBV).

[38] The above properties can be summarized in the following important relationship between the surface width and the spatial correlation length: $w(t) \sim t^{1/z} \sim \xi(t)$. If we consider the scaling and saturated regimes we have the following relationship, which characterizes the growth of the fluctuations:

$$w(t) \sim \begin{cases} \xi(t)^\alpha & \text{for } t \leq t_s \\ w_i = \min(L^\alpha, \xi_0) & \text{for } t \geq t_s. \end{cases}$$

(7)

Thus Figure 7 also shows the evolution of the spatial correlation of the bred vectors. In the standard case all the bred vectors exhibit the same behavior, saturating at a fixed value $\sim L^{2\alpha}$ corresponding to the finite-size saturation of the system; however, in the case of LBV the width saturates at a specific value $\sim \xi_0^\alpha$, depending on the parameter $\rho_0$ (indicated with horizontal dashed lines in panel a), which also determines the specific saturation time $t_s$ (indicated with vertical dashed lines in panel b).

[39] The theory of kinetic rough interfaces provides the analytical description of the LVB growth and the relationship between saturation time and space scales. LBV verify different correlation lengths depending on the initial amplitude value and thus, different spatial patterns with characteristic scales. Therefore the logarithmic breeding process can be viewed as a practical mechanism for controlling the spatial correlation of bred vectors by saturation of the correlation length due to nonlinear effects appearing when $\rho$ reaches a given threshold.

6. Analysis of the LBV Ensembles

[40] In the previous sections, the logarithmic breeding method was characterized in terms of the width of the fluctuations (the variance of the logarithms of errors) or, equivalently, the spatial correlation length, which is an indicator or the spatial structure. In this section we analyze the influence of these magnitudes on the spread and calibration of the resulting ensemble. In all cases, the bred vectors are renormalized to a value equal to 5% of the system amplitude before using them to perturb the initial condition generating the members of the ensemble. Thus all the ensembles compared in this section will have the same initial dispersion.

6.1. Tuning the Ensemble Spreads

[41] The most intuitive magnitudes associated with the dynamics and performance of the ensemble are the spread and the Root Mean Square Error (RMSE) along time. On one hand, the spread is defined as the standard deviation of a characteristic magnitude of the system (a variable in a point or a region of the space) over the ensemble. On the other hand, the RMSE evaluates the performance of the ensemble forecast by considering the mean error between the ensemble mean and the “real trajectory” or “verifying analysis”. In an ideal example, the real trajectory should evolve as a member of the ensemble. In other words, the spread of the ensemble and the RMSE should evolve in a similar way.

[42] The perturbations added to generate the ensemble of forecasts are generated to simulate the analysis error in the best possible way. However, in a real system the spatial structure of the analysis error is unknown and it depends on both the dynamics of the model and the spatial distribution of the observations. In an attempt to simulate the spatial structure variability of the analysis error, in this example the “real” trajectory in each realization is considered as a trajectory obtained from perturbing the initial state with a perturbation chosen from a sample of random perturbations. These sample consists of random perturbations from a Gaussian distribution, standard bred vectors and logarithmic
bred vectors. All the perturbations of the sample are equiprobable to be chosen. As shown in previous sections, the spatial structure of the bred vectors depends on the breeding method used and the initial amplitude/norm considered. To generate the sample of perturbations we have considered amplitudes/norms equal to $1, 0.5, 10^{-1}, 10^{-2}, 10^{-3}, 10^{-4}, 10^{-5}$. Thus the spatial structure of the analysis error considered in this example goes from very localized to random spatial structures. The sample of perturbations is generated in addition to the ensemble of perturbations that generate the forecasts but with the same amplitude (all of them are renormalized to 5% of the system amplitude). Hence in the initial time the real state behaves as a member of the ensemble, and so at $t = 0$ the spread and the RMSE is equal for all the methods. Then, the real state and perturbed members evolve following the same model equations (model (1)). That is, perfect model scenario is assumed and only errors in the initial conditions are taken into account.

[43] We generated 60 realizations of different 10-member ensembles for a chain of Lorenz systems with diffusion $D = 0.5$ using positive and negative perturbations from standard and logarithmic bred vectors (symmetric ensembles with pairs of equal perturbations with opposite sign). Figure 8 shows the evolution of both the spread (black and dark blue) and RMSE (gray and light blue) of the ensembles generated using standard (BV) and logarithmic Bred Vectors (LVB) with values $0.5, 10^{-1}, 10^{-2}, 10^{-3}, 10^{-4}, 10^{-5}$ for $\mu_0$ and $\rho_0$, respectively. This figure shows that the ensemble dispersion (spread) can be increased or decreased by changing the amplitude $\mu_0$ of the LBV. The RMSE of the LBV always keeps smaller than the RMSE obtained by the standard breeding technique. Thus by using the LVB technique bigger spread can be obtained without increasing the RMSE and so with no significant change in the performance of the model. The behavior of the spread in the standard BV is very similar when varying $m_0$.

6.2. Calibration of the Ensembles

[44] In a calibrated ensemble the members can be considered to be equiprobable; thus, the verifying analysis is

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**Figure 8.** Evolution of the spread and RMSE of the ensembles generated using the standard (BV) and logarithmic Bred Vectors (LVB) corresponding to a chain of $L = 50$ Lorenz systems ($D = 0.5$) using different perturbation techniques: LBV and standard BV with initial amplitude/norm equal to $0.5, 10^{-1}, 10^{-2}, 10^{-3}, 10^{-4}, 10^{-5}$, respectively.

**Figure 9.** Evolution of the Talagrand histograms for different 10-member ensembles for a chain of Lorenz systems with diffusion $D = 0.5$. Gray histograms correspond to ensembles generated from standard BV with different initial norm $\mu_0 = 10^{-5}$ and 0.5. In this case the diagram shape does not depend on the initial norm value. Black histograms show the evolution of the Talagrand diagram for LBV with different initial amplitude values: $\rho_0 = 10^{-5}, 10^{-2},$ and 0.5. All perturbations have the same initial norm (5% of the system amplitude). Using LBV, well-calibrated ensembles are attained in different times depending on the initial amplitude.
equally likely to lie between any two adjacent members, including the cases when the analysis lies outside the ensemble range. Since the spread of the ensemble increases in time, the members become equiprobable only for some lead-time window which determines the operative range of the ensemble. The principal method for assessing the calibration of an ensemble is the verification rank histogram, or Talagrand diagram [see, e.g., Anderson, 1996], obtained as the histogram of ranks produced by pooling the verifying analysis within the ordered ensemble values.

We computed the Talagrand diagram of the ensembles described in the previous section for times $t = 1, 5, 20,$ and 50 (vertical dashed lines are shown in Figure 8 to represent these time values). Figure 9 shows the evolution of the Talagrand diagram for standard BV (gray histograms) and LBV (black histograms). In the case of standard BV, only the cases $\mu_0 = 0.5$ and $10^{-5}$ are presented, since the rest of the $\mu$ values exhibit the same behavior. However, the logarithmic BV present different behaviors for different values of $\rho_0$. Note that in an operative setting a calibrated ensemble is given by a uniform diagram, obtained as the transition between “∪”- and “∩”-shape ensembles, which indicate under- and over-dispersion, respectively. Figure 8 shows that the calibration time varies with $\rho_0$, allowing to obtain ensembles calibrated at shorter lead times preserving the performance. Thus the new logarithmic breeding technique improves the ability to identify a wider set of uncertainties in the initial conditions.

7. Evidence of Lognormality in Operative EPS

[46] In the previous sections we have considered a simple model to introduce the ideas of the new logarithmic breeding method. In this section, we give some evidence of lognormal behavior of fluctuations in operative numerical global circulation models. In particular, we have analyzed the ECMWF medium-range EPS system during the period December–February 1997–1998 in the NAO area (80$^\circ$N, 10$^\circ$S, −90$^\circ$W, 50$^\circ$E). Fluctuations $\delta x(t)$ of the evolved perturbed condition from the control trajectory have been computed for different lead times (0, 1 and 6 days) for each of the 50 ensemble members, obtaining lognormal-like distributions of the fluctuations for different variables; in particular we considered geopotential at 500 hPa (Figures 10a–10c) and relative humidity at 1000 hPa (Figures 10d–10f). The insets represent the histograms of the logarithmic fluctuations $\log(\delta x(t))$. 

![Figure 10. Histograms of the spatial finite fluctuation $\delta x(t)$ obtained for the ECMWF medium-range EPS system for the period December–February 1997–1998 in the NAO area for different lead times. The inset represent the histograms of the logarithmic fluctuations $\log(\delta x(t))$.](image-url)
perturbations: $\log|\delta x(t)|$. In this real example, the perturbations are initially localized since they already represent the fastest growing modes.

[47] Note that although this EPS system does not use the bred vector technique, but singular vectors, the finite fluctuations are localized growing toward exponential distributions. This result confirms that the technique presented in this paper could be also of interest for the operative numerical weather forecasting community.

8. Conclusions

[48] In this paper we have shown that the dynamics of bred vectors depend not only on the exponential divergence in time due to chaos, but also on the growth of the spatial correlation in the extended system. Using this idea, a new breeding method has been introduced, which is a simple, but important, modification of the standard breeding technique. The full algorithm required to apply this new method has been detailed in the paper. This paper also presents the possibility to obtain bred vectors with different spatial patterns leading to ensembles with different spread growth preserving the model performance. The possibility of switching the spread of the ensemble also gives an opportunity to design calibrated EPS systems for different lead-time windows.

[49] Further research is needed to study the application of this analysis and this new breeding method in more complex inhomogeneous models, where different characteristic scales may exist simultaneously, complicating the study.

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