Model for fermion masses and lepton mixing in $SO(10) \times A_4$

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The discrete flavor symmetry $A_4$ explains very well neutrino data at low energy, but it seems difficult to extend it to grand unified models since, in general, left-handed and right-handed fields belong to different $A_4$ representations. Recently a model has been proposed where all the fermions equally transform under $A_4$. We study here a concrete $SO(10)$ realization of such a model providing small neutrino masses through the see-saw mechanism. We fit the charged fermion masses run up to the unification scale. Some fermion masses properties come from the $SO(10)$ symmetry while lepton mixing angles are a consequence of the $A_4$ properties. Moreover, our model predicts the absolute value of the neutrino masses; these are in the range $m_\nu \approx 0.005$--0.052 eV.

I. INTRODUCTION

The existence of a grand unified theory (GUT) [1,2] has continued to be an attractive idea for physics beyond the standard model (SM) since the 70’s. Among indications toward GUTs is the phenomenological tendency to unify of the gauge couplings, and the theoretical implicit possibility to explain charge quantization and anomaly cancellation. One of the main features of GUTs is their potential to unify the particle representations and the fundamental parameters in a hopefully predictive framework. There are many gauge groups that can accommodate the SM ($SU(5)$, $SU(6)$, $SO(10)$, $E_6$, etc.). Among them $SO(10)$ is the smallest simple Lie group for which a single anomaly-free irreducible representation (namely the spinor $16$ representation) can accommodate the entire SM fermion content of each generation.

Once we fix the unification group, we deal with the flavor physics. The introduction of an extra horizontal symmetry acting on the fermion families may further constrain the neutrino mixing parameters and hopefully explain large mixing angles. After the recent neutrino evidence [3–13] we know very well almost all the parameters both in the quark [14] and lepton [15–33] sectors. We know all the quark and charged lepton masses and the value of the difference between the square of the neutrino masses: $\delta m^2_{12} = m_1^2 - m_2^2$ and $\delta m^2_{23} = |m_3^2 - m_2^2|$. We also know the value of the quark mixing angles and phases, and the two mixing angles $\theta_{12}$ and $\theta_{23}$ in the lepton sector.

Moreover we have an upper bound for the $\theta_{13}$ mixing angle in the lepton sector. All these experimental informations seem to indicate a discrete flavor symmetry such as $2$–$3$ [34–36], $S_3$ [37–40], $S_4$ [41,42], $D_3$, $D_4$ [43], $A_4$ [44–51], $T^\prime$ [52], etc., in the lepton sector. In particular, models with $A_4$ flavor symmetry, the case studied here, very easily give the tri-bi-maximal mixing matrix [53] that fits well the neutrino data. Non-Abelian discrete symmetries could arise from superstring theory, in particular, from the compactification of heterotic orbifolds [54], the case for $A_4$ is reported in [55]. Models with $SU(5) \times A_4$ [50] and $SU(2) \times SU(2) \times SU(4) \times A_4$ [51] symmetries have already been studied in literature. In these previous studies, fermion singlets and $SU(2)$ doublets do not equally transform under $A_4$. Thus this family symmetry seems not to be compatible with $SO(10)$ models where all the matter fields belong to the same representation. Only recently it has been proposed a generic phenomenological model with $A_4$ [56] which is suitable, as we will see in this work, for a $SO(10)$ GUT generalization.

The purpose of the paper is to construct an explicit $SO(10) \times A_4$ GUT model and to fit, at tree level, fermion masses and mixing. We propose here a non-SUSY GUT model with a Lagrangian invariant under $SO(10) \times A_4$. The matter fields are in a $16$, triplet of $A_4$. In the Higgs sector, we introduce a $10$, a $126$, and three $45$s singlets of $A_4$, a $45$ and a $126$, triplets of $A_4$. The $A_4$ symmetry is dynamically broken by the vacuum expectation value (vev) of the Higgs $A_4$-triplets. The study of the problem of the vacuum alignment in $A_4$ just studied in the context of extra dimensions [48] and the MSSM [57] is beyond the scope of this work. The direction of the four vevs of the $45$s in the $SO(10)$ are simply assumed to be $T_{3R}$, $Y$ and two other...
combinations of them. The 10 gives contributions to the Dirac mass matrices proportional to the identity. Because of the chosen vev directions and the fact that the 45s appear only in a given combination, we get contributions to $M^a$, $M^d$, $M^l$, but not to $M^\text{Dirac}$ from higher dimension operators. The 126 gives contributions only to the Majorana neutrino mass matrix. The low energy neutrino mass matrix is obtained with the see-saw mechanism (for a phenomenological realization in $A_4$ see [49]).

The paper is organized as follows. In Sec. II we define the matter and Higgs fields transformations under the $SO(10)$ and $A_4$ groups. In Sec. III we write the Lagrangian of our model. In Sec. IV we show the relations between the Dirac mass matrices and the Higgs vevs. We show similar relations for the Majorana mass matrix of the neutrinos. In Sec. V we write the mixing matrices and masses as function of the Higgs vevs. In Sec. VI we show how the experimental data constrain our model. In Sec. VI A, we perform a numerical analysis of the experimental data by using a Monte Carlo minimization fit. In Sec. VI B we investigate some predictions of our model. Section VII is devoted to conclusions. We list some relevant, well known, $A_4$ group and representation properties in Appendix A.

II. MATTER AND HIGGS FIELDS

The smaller spinorial representation of $SO(10)$ is the 16 dimensional one. All the fermionic matter fields of one family can be accommodated within the 16 by including the right-handed neutrino. The Higgs electroweak doublet can be taken in the 10 as well as one of the 126 representations. For simplicity we assume that the electroweak doublet Higgs belongs to the 16 representation. Since leptons and quarks mass matrices cannot be symmetric, we need to break the $SO(10)$ left-right symmetry at the unification scale. We perform this job by introducing sets of fields in the 45 representation. The scalar 45 representations can get vev in any combination of the extra Abelian factors $Y$ and $T_{3R}$. The matter fields and scalar fields transform under $A_4$ as in Table I, where the index of the 45s refers to the vev’s direction. $C$ and $D$ are linear combinations of $Y$ and $T_{3R}$. We will determine these combinations latter, by using the experimental constraints.

III. THE LAGRANGIAN

Let us write our Lagrangian as,

\[ L_Y = h^{i\bar{j}}_{0}(16^i 16^j, 10 16^i, 45^i_{T_{3R}} 45^j_{T_{3R}}, 45^i_{T_{3R}} 16^j, \
+ \hat{h}^{i\bar{j}}_{0}(16^i 16^j, 10 45^i_{T_{3R}} 45^j_{T_{3R}}, 45^i_{T_{3R}} 16^j, \
+ \sigma^{i\bar{j}}(16^i 126^j, 16^i 45^j_{T_{3R}}, 16^j, \
+ \lambda^{i\bar{j}}(16^i 126^j, 45^j_{T_{3R}}, 16^j) = L_{\text{Dirac}} + L_{\text{Majo}} \]  

(1)

where the indices \{i, j, k, l\} are $A_4$ indices and the sum over the gauge indices is understood. As shown in [58] any Lagrangian of the form in Eq. (1) can be easily obtained from a renormalizable Lagrangian, by including a set of heavy spinor fields, with the inclusion of an $U(1)$ charge and/or supersymmetry. We reserve to a further investigation the question of how general our Lagrangian is, and how it can be obtained in a renormalizable theory.

As we will better clarify in the Appendix A, in the second and in the last terms of Eq. (1) there are two ways of contracting the three $A_4$ indices in an invariant way. We have to choose to which representation of $A_4$ the 10 scalar field belongs. Because we want only one Higgs, we excluded the triplet possibility but we still have three possibilities that correspond to how the 10 transforms with respect to $A_4$: as 1, 1', 1''. The fermion mass matrices $M_f$ (with $f = u, d, l, \nu$) coming from the first term in $L_Y$ will be, respectively

\[ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \omega & 0 & 0 \\ 0 & 0 & \omega & 0 \end{pmatrix}. \]  

(3)

In any case we have three degenerate eigenvalues, namely $m_u = m_d = m_t$, that are corrected by the additional terms in Eq. (1). Let us assume that the $A_4$ triplets $45_C$ and $126_C$ get vevs, respectively, in the following directions of $A_4$

\[ \langle 45_C \rangle = v_{45_C}(1, 1, 1), \quad \langle 126_C \rangle = v_{126}(1, 0, 0), \]  

(4)

where the $SO(10)$ indices are understood on both left and right sides. After symmetry breaking, once the Higgs acquire vevs, the quadratic part for the fermions of the Lagrangian in Eq. (1) can be rewritten as

\[ L_{\text{Dirac}} = h(16^i 16^j, 126^i, 16^j) + h'(16' 16'^j, 126^i, 16'^j) + h''(16'^i 16'^j, 126^i, 16'^j) + h'''(16'^i 16'^j, 126^i, 16'^j), \]  

(5a)

\[ \begin{align*}
L_{\text{Majo}} &= \sigma^{i\bar{j}}(16'^i 16'^j, 126^i, 16'^j) + \lambda^{i\bar{j}}(16'^i 126^j, 16'^j) \] 

(5b)

where

\[ \begin{align*}
16'^i &= v_{45_{T_{3R}}} v_{45_C} v_{45_{T_{3R}}} 16^i, \quad 16'^j = v_{45_{T_{3R}}} 16^j, \\
16'^i &= v_{45_{T_{3R}}} v_{45_C} 16^i, \quad \text{with } i = 1, 2, 3 \] 

(6)

We obtain the following expression by absorbing the vevs of the 45s into the coupling constants.

<table>
<thead>
<tr>
<th>TABLE I. Matter and Higgs field representations.</th>
</tr>
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<tbody>
<tr>
<td>$SO(10)$</td>
</tr>
<tr>
<td>$A_4$</td>
</tr>
</tbody>
</table>
TABLE II. Quantum numbers for the low energy matter fields.

<table>
<thead>
<tr>
<th></th>
<th>X</th>
<th>Y</th>
<th>B − L</th>
<th>T3R</th>
</tr>
</thead>
<tbody>
<tr>
<td>q</td>
<td>1</td>
<td>1/3</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>u^c</td>
<td>1</td>
<td>-4/3</td>
<td>-1</td>
<td>1/2</td>
</tr>
<tr>
<td>d^c</td>
<td>-3</td>
<td>2/3</td>
<td>-1</td>
<td>-1/2</td>
</tr>
<tr>
<td>e^c</td>
<td>-3</td>
<td>-1</td>
<td>-3</td>
<td>0</td>
</tr>
<tr>
<td>v^c</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>-1/2</td>
</tr>
<tr>
<td></td>
<td>5</td>
<td>0</td>
<td></td>
<td>1/2</td>
</tr>
</tbody>
</table>

\[ 16' = (x_qL, x_uR, u^c, x_dR, d^c, x_IL, l, e^c, x_vR)^T, \]
\[ 16'' = (x_qL, x_uR, u^c, x_dR, d^c, x_IL, l, e^c, x_vR)^T, \]
\[ 16''' = (x_qL, x_uR, u^c, x_dR, d^c, x_IL, l, e^c, x_vR)^T, \]

where \( x_{LR}, x_{fLR} \), and \( x_{jLR} \) are the quantum numbers, respectively, of the product of the charges \( T_{3R} \), \( Y \), of the product of the charges \( T_{3R}, Y, C, \) and \( D \), and of the charge \( T_{3R} \) reported in Table II [58].

### IV. FROM VEVS TO MASS MATRICES

From Table II we observe that \( x^R_{eR} = 0 \) (because \( Y \) of the right-handed neutrino is zero) and \( x^R_{IL} = 0 \) (because \( T_{3R} \) of the lepton doublet is zero). These two conditions imply that the terms \( 16, 16'' \nu_{10} \) in the Lagrangian \( L_{\text{Dirac}} \) do not contribute to the Dirac neutrino mass term. Therefore, once the \( 45s \) get a vev, from Eq. (5b) we have that the Dirac neutrino mass matrix \( M_{\text{Dirac}} \) is proportional to the identity.

\[ M_{\text{Dirac}} = h_0 \nu^u I, \quad (8) \]

where \( I \) is the identity matrix and \( \nu^u \) is the vev of the up component of the 10 [59]. The fact that the \( M_{\text{Dirac}} \) is proportional to the identity will be important in order to realize the see-saw mechanism and not spoiling the main consequence of the \( A_4 \) symmetry; the explanation of the appearance of a tri-bi-maximal mixing matrix in the lepton sector. With the conventions \( x_{dL} = x_{dL} = x_{QL} = x_{QL}, \) \( x_{eL} = x_{eL} = x_{eL} = x_{eL}, \) and \( \nu^e = \nu^d, \) the interactions \( h_1 16, 16'' \) and \( h_2 16_2 16_2'' \) in Eqs. (5c) and (5d) give the following mass terms

\[ h_1 \nu^d (x_{1L}^T \bar{u}_{1L} \psi_{2R} + x_{1R}^T \bar{u}_{2L} \psi_{1R}) + h_2 \nu^d (x_{1L}^T \bar{u}_{2L} \psi_{1R} + x_{1R}^T \bar{u}_{1L} \psi_{2R}) + \text{H.c.} \quad (9) \]

namely,

\[ \nu^d \begin{pmatrix} 0 & h_1 x_{jL}^r + h_2 x_{jR}^r \\ h_1 x_{jL}^r + h_2 x_{jR}^r & 0 \end{pmatrix} \]

and so on for the other interactions (in the flavor planes 31 and 23). If we introduce

\[ A^f = (h_1 x_{fL}^r + h_2 x_{fR}^r) \quad \text{and} \quad B^f = (h_1 x_{fR}^r + h_2 x_{fL}^r) \]

\[ (10) \]

The full contribution to the Dirac mass matrices, coming from the operators proportional to the 45 representations, is

\[ \nu^d \begin{pmatrix} 0 & A^f & B^f \\ B^f & 0 & A^f \\ A^f & B^f & 0 \end{pmatrix}. \quad (11) \]

The charged fermion mass matrices are then

\[ M^u = \nu^d \begin{pmatrix} h^u_0 & A^u & B^u \\ A^u & h^u_0 & A^u \\ B^u & A^u & h^u_0 \end{pmatrix}; \]

\[ M^d = \nu^d \begin{pmatrix} h^d_0 & A^d & B^d \\ A^d & h^d_0 & A^d \\ B^d & A^d & h^d_0 \end{pmatrix}; \quad (12) \]

\[ M^l = \nu^d \begin{pmatrix} h^l_0 & A^d & B^d \\ A^d & h^l_0 & A^d \\ B^d & A^d & h^l_0 \end{pmatrix}. \]

where \( \nu^u \) and \( \nu^d \) are the vevs of the up and down components of the 10, while the \( A \) and \( B \) coefficients are defined in Eq. (10). The \( h^u_0 \) are defined by the combinations of \( h_0 \) and \( h_0^r \) with the weight corresponding to the charge \( x_{fR} \)

\[ h^u_0 = h_0 + x_{uR} h_0^r, \]

\[ h^d_0 = h_0 + x_{dR} h_0^r, \]

\[ h^l_0 = h_0 + x_{eR} h_0^r. \]

We observe that the general form of the mass matrices \( M^{u,d,l} \), are of the same type of the one reported in Ref. [60] (see Eq. (20)). Moreover the Majorana mass matrix for the right-handed neutrino is given by

\[ M_R = \begin{pmatrix} a & 0 & 0 \\ 0 & a & b \\ 0 & b & a \end{pmatrix} \quad (14) \]

where \( a = \sigma \nu_{126} \), and \( b = \lambda \nu_{126} \). The Dirac neutrino mass matrix has been previously given in Eq. (8).

### V. MASSES AND MIXING

It has been recently shown in [60] that, if the Dirac mass matrices are given by Eq. (12), the charged fermion mass matrices are diagonalized by

\[ U = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{pmatrix} \quad (15) \]

and then we have
and it is diagonalized by a tri-bi-maximal mixing matrix.

From the Lagrangian in Eq. (1), the light neutrino mass matrix comes from a type-I see-saw mechanism as below

$$M^{\nu} = \frac{M^T}{M_R} (M^T_R)^T$$

where the Dirac neutrino mass matrix $M^\nu$ is proportional to the identity (see Eq. (8)), while $M_R$ is the right-handed Majorana neutrino matrix. We observe that our Lagrangian does not give the left-handed Majorana neutrino mass matrix since we have introduced the $T_{3R}$ fields. In the basis where the charged leptons are diagonal, the mass matrix of the low energy neutrino $M^\nu$ is given by

$$\bar{M}_\nu = U^T M^\nu U = \frac{M^\nu}{M_R} (M^T_R)^T$$

where we used the fact that $M^\nu$ is proportional to the identity. We have

$$U^T M_R U^* = \begin{pmatrix} a + \frac{2b}{3} & -b/3 & -b/3 \\ -b/3 & 2b/3 & a - b/3 \\ -b/3 & a - b/3 & 2b/3 \end{pmatrix}$$

and it is diagonalized by a tri-bi-maximal mixing matrix. Consequently $\bar{M}_\nu$ is diagonalized by the same tri-bi-maximal mixing matrix too. The eigenvalues of $\bar{M}_\nu$ are

$$m_1 = \frac{(h_0 \nu \nu^\dagger)^2}{a + b},$$

$$m_2 = \frac{(h_0 \nu \nu^\dagger)^2}{a},$$

$$m_3 = \frac{(h_0 \nu \nu^\dagger)^2}{b - a}.$$
\[ \frac{A^f}{h_0} \simeq \omega \quad \text{and} \quad \frac{B^f}{h_0} \simeq \omega^2. \] 

(25)

From the solution in Eq. (22) and by using the definitions of \( A^f, B^f \) in Eq. (10) we obtain

\[ x'_+ = \frac{1}{3v^f} \frac{m_i^f + m_e^f - 2m_u^f}{h_1 + h_2} \quad \text{and} \]
\[ x'_- = \frac{1}{\sqrt{3v^f}} \frac{m_i^f - m_e^f}{h_1 - h_2}. \]

(26)

where we have introduced the notation \( x'_L \equiv x'_R \). In Eq. (26) we must remember that each mass includes an undetermined phase. We notice that the ratios \( x'_+/x'_+ \) and \( x'_-/x'_- \) do not depend on \( h_i \), then they are experimentally determined (up to the undetermined phases). In fact we have

\[ \frac{x'_+}{x'_-} = \frac{\nu^d}{\nu^u} \frac{m_i + m_e - 2m_u}{m_b + m_s - 2m_d}, \]

(27a)

\[ \frac{x'_-}{x'_-} = \frac{\nu^d}{\nu^u} \frac{m_i - m_e}{m_b - m_s}, \]

(27b)

\[ \frac{x'_+}{x'_+} = \frac{\nu^d}{\nu^u} \frac{m_i + m_e - 2m_u}{m_\mu + m_{\tau} - 2m_e}, \]

(27c)

\[ \frac{x'_-}{x'_-} = \frac{\nu^d}{\nu^u} \frac{m_i - m_e}{m_\mu - m_{\tau}}. \]

(27d)

By using the masses run up to the \( 2 \cdot 10^{16} \) GeV scale in the (non-SUSY) standard model given in Table III, we performed a Monte Carlo analysis of Eq. (27). For the masses we took two sided Gaussian distributions with central values and standard deviations taken from Table III. For the unknown phases we took flat random distributions in the interval \([0, 2\pi]\). Our results can be summarized as

| TABLE III. Quark masses run at the \( 2 \cdot 10^{16} \) GeV scale in non-SUSY standard model (see Ref. [61]). |
|-------------|-----------------|
| \( m_u \) (MeV) | \( 0.8351^{+0.1636}_{-0.1700} \) |
| \( m_c \) (MeV) | \( 242.6476^{+23.5536}_{-24.7026} \) |
| \( m_t \) (GeV) | \( 75.4348^{+9.9647}_{-8.5401} \) |
| \( m_d \) (MeV) | \( 1.7373^{+0.4846}_{-0.2636} \) |
| \( m_s \) (MeV) | \( 34.5971^{+4.8857}_{-3.1971} \) |
| \( m_\mu \) (GeV) | \( 0.9574^{+0.0337}_{-0.0159} \) |
| \( m_\tau \) (MeV) | \( 93.1431^{+9.0464}_{-13.6336} \) |

Notice that, if we neglect the undetermined phases in the masses, we get similar central values but wrong errors in the constraints. For example we would obtain in such a case

\[ \frac{x'_+}{x'_+} = 0.972 \pm 0.005, \quad \frac{x'_-}{x'_-} = 1.034 \pm 0.006. \]

(29)

**B. The theoretical prediction**

In our model we are able to fit all the masses of quarks and leptons. Moreover we obtain, thanks to the \( A_1 \) structure of the model, a tri-bi-maximal lepton mixing matrix. Let us investigate the fermion masses. As obtained in the previous section, the quantities to be fitted are the ratios in Eq. (28).

The theoretical result for the ratios \( x'_+/x'_+ \) and \( x'_-/x'_- \) are determined from Table I and the definitions of \( x'_L \). By using, for example, the direction \( C = (28X - 249Y) \) and \( D = (238X - 9Y) \) we get

\[ \frac{x'_+}{x'_+} = 1 \quad \text{and} \quad \frac{x'_-}{x'_-} = 1; \]

(30a)

\[ \frac{x'_+}{x'_+} = \frac{300}{517} \quad \text{and} \quad \frac{x'_-}{x'_-} = \frac{300}{517}. \]

(30b)

in good agreement with the experimental values in Eq. (28). The absolute neutrino mass scale is fixed, because the presence of, essentially, only two free parameters, \( a \) and \( b \), in the neutrino sector. If we impose the experimental constraints on \( \delta m^2_{12} = 7.92(1 \pm 0.09) \times 10^{-5} \text{ eV}^2 \) and \( |\delta m^2_{13}| = 2.41(1^{+0.21}_{-0.36}) \times 10^{-3} \text{ eV}^2 \) we get the following neutrino masses:

\[ m_1 = 0.052 \pm 0.005 \text{ eV}, \]
\[ m_2 = 0.052 \pm 0.005 \text{ eV}, \]
\[ m_3 = 0.017 \pm 0.002 \text{ eV}, \]

(31)

\[ m_1 = 0.0051 \pm 0.0005 \text{ eV}, \]
\[ m_2 = 0.0102 \pm 0.0005 \text{ eV}, \]
\[ m_3 = 0.049 \pm 0.004 \text{ eV}, \]

(32)

where the first results correspond to an Inverted Hierarchy case, while the second ones would correspond to the Normal Hierarchy.
VII. CONCLUSIONS

Neutrino data at low energy are well explained by an $A_4$ symmetry, nevertheless it is difficult to include this symmetry in grand unified theories. In this paper we investigate the possibility to construct an explicit model with a Lagrangian invariant under $SO(10) \times A_4$. We assumed that the matter fields are in a 16 dimensional $SO(10)$ representation, triplet of $A_4$. In the Higgs sector, we introduced a 10, a 126 and three 45s singlets of $A_4$, a 45 and a 126 triplets of $A_4$. The $A_4$ symmetry is dynamically broken by the vevs of the Higgs $A_4$-triplets. The direction of the vevs of the 45s in the $SO(10)$ are assumed to be $T_{3R}$, $Y$ and two other combinations of them, $C$ and $D$. The Lagrangian contains three terms with the 10 that give contributions to the Dirac mass matrices, and two terms with the 126s that determine the Majorana neutrino mass matrix. The first two terms containing the 10 give a contribution to the Dirac mass matrices which is proportional to the identity (the second term is used to avoid the $\tau$ bottom unification). The third term, because of the fact that the 45s appear only in the given combination, provides contributions to $M^a$, $M^b$, $M^c$, but not to $M^d_{\text{Dirac}}$. For these reasons $M^d_{\text{Dirac}}$ results to be proportional to the identity. Finally the 126 terms give contribution only to the right-handed Majorana mass matrix $M_R$. The low energy neutrino mass matrix is then obtained with the see-saw mechanism.

The mixing angle structure of the charged fermion mass matrices are fixed by the $A_4$ structure of the model. They are diagonalized by the mixing matrix in Eq. (15). The $A_4$ direction of the vev of the triplet 126 implies a particular form for $M_R$. This particular form of $M_R$ and the fact that $M_{\text{Dirac}}^d$ is proportional to the identity, imply that the low energy neutrino mass matrix, in the base with diagonal charged lepton, is diagonalized by the tri-bi-maximal mixing matrix.

We show that at tree level our model fits with great precision (within 1 standard deviation) the values of the fermion masses, run at $2 \times 10^{16}$ GeV scale in the (non-SUSY) standard model, if particular directions of the vevs of the 45$_C$ and 45$_D$ are assumed ($C = (28X - 249Y)$ and $D = (238X - 9Y)$).

One important consequence of the structure of this model is the prediction of an absolute scale for low mass neutrinos. We predict the absolute scale of the neutrino mass to be close to $\sim 0.05$ eV. Normal or inverted hierarchies are allowed by the model.

In the model presented here, both up and down sector are diagonalized by the same mixing matrix. For this reason the resulting quark mixing matrix, the CKM matrix is proportional to the identity, in agreement with evidence only at first order. The explanation of the correct CKM matrix is beyond the scope of this work. However a deeper study of radiative corrections to the potential could possibly shed light on the right CKM structure.

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APPENDIX A: THE $A_4$ PROPERTIES

The group $A_4$ is the finite group of the even permutations of four object and contains 12 elements. Every finite group can be generated by a subset of elements, called generators. A set of elements is independent if none of them can be expressed in terms of the other. The group $A_4$ has two independent generators denoted as $S$ and $T$, which can be chosen to verify the following defining relations:

$$S^2 = T^3 = (ST)^3 = 1.$$

There are four irreducible representations for the $A_4$ group: denoted as 1, 1', 1'' and the 3. In each of these representations the generators are explicitly written as follows:

$$1: S = 1, T = 1,$$

$$1': S = 1, T = \omega,$$

$$1'': S = 1, T = \omega^2,$$

$$3: S = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, T = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix},$$

where $\omega = e^{2\pi i/3}$ and then $\omega^3 = 1$ and $1 + \omega + \omega^2 = 0$. If $a = (a_1, a_2, a_3)$ is a triplet, then the action of the $S$ and $T$ operators is $Sa = (a_1, -a_2, -a_3)$ and $Ta = (a_2, a_3, a_1)$. If $b$ is another analogous $A_4$ triplet, their tensorial product decomposes in irreducible representations as

$$3 \times 3 = 1 + 1' + 1'' + 3 + 3.$$

In order to explicitly construct a singlet from these quantities we first impose the invariance under $S$, the most generic term will be

$$xa_1b_1 + ya_2b_2 + za_3b_3 + ta_2b_3 + ra_3b_2,$$

where $x, y, z, r$ and $t$ are parameters. If we impose also the invariance under $T$, we have that the above term transforms like a 1 single, if and only if $x = y = z$ and $r = t = 0$. Then we have

$$1 = (ab) = (a_1b_1 + a_2b_2 + a_3b_3).$$

Similarly one can check that
from which we get
\[ x = y = \tilde{x} = \tilde{z} = z = 0, \quad t = \tilde{t}, \quad r = \tilde{y}. \]

The final result is
\[ 3 = (a_2b_3, a_3b_1, a_1b_2) \quad \text{and} \quad 3 = (a_3b_2, a_1b_3, a_2b_1) \]
where the first line comes from terms proportional to \( t \) while the second line is proportional to \( r \). In summary, with this notation, if \( \nu = (v_1, v_2, v_3) \) is an additional triplet, the product of the three triplet \( a, b \) and \( \nu \) that transform as a singlet 1 in \( A_4 \) is given by
\[
\begin{align*}
    h_1(a_2b_3v_1 + a_3b_1v_2 + a_1b_2v_3) \\
    + h_2(a_3b_2v_1 + a_1b_3v_2 + a_2b_1v_3)
\end{align*}
\]
where \( h_1 \) and \( h_2 \) are arbitrary parameters. The term in Eq. (A2) is invariant under \( A_4 \).
[59] Because the presence of the product $45T_{\nu\nu}$ in the $h_0^0$ term, the only contribution to $M_{\text{Dirac}}^0$ comes from $h_0$.