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Higher Order Integrability in Generalized Holonomy

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ABSTRACT

Supersymmetric backgrounds in M-theory often involve four-form flux in addition to pure geometry. In such cases, the classification of supersymmetric vacua involves the notion of generalized holonomy taking values in $SL(32,\mathbb{R})$, the Clifford group for eleven-dimensional spinors. Although previous investigations of generalized holonomy have focused on the curvature $\mathcal{R}_{MN}(\Omega)$ of the generalized $SL(32,\mathbb{R})$ connection Ω_M, we demonstrate that this local information is incomplete, and that satisfying the higher order integrability conditions is an essential feature of generalized holonomy. We also show that, while this result differs from the case of ordinary Riemannian holonomy, it is nevertheless compatible with the Ambrose-Singer holonomy theorem.

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1 Introduction

The connection between holonomy and supersymmetry is a close and important one. The notion of a supersymmetric background is simply one where some fraction of the supersymmetry variations vanish. In particular, for bosonic field configurations, we are invariably led to the vanishing of the gravitino transformation

$$\delta\psi_M \equiv \mathcal{D}_M \epsilon = 0, \tag{1.1}$$

where $\mathcal{D}_M$ is the supercovariant derivative. The number of preserved supersymmetries is then equal to the number of linearly independent solutions of the Killing spinor equation (1.1). Thus the goal of enumerating supersymmetric vacua is essentially one of classifying all solutions to the above Killing spinor equation.

A necessary condition for the existence of Killing spinors is obtained from the integrability of the Killing spinor equation (1.1)

$$M_{MN}\epsilon \equiv [\mathcal{D}_M, \mathcal{D}_N]\epsilon = 0. \tag{1.2}$$

However, it ought to be evident that this simply measures the effect of parallel transportation of a spinor around an infinitesimal loop along the M and N directions of the base manifold. By the Ambrose-Singer theorem [1], this is in general related to the Lie algebra of some holonomy group $\mathrm{Hol}(\mathcal{D})$ acting on the spinors. For the case of a Riemannian connection, so that $\mathcal{D}_M$ is identified with ∇_M, the gravitational covariant derivative, the first order integrability condition directly yields the conventional Riemannian holonomy group $\mathrm{Hol}(\nabla) \subseteq \mathrm{SO}(n)$ where $\mathrm{SO}(n)$ is the Riemannian structure group for an n-dimensional orientable Euclidean manifold with a metric. In this case, the analysis is quite familiar, and holonomy groups have been classified by Berger in [2] for the Euclidean case, and partially extended to the Lorentzian case by Bryant in [3].

In practice, in order to obtain Killing spinors, one often starts with the integrability condition (1.2) and not directly with the Killing spinor equation (1.1), as the integrability condition is only algebraic in ϵ. For the case of a Riemannian connection, use of the Ambrose-Singer theorem demonstrates that the integrability condition (1.2) is also a sufficient condition for the existence of Killing spinors. However, it has been observed that this is no longer the case for more general connections. This is perhaps most evident in the squashed 7-sphere compactification of M-theory [4, 5], where left-squashing preserves

$N = 1$ supersymmetry in four dimensions while right-squashing leaves no unbroken supersymmetries at all. Yet, at the same time, first order integrability of the form (1.2) cannot distinguish between the two cases; only by going to second order integrability can the issue of the existence or non-existence of Killing spinors be resolved [6].

This question of whether the algebra generated by the curvature [expressed in the first order integrability condition (1.2)] agrees or does not agree with the algebra of the holonomy group has until now been mostly ignored in the study of generalized holonomy [7, 8, 9]. At present we will focus on generalized holonomy in eleven-dimensional supergravity. In this case, the bosonic fields are the metric g_{MN} and 3-form potential with 4-form field strength $F_{(4)}$. The fermionic superpartner is simply the gravitino, with transformation

$$\mathcal{D}_M \equiv \partial_M + \tfrac{1}{4}\Omega_M = \nabla_M - \tfrac{1}{288}(\Gamma_M{}^{NPQR} - 8\delta_M^N \Gamma^{PQR})F_{NPQR}. \tag{1.3}$$

Here Ω_M is considered to be a generalized connection, consisting of the conventional Riemannian connection as well as the flux-induced term and taking values in the space of forms $\Lambda^*(\mathbb{R}^{1,10})$ (which is identical to the Clifford algebra of the Dirac matrices). Actually it is only the even part of the Clifford group that is relevant; as a result the generalized structure group is $\mathrm{SL}(32,\mathbb{R})$ [8], which is considerably larger than the Riemannian structure group $\mathrm{SO}(1,10)$.

The idea behind generalized holonomy is simply to consider the holonomy of the generalized connection Ω_M. Since Ω_M takes values in the generalized structure group, we see that the generalized holonomy group is a subgroup of $\mathrm{SL}(32,\mathbb{R})$. Furthermore, as shown in [8], for a background preserving n supersymmetries, the generalized holonomy must be contained in $\mathrm{SL}(32-n,\mathbb{R}) \ltimes n\mathbb{R}^{32-n}$. As a result, the issue of classifying supersymmetric vacua may be mapped into one of classifying the generalized holonomy groups as subgroups of $\mathrm{SL}(32,\mathbb{R})$. Expressions for the generalized curvature of a background preserving n supersymmetries were given in [10] (including the conjectured preonic case [11], with $n = 31$) by relating the notions of Killing and preonic spinors, and an investigation of basic supersymmetric configurations of M-theory was performed in [12], where a large variety of generalized holonomy groups were obtained. However, one of the striking results of the analysis of [12] was the fact that identical generalized holonomies may yield different amounts of supersymmetries. This shows that knowledge of the holonomy group is insufficient to fully classify the background, and that knowledge of the decomposition of the 32-component spinor under $\mathrm{Hol}(\mathcal{D})$ is also needed.

At a more technical level, it was also seen in [12] that in many cases the complete Lie algebra of $\mathrm{Hol}(\mathcal{D})$ was not obtained from first order integrability (1.2), so that in particular the algebra had to be closed by hand. This issue is rather suggestive that the generalized curvature at a local point carries incomplete information of the generalized holonomy group, in apparent violation of the Ambrose-Singer theorem (but in agreement with the issue of left- versus right-squashing of S^7 mentioned above). However a careful reading of the Ambrose-Singer theorem indicates that $\mathrm{Hol}_p(\mathcal{D})$ at a point p is spanned by elements of the generalized curvature (1.2) not just at point p, but at all points q connected to p by parallel transport (see *e.g.* [13], p. 388 or [14], p. 33). Thus there is in fact no contradiction. Furthermore, this is rather suggestive that satisfying higher order integrability (representing motion from p to q) is in fact a necessary condition for identifying the proper generalized holonomy group, and that it is the Riemannian case that is the exception.

These issues have led us to the present work, where we explore the interplay of higher order integrability and generalized holonomy. We begin by revisiting the generalized holonomy of the M5-brane and M2-brane solutions of supergravity, and show that higher order integrability yields precisely the 'missing' generators that were needed to close the algebra. Other than this, however, the generalized holonomy groups $\mathrm{SO}(5) \ltimes 6\mathbb{R}^{4(4)}$ for the M5-brane and $\mathrm{SO}(8) \ltimes 12\mathbb{R}^{2(8_s)}$ for the M2-brane identified in [12] are unchanged. Following this, we turn to the squashed S^7 [4, 5], where the situation is considerably different.

The importance of higher order integrability was of course previously recognized in [6] for the case of the squashed S^7. Here, we reinterpret the result of [6] in the language of generalized holonomy, and confirm the statement of [15] that while first order integrability yields the incorrect result $\mathrm{Hol}^{(1)}(\mathcal{D}) = G_2 \subset \mathrm{SO}(7) \subset \mathrm{SO}(8)$, higher order integrability corrects this to $\mathrm{Hol}(\mathcal{D}) = \mathrm{SO}_\pm(7) \subset \mathrm{SO}(8)$, where the two distinct possibilities $\mathrm{SO}_-(7)$ and $\mathrm{SO}_+(7)$ arise from left- and right-squashing, respectively. Since the spinor decomposes as either $\mathbf{8}_s \rightarrow \mathbf{7} + \mathbf{1}$ or $\mathbf{8}_s \rightarrow \mathbf{8}$ in the two cases, this explains the resulting $N = 1$ or $N = 0$ supersymmetry in four dimensions [15].

In the following section, we provide a brief review of higher order integrability and then proceed to reexamine the generalized holonomy of the M5-brane and M2-brane solutions of supergravity. For both cases, we find that the higher order conditions close the holonomy algebra but otherwise do not affect the results of [12]. In section 3 we turn to the squashed

S^7 example and show that for this case higher order integrability is essential in obtaining the correct holonomy group from the curvature. Some of the details of this example are relegated to an Appendix.

2 Generalized curvature and higher order integrability

We start by defining the generalized curvature and n-th order integrability conditions. For a generalized covariant derivative of the form (1.3)

$$\mathcal{D}_M \equiv \partial_M + \tfrac{1}{4}\Omega_M, \tag{2.1}$$

first order integrability (1.2) yields

$$M_{MN} \equiv [\mathcal{D}_M(\Omega), \mathcal{D}_N(\Omega)] = \tfrac{1}{4}(\partial_M\Omega_N - \partial_N\Omega_M + \tfrac{1}{4}[\Omega_M, \Omega_N]) \equiv \tfrac{1}{4}\mathcal{R}_{MN}(\Omega), \tag{2.2}$$

where $\mathcal{R}_{MN}(\Omega)$ is the curvature of the generalized connection Ω; in particular, $\mathcal{R}_{MN}(\Omega) = R_{MNPQ}\Gamma^{PQ} + \cdots$. It is a familiar result that, when contracted with Γ^M, the first order integrability condition $\Gamma^M\mathcal{R}_{MN}(\Omega)\epsilon = 0$ yields an expression compatible with the bosonic equations of motion [16, 10], which for eleven-dimensional supergravity read

$$R_{MN} = \tfrac{1}{12}\left(F_{MPQR}F_N{}^{PQR} - \tfrac{1}{12}g_{MN}F^{PQRS}F_{PQRS}\right), \tag{2.3}$$

$$d * F_{(4)} + \tfrac{1}{2}F_{(4)} \wedge F_{(4)} = 0. \tag{2.4}$$

Higher order integrability expressions may be obtained by taking generalized covariant derivatives of (2.2). Here we make the definition precise by taking the chain of expressions

$$\begin{aligned} M_{AMN} &\equiv [\mathcal{D}_A, M_{MN}] = \tfrac{1}{4}\mathcal{D}_A(\Omega)\mathcal{R}_{MN}(\Omega), && (2.5)\\ M_{ABMN} &\equiv [\mathcal{D}_A, M_{BMN}] = \tfrac{1}{4}\mathcal{D}_A(\Omega)\mathcal{D}_B(\Omega)\mathcal{R}_{MN}(\Omega). && (2.6)\\ &\vdots \end{aligned}$$

It ought to be evident that the higher order integrability conditions correspond to measuring the generalized curvature $\mathcal{R}_{MN}(\Omega)$ parallel transported away from the original base point p. In this sense, the information obtained from higher order integrability is precisely that required by the Ambrose-Singer theorem in making the connection between $\mathrm{Hol}_p(\mathcal{D})$ and the curvature of the generalized connection.

Note that we take Ω_M in (2.1) to include the Levi-Civita connection on the base in addition to the generalized connection on the Clifford bundle. In this sense, we actually work with a generalized connection in $\mathcal{T}M \times \text{Cliff}(M)$. Writing

$$\mathcal{D}_M = \partial_M + \tfrac{1}{4}\Omega_M = \nabla_M + \tfrac{1}{4}\widetilde{\Omega}_M, \tag{2.7}$$

where

$$\widetilde{\Omega}_M = -\tfrac{1}{72}(\Gamma_M{}^{NPQR} - 8\delta_M^N\Gamma^{PQR})F_{NPQR} \tag{2.8}$$

in the case of eleven-dimensional supergravity, the integrability expressions (2.2), (2.5) and (2.6) are equivalent to

$$\begin{aligned}
M_{MN} &= \tfrac{1}{4}\mathcal{R}_{MN}(\Omega) = \tfrac{1}{4}R_{MNAB}\Gamma^{AB} + \tfrac{1}{2}\nabla_{[M}\widetilde{\Omega}_{N]} + \tfrac{1}{9}\widetilde{\Omega}_{[M}\widetilde{\Omega}_{N]},\\
M_{AMN} &= \mathcal{D}_A(\Omega)M_{MN} = \nabla_A M_{MN} + \tfrac{1}{4}[\widetilde{\Omega}_A, M_{MN}],\\
M_{ABMN} &= \mathcal{D}_A(\Omega)M_{BMN} = \nabla_A M_{BMN} + \tfrac{1}{4}[\widetilde{\Omega}_A, M_{BMN}],\\
&\vdots
\end{aligned} \tag{2.9}$$

We occasionally find these expressions useful for direct computation.

2.1 Generalized holonomy of the M5-brane

As examples of how higher order integrability may affect determination of the generalized holonomy group, we first revisit the case of the M5- and M2-brane solutions of supergravity. The generalized holonomy of these solutions, as well as several others, was originally investigated in [12]. For vacua with non-vanishing flux, including the brane solutions, it was seen that the Lie algebra generators obtained from first order integrability, (2.2), are insufficient for closure of the algebra. In particular, additional generators must be obtained by further commutators. In [12], this was done by closing the algebra by hand. In the present context, however, additional commutators are readily available from the higher order integrability expressions, (2.9).

It turns out that, for the M5-brane, working up the third order integrability is sufficient to close the algebra. To see this, we recall the M5-brane solution is given in isotropic coordinates as

$$\begin{aligned}
&ds_{11}^2 = H_5^{-1/3}dx_\mu^2 + H_5^{2/3}d\vec{y}^{\,2},\\
&F_{ijkl} = \epsilon_{ijklm}\partial_m H_5,
\end{aligned} \tag{2.10}$$

where $H_5(\vec{y})$ is harmonic in the six-dimensional transverse space spanned by the $\{y^i\}$, and $\epsilon_{ijklm} = \pm 1$ is a density. Computation of the supercovariant derivative (1.3) in this background yields a generalized connection of the form [12]

$$\Omega_\mu = \Omega_\mu^{\nu i} K_{\mu i}, \qquad \Omega_i = -\tfrac{1}{3}\partial_i \ln H_5 \Gamma^{(5)} + \tfrac{1}{2}\Omega_i^{jk}\hat{T}_{jk}, \tag{2.11}$$

when acting on spinors. Here we have highlighted the Lie algebra structure by introducing a set of generators

$$\hat{T}_{ij} = \Gamma_{\bar{i}\bar{j}} P_5^+, \qquad K_\mu = \Gamma_\mu P_5^+, \qquad K_{\mu i} = \Gamma_{\bar{\mu}\bar{i}} P_5^+, \qquad K_{\mu ij} = \Gamma_{\bar{\mu}\bar{i}\bar{j}} P_5^+, \tag{2.12}$$

where $\Gamma^{(5)} \equiv \frac{1}{5!}\epsilon_{ijklm}\Gamma^{\bar{i}\bar{j}\bar{k}\bar{l}\bar{m}}$ and $P_5^+ \equiv \frac{1}{2}(1+\Gamma^{(5)})$ is the half-BPS projector, and the overlined indices refer to local frame indices. The component expressions for $\Omega_\mu^{\nu i}$ and Ω_i^{jk} are

$$\Omega_\mu^{\nu i} = -\tfrac{2}{3}H_5^{-1/2}\partial_i \ln H_5, \qquad \Omega_i^{jk} = \tfrac{8}{3}\delta_{i[j}\partial_{k]}\ln H_5. \tag{2.13}$$

Even before addressing integrability, we see that Ω_M includes the generator $\Gamma^{(5)}$ in addition to $\hat{T}_{ij}$ and $K_{\mu i}$. However, the connection itself is not physical, and we see below that the terms proportional to $\Gamma^{(5)}$ drop out in generalized curvatures (and hence do not contribute to generalized holonomy).

The first order integrability of the generalized connection, given by (2.2), was computed in [12]. The result was

$$\begin{aligned}
M_{\mu\nu} \equiv \tfrac{1}{4}\mathcal{R}_{\mu\nu} &= 0, \\
M_{\mu i} \equiv \tfrac{1}{4}\mathcal{R}_{\mu i} &= H_5^{-1/2}\left[\tfrac{1}{6}(\partial_i\partial_j \ln H_5 - \tfrac{2}{3}\partial_i \ln H_5 \partial_j \ln H_5) + \tfrac{1}{18}\delta_{ij}(\partial \ln H_5)^2\right] K_{\mu j}, \\
M_{ij} \equiv \tfrac{1}{4}\mathcal{R}_{ij} &= \left[\tfrac{2}{3}(\partial_l\partial_{[i} \ln H_5 - \tfrac{2}{3}\partial_l \ln H_5 \partial_{[i} \ln H_5)\delta_{j]k} - \tfrac{2}{9}(\partial \ln H_5)^2 \delta^k_{[i}\delta^l_{j]}\right]\hat{T}_{kl}.
\end{aligned} \tag{2.14}$$

At this point, it was noted that the generators $\hat{T}_{ij}$ and $K_{\mu i}$ do not form a closed algebra, as both K_μ and $K_{\mu ij}$ are missing. Thus the algebra of the holonomy group, with generators (2.12), was obtained only after closing the algebra by hand. In fact we now see that the higher order integrability relations, expressed as (2.9), give rise to a sequence of additional commutators which are precisely the ones necessary to ensure closure of the algebra.

For the M5-brane, the second order integrability conditions, defined by (2.5), take on the form

$$\begin{aligned}
&M_{\mu\nu\lambda} = M_{\mu\nu\lambda}^{\rho i} K_{\rho i}, \qquad M_{\mu\nu i} = \tfrac{1}{2}M_{\mu\nu i}^{jk}\hat{T}_{jk}, \qquad M_{\mu ij} = M_{\mu ij}^{\nu k}K_{\nu k} + \tfrac{1}{2}M_{\mu ij}^{\nu kl}K_{\nu kl}, \\
&M_{i\mu\nu} = 0, \qquad M_{i\mu j} = M_{i\mu j}^{\nu k}K_{\nu k} + \tfrac{1}{2}M_{i\mu j}^{\nu kl}K_{\nu kl}, \qquad M_{ijk} = \tfrac{1}{2}M_{ijk}^{lm}\hat{T}_{lm},
\end{aligned} \tag{2.15}$$

where the component factors $M_{AMN}^{\cdot\cdot\cdot}$ are functions of H_5 and its derivatives. For example,

$$\begin{aligned} M_{\mu\nu\lambda}^{\rho i} &= \tfrac{1}{36} H^{-3/2} [\partial_j \ln H_5 \partial_j \partial_i \ln H_5 - \tfrac{1}{3} \partial_i \ln H_5 (\partial H_5)^2] \eta_{\mu[\nu} \delta_{\lambda]}^{\rho}, \\ M_{\mu\nu i}^{jk} &= \tfrac{4}{9} H^{-1} [\partial_{[j} \ln H_5 \partial^i \partial_{k]} \ln H_5 - \delta_{i[j} \partial^l \ln H_5 \partial^l \partial_{k]} \ln H_5] \eta_{\mu\nu}. \end{aligned} \tag{2.16}$$

The other factors arising in (2.15) are similar. However, we will not need their explicit forms. We simply note that one additional generator $K_{\mu ij}$ arises at second order. However, the algebra is not closed until third order integrability (2.6) is taken into account, since one additional commutator is necessary to provide the K_μ generator. In the case of the M5-brane, no additional information is gained beyond the third order integrability level. In fact, the identification of the proper generalized holonomy group

$$\mathrm{Hol}_{M5} = SO(5)_+ \ltimes 6\mathbb{R}^{4(4)}, \tag{2.17}$$

is unchanged from the presentation of [12]. All that has arisen from higher order integrability of the M5-brane is closure of the algebra on the same set of generators that were present at first order.

2.2 Generalized holonomy of the M2-brane

The analysis of the M2-brane is similar to that of the M5-brane. The supergravity solution takes the form

$$\begin{aligned} & ds^2 = H_2^{-2/3} dx_\mu^2 + H_2^{1/3} d\vec{y}^{\,2}, \\ & F_{\mu\nu\rho i} = \epsilon_{\mu\nu\rho} \partial_i H_2^{-1}, \end{aligned} \tag{2.18}$$

where $\epsilon_{\mu\nu\rho} = \pm 1$ and $H_2(\vec{y})$ is a harmonic function in the transverse space. Following [12], we introduce a set of generators

$$\hat{T}_{ij} = \Gamma_{\bar{i}\bar{j}} P_2^+, \qquad K_{\mu i} = \Gamma_{\bar{\mu}\bar{i}} P_2^+, \qquad K_{\mu ijk} = \Gamma_{\bar{\mu}\bar{i}\bar{j}\bar{k}} P_2^+, \tag{2.19}$$

where $P_2^+ = \frac{1}{2}(1 + \Gamma^{(2)})$ with $\Gamma^{(2)} = \frac{1}{3!}\epsilon_{\mu\nu\rho}\Gamma^{\bar{\mu}\bar{\nu}\bar{\rho}}$. In this case, the generalized connection takes the form

$$\Omega_\mu = \Omega_\mu^{\nu i} K_{\nu i}, \qquad \Omega_i = \tfrac{2}{3} \partial_i \ln H_2 \Gamma^{(2)} + \tfrac{1}{2} \Omega_i^{jk} \hat{T}_{jk}, \tag{2.20}$$

where

$$\Omega_\mu^{\nu i} = -\tfrac{4}{3} H_2^{-1/2} \partial_i \ln H_2 \delta_\mu^\nu, \qquad \Omega_i^{jk} = \tfrac{4}{3} \delta_{i[j} \partial_{k]} \ln H_2. \tag{2.21}$$

First order integrability yields the generalized curvature

$$\begin{aligned} M_{\mu\nu} \equiv \tfrac{1}{4}\mathcal{R}_{\mu\nu} &= 0, \\ M_{\mu i} \equiv \tfrac{1}{4}\mathcal{R}_{\mu i} &= \tfrac{1}{18}H_2^{-1/2}\left[6(\partial_i\partial_j \ln H_2 + 2\partial_i \ln H_2 \partial_j \ln H_2) - (\partial \ln H_2)^2\delta_{ij}\right] K_{\mu j}, \\ M_{ij} \equiv \tfrac{1}{4}\mathcal{R}_{ij} &= \left[-\tfrac{1}{3}(\partial_l\partial_{[i} \ln H_2 - \tfrac{1}{3}\partial_l \ln H_2 \partial_{[i} \ln H_2)\delta_{j]k} - \tfrac{1}{18}(\partial \ln H_2)^2 \delta^k_{[i}\delta^l_{j]}\right] \hat{T}_{kl}, \end{aligned} \quad (2.22)$$

which is similar in structure to that of the M5-brane (2.14). The 'missing' generator $K_{\mu ijk}$ of (2.19) is obtained by the commutation of $\hat{T}_{ij}$ with $\hat{K}_{\mu k}$. This arises in second order integrability via either $\mathcal{D}_i\mathcal{R}_{\mu j}$ or $\mathcal{D}_\mu\mathcal{R}_{ij}$. The general structure of the second order integrability expressions are as follows:

$$\begin{aligned} &M_{\mu\nu\lambda} = M^{\rho i}_{\mu\nu\lambda}K_{\rho i}, \qquad M_{\mu\nu i} = \tfrac{1}{2}M^{jk}_{\mu\nu i}\hat{T}_{jk} + M^{\nu k}_{\mu\nu i}K_{\nu k}, \qquad M_{\mu ij} = M^{\nu k}_{\mu ij}K_{\nu k} + \tfrac{1}{6}M^{\nu klm}_{\mu ij}K_{\nu klm}, \\ &M_{i\mu\nu} = 0, \qquad M_{i\mu j} = M^{\nu k}_{i\mu j}K_{\nu k} + \tfrac{1}{6}M^{\nu klm}_{i\mu j}K_{\nu klm} + \tfrac{1}{2}M^{kl}_{i\mu j}\hat{T}_{kl}, \qquad M_{ijk} = \tfrac{1}{2}M^{lm}_{ijk}\hat{T}_{lm}. \end{aligned} \quad (2.23)$$

In this case, working to second order in integrability is sufficient to guarantee closure of the holonomy algebra. The group generated by (2.19) was identified in [12] to be

$$\mathrm{Hol}_{M2} = SO(8)_+ \ltimes 12\mathbb{R}^{2(8_s)}. \quad (2.24)$$

It ought to be noted that the generalized connection Ω_M contains complete information about the generalized holonomy of the spacetime, as the complete set of integrability conditions (2.9) may be obtained through commutators and derivatives of Ω_M. In this sense, the algebra of the holonomy group can never be larger than the algebra obtained through the generators in Ω_M itself. However it can certainly be smaller. This is apparent for the M5-brane, where the $\Gamma^{(5)}$ generator is absent in the generalized curvature $\mathcal{R}_{MN}(\Omega)$ and its derivatives and also for the M2-brane, where $\Gamma^{(2)}$ is absent. For these examples, and in fact for all vacua considered in [12, 17], the generators appearing in Ω_M and those appearing in $\mathcal{R}_{MN}(\Omega)$ are nearly identical. As a result, the generalized holonomy group may be correctly identified at first order in integrability, and the higher order conditions only serve to complete the set of generators needed for closure of the algebra.

A different situation may arise, however, if for some reason (such as accidental symmetries) a greatly reduced set of generators appear in $\mathcal{R}_{MN}(\Omega)$. In such cases, examination of first order integrability may result in the misidentification of the actual generalized holonomy group. What happens here is that the algebra of the curvature $\mathcal{R}_{MN}(\Omega)$ at a single point p forms a subalgebra of the holonomy algebra. It is then necessary to explore the curvature at all points q connected by parallel transport to p in order to determine the actual holonomy algebra itself. Although this never occurs for Riemannian connections, we demonstrate below that this incompleteness of first order integrability does arise in the case of generalized holonomy.

3 Higher order integrability and the squashed S^7

For an example of the need to resort to higher order integrability to characterize the generalized holonomy group Hol($\mathcal{D}$), we turn to Freund-Rubin compactifications of eleven-dimensional supergravity. With vanishing gravitino, the Freund-Rubin ansatz for the 4-form field strength [18]

$$F_{\mu\nu\rho\sigma} = 3m\epsilon_{\mu\nu\rho\sigma}, \quad \mu = 0, 1, 2, 3, \tag{3.1}$$

with m constant and all other components vanishing, leads to spontaneous compactifications of the product form $\mathrm{AdS}_4 \times X^7$. Here X^7 is a compact, Einstein, Euclidean 7-manifold. Decomposing the eleven-dimensional Dirac matrices Γ_M as

$$\Gamma_M = (\gamma_\mu \otimes 1, \gamma_5 \otimes \Gamma_m), \qquad \mu = 0, 1, 2, 3, \qquad m = 1, \ldots, 7, \tag{3.2}$$

where γ_μ and Γ_m are four- and seven-dimensional Dirac matrices, respectively, and assuming the usual direct-product split $\epsilon(x^\mu) \otimes \eta(y^m)$ for eleven-dimensional spinors, the Killing spinor equation (1.1) splits as

$$\mathcal{D}_\mu \epsilon \;=\; \left(\partial_\mu \;+ \tfrac{1}{4}\omega_\mu{}^{\alpha\beta}\gamma_{\alpha\beta} + m\gamma_\mu\gamma_5\right)\epsilon = 0, \tag{3.3}$$

$$\mathcal{D}_m \eta \;=\; \left(\partial_m \;+ \tfrac{1}{4}\omega_m{}^{ab}\Gamma_{ab} - \tfrac{i}{2}m\Gamma_m\right)\eta = 0. \tag{3.4}$$

Since AdS_4 admits the maximum number of Killing spinors (four in this case), the number N of supersymmetries preserved in the compactification coincides with the number of Killing spinors of the internal manifold X^7, that is, with the number of solutions to the Killing spinor equation (3.4). Therefore we only need to concern ourselves with the Killing spinors on X^7.

An orientation reversal of X^7 or, alternatively, a sign reversal of $F_{(4)}$, provides another solution to the equations of motion (2.3), (2.4) and, hence, another acceptable Freund-Rubin vacuum [5, 19]. For definiteness, we shall call *left*-orientation the solution corresponding to the choice of sign of $F_{(4)}$ in (3.1), that leads to the Killing spinor equation (3.4), and *right*-orientation the solution corresponding to the opposite choice of sign of $F_{(4)}$:

$$\text{(right)} \qquad F_{\mu\nu\rho\sigma} = -3m\epsilon_{\mu\nu\rho\sigma}, \quad \mu = 0,1,2,3, \tag{3.5}$$

leading to the Killing spinor equation

$$\text{(right)} \qquad \mathcal{D}_m\eta = \left(\partial_m \; + \tfrac{1}{4}\omega_m{}^{ab}\Gamma_{ab} + \tfrac{i}{2}m\Gamma_m\right)\eta = 0. \tag{3.6}$$

From either (3.4) or (3.6), we see that the generalized connection $\mathcal{D}_m$ takes values in the algebra spanned by $\{\Gamma_{ab}, \Gamma_a\}$ and therefore the generalized structure group is SO(8). Notice, however, that both Killing spinor equations (3.4) and (3.6) share the same first order integrability condition [4, 19]

$$M_{mn}\eta \equiv [\mathcal{D}_m, \mathcal{D}_n]\eta = \tfrac{1}{4}\mathcal{R}_{mn}\eta \equiv \tfrac{1}{4}\mathcal{C}_{mn}\eta = \tfrac{1}{4}C_{mn}{}^{ab}\Gamma_{ab}\eta = 0, \tag{3.7}$$

where $C_{mn}{}^{ab}$ is the Weyl tensor of X^7 (thus demonstrating that, in this case the generalized curvature tensor is simply the Weyl tensor). Thus first order integrability is unable to distinguish between left and right orientations on the sphere. Then it might be possible that spinors η solving the integrability condition (3.7) will only satisfy the Killing spinor equation for one orientation, that is, satisfy (3.4) but not (3.6) (or the other way around). In fact, the skew-whiffing theorem [5, 19] for Freund-Rubin compactifications proves that this will, in general, be the case: it states that at most one orientation can give $N > 0$, with the exception of the round S^7, for which both orientations give maximal supersymmetry, $N = 8$. Since the preserved supersymmetry N is given by the number of singlets in the decomposition of the $\mathbf{8}_s$ of SO(8) (the generalized structure group) under the generalized holonomy group Hol($\mathcal{D}$), it is then clear that, in general, each orientation must have either a different generalized holonomy, or the same generalized holonomy but a different decomposition of the $\mathbf{8}_s$.

To illustrate this feature, consider compactifications on the squashed S^7 [5, 4]. This choice for X^7 has the topology of the sphere, but the metric is distorted away from that of the round S^7; it is instead the coset space SO(5) × SU(2)/SU(2) × SU(2) endowed with its Einstein metric [5, 4]. The compactification on the left-squashed S^7 preserves $N = 1$ supersymmetry whereas that on the right-squashed S^7 has $N = 0$; put another way, the integrability condition (3.7) has one non-trivial solution, corresponding in turn to a solution to the Killing spinor equation (3.4) (making the left-squashed S^7 preserve $N = 1$), but not to a solution to (3.6), which in fact has no solutions (yielding $N = 0$ for the right-squashed S^7). On the other hand, an analysis of the Weyl tensor of the squashed S^7 shows that there are only 14 linear combinations $\mathcal{C}_{mn}$ of gamma matrices in (3.7), corresponding to the generators of G_2 [4, 19]. Though appealing, G_2 cannot be, however, the generalized holonomy since the $\mathbf{8}_s$ of SO(8) would decompose as $\mathbf{8}_s \to \mathbf{8} \to \mathbf{7} + \mathbf{1}$ under SO(8) ⊃ SO(7) ⊃ G_2 regardless of the orientation, giving $N = 1$ for both left- and right-squashed solutions. We thus conclude that in this case the first order integrability condition (3.7) is insufficient to determine the generalized holonomy.

The resolution to this puzzle is naturally given by higher order integrability. In the case of the squashed S^7, it turns out that the second order integrability condition (2.5) is sufficient. For a general Freund-Rubin internal space X^7 this condition reads [6]

$$M_{lmn}\eta \equiv \tfrac{1}{4}[\mathcal{D}_l, \mathcal{C}_{mn}]\eta = \tfrac{1}{4}\left(\nabla_l C_{mn}{}^{ab}\Gamma_{ab} \mp 2imC_{mnl}{}^{a}\Gamma_a\right)\eta = 0\ , \qquad (3.8)$$

the $-$ sign corresponding to the left solution, and the $+$ to the right. For the squashed S^7, we find that only 21 of the M_{lmn} are linearly independent combinations of the Dirac matrices. The details are provided in the Appendix. Following the notation of [4, 19], we split the index m as $m = (0, i, \hat{\imath})$, with $i = 1, 2, 3$, $\hat{\imath} = 4, 5, 6 = \hat{1}, \hat{2}, \hat{3}$; then, with a suitable normalization, the linearly independent generators may be chosen to be

$$\mathcal{C}_{0i} = \Gamma_{0i} + \tfrac{1}{2}\epsilon_{ikl}\Gamma^{\hat{k}\hat{l}}, \quad \mathcal{C}_{ij} = \Gamma_{ij} + \Gamma_{\hat{\imath}\hat{\jmath}}, \quad \mathcal{C}_{i\hat{\jmath}} = -\Gamma_{i\hat{\jmath}} - \tfrac{1}{2}\Gamma_{j\hat{\imath}} + \tfrac{1}{2}\delta_{ij}\delta^{kl}\Gamma_{k\hat{l}} - \tfrac{1}{2}\epsilon_{ijk}\Gamma^{0\hat{k}}, \qquad (3.9)$$

$$M_{ij} = \Gamma_{\hat{\imath}\hat{\jmath}} \mp \tfrac{2}{3}\sqrt{5}im\epsilon_{ijk}\Gamma^{\hat{k}}, \quad M_i = \Gamma_{0\hat{\imath}} \mp \tfrac{2}{3}\sqrt{5}im\Gamma_i, \quad M = \delta^{kl}\Gamma_{k\hat{l}} \pm 2\sqrt{5}im\Gamma_0, \qquad (3.10)$$

the $-$ sign in front of m corresponding to the left solution and the $+$ to the right. Notice that there are 8 linearly independent generators in $\mathcal{C}_{i\hat{\jmath}}$ of (3.9), since $\delta^{kl}\mathcal{C}_{k\hat{l}} \equiv \mathcal{C}_{1\hat{1}} + \mathcal{C}_{2\hat{2}} + \mathcal{C}_{3\hat{3}} = 0$. The 14 generators $\mathcal{C}_{0i}$, $\mathcal{C}_{ij}$, $\mathcal{C}_{i\hat{\jmath}}$ span G_2 [4, 19], and are the same as those obtained from the first integrability condition (3.7), while the 7 additional generators M_{ij}, M_i, M were not contained in (3.7). Taken together, they generate the 21 dimensional algebra of SO(7), regardless of the orientation, provided

$$m^2 = \frac{9}{20}, \qquad (3.11)$$

in agreement with the Einstein equation for the squashed S^7 [19].

The embedding of SO(7) into SO(8) is, however, different for each orientation. We use $SO(7)_-$ to denote the embedding corresponding to the left solution and $SO(7)_+$ the right. While the spinor η transforms as an $\mathbf{8}_s$ of the generalized structure group SO(8), the decomposition of the $\mathbf{8}_s$ is different under left- and right-squashing. With our Dirac conventions, it turns out that $\mathbf{8}_s \rightarrow \mathbf{7} + \mathbf{1}$ under SO(8) $\supset$ SO(7)$_-$, giving $N = 1$ for the left-squashed S_7, while $\mathbf{8}_s \rightarrow \mathbf{8}$ under SO(8) $\supset$ SO(7)$_+$, giving $N = 0$ for the right-squashed S^7.

Since SO(7) is the subgroup of SO(8) that yields the correct branching rules of the $\mathbf{8}_s$ of SO(8), we conclude that second order integrability is sufficient in this case to identify all generators of the Lie algebra of Hol($\mathcal{D}_m$). Hence the generalized holonomy group of the Freund-Rubin compactification on the squashed S^7 is precisely SO(7). In this case, it is the embedding of SO(7) in SO(8) (with corresponding spinor decomposition $\mathbf{8}_s \rightarrow \mathbf{7} + \mathbf{1}$ or $\mathbf{8}_s \rightarrow \mathbf{8}$) that determines the number of preserved supersymmetries. This indicates that, for generalized holonomy, knowledge of the holonomy group *and* the embedding are both necessary in order to understand the number of preserved supersymmetries. While this was already observed in [12, 8] for non-compact groups, here we see that this is also true when the generalized holonomy group is compact.

The analysis of the squashed S^7, along with the brane solutions of the previous section, highlights several features of generalized holonomy. For the squashed S^7, the generalized holonomy algebra is in fact larger than that generated locally by the Weyl curvature at a point p. In this case, the algebra arising from lowest order integrability is already closed, but is only a subalgebra of the correct holonomy algebra. It is then mandatory to examine the second order integrability expression (3.8) in order to identify the generalized holonomy group. On the other hand, for the M2- and M5-branes, lowest order integrability, while lacking a complete set of generators, nevertheless closes on the correct holonomy algebra, and no really new information is gained at higher order.

Of course, in all cases, complete information is contained in the generalized connection Ω_M itself. However, examination of Ω_M directly can be misleading, as it may contain gauge degrees of freedom, which are unphysical. This is most clearly seen in the case of the round S^7, where $\Omega_m = \omega_m^{ab}\Gamma_{ab} - 2im\Gamma_m$ is certainly non-vanishing, while the generalized curvature $\mathcal{R}_{mn}$, given by the Weyl tensor, is completely trivial.

For generalized holonomy to be truly useful, we believe it ought to go beyond simply a classification scheme, and must yield methods for constructing new supersymmetric backgrounds. In much the same way that the rich structure of Riemannian holonomy teaches us a great deal about the geometry of Killing spinors on Riemannian manifolds, we anticipate that the formal analysis of generalized holonomy via connections on Clifford bundles may one day lead to a similar expansion of knowledge of supergravity structures and manifolds with fluxes. While much remains to be done, as we have only highlighted a few examples, we hope that a more complete understanding of higher order integrability for the curvature of generalized connections will soon lead to a better appreciation of the geometry behind generalized holonomy.

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A Second order integrability for the squashed S^7

In this Appendix we present the details of the derivation of the linearly independent generators (3.9) and (3.10) of the generalized holonomy group $\mathrm{Hol}(\mathcal{D}_m) = SO(7)$ of the squashed S^7, associated to the second order integrability condition (3.8). For convenience, we rewrite (3.8) with a modified normalization

$$M_{abc} = 5\left(\sqrt{5}\nabla_a C_{bcde}\Gamma^{de} \;\; - m' C_{bcad}\Gamma^d\right) \,, \tag{A.1}$$

where we have defined

$$m' = 2\sqrt{5}im, \tag{A.2}$$

and have chosen the $-$ sign in front of m' for definiteness.

To obtain M_{abc}, we have computed both the Weyl tensor C_{bcad} (given in [19]) and its covariant derivative $\nabla_a C_{bcde}$. We obtain, for the non-vanishing generators:

$$
\begin{aligned}
M_{00j} &= 4\Gamma_{0\hat{j}} - \epsilon_{jkl}\Gamma^{k\hat{l}} - 2m'\Gamma_j \ , && \text{(A.3)}\\
M_{00\hat{j}} &= 4\Gamma_{0j} + \epsilon_{jkl}\Gamma^{kl} + 2m'\Gamma_{\hat{j}} \ , && \text{(A.4)}\\
M_{0ij} &= 2\epsilon_{ijk}\Gamma^{0\hat{k}} + \Gamma_{i\hat{j}} - \Gamma_{j\hat{i}} \ , && \text{(A.5)}\\
M_{0i\hat{j}} &= -\epsilon_{ijk}\Gamma^{0k} + \Gamma_{ij} - 3\Gamma_{\hat{i}\hat{j}} + m'\epsilon_{ijk}\Gamma^{\hat{k}} \ , && \text{(A.6)}\\
M_{0\hat{i}\hat{j}} &= -3\Gamma_{i\hat{j}} + 3\Gamma_{j\hat{i}} - 2m'\epsilon_{ijk}\Gamma^{k} \ , && \text{(A.7)}\\
M_{h0j} &= \epsilon_{hjk}\Gamma^{0\hat{k}} + 2\Gamma_{h\hat{j}} + \delta_{hj}\delta^{kl}\Gamma_{k\hat{l}} + \Gamma_{j\hat{h}} + 2m'\delta_{hj}\Gamma_0 \ , && \text{(A.8)}\\
M_{h0\hat{j}} &= -\epsilon_{hjk}\Gamma^{0k} + \Gamma_{hj} + 3\Gamma_{\hat{h}\hat{j}} - m'\epsilon_{hjk}\Gamma^{\hat{k}} \ , && \text{(A.9)}\\
M_{hij} &= \delta_{hi}\Gamma_{0\hat{j}} - \delta_{hj}\Gamma_{0\hat{i}} + 4\epsilon_{ij}{}^{k}\Gamma_{h\hat{k}} - \epsilon_{hij}\delta^{kl}\Gamma_{k\hat{l}} - \epsilon_{ij}{}^{k}\Gamma_{k\hat{h}} + 2m'(\delta_{hj}\Gamma_i - \delta_{hi}\Gamma_j) \ , && \text{(A.10)}\\
M_{hi\hat{j}} &= 2\delta_{hi}\Gamma_{0j} + \delta_{ij}\Gamma_{0h} + \delta_{hj}\Gamma_{0i} + (2\epsilon_{jkl}\delta_{hi} - \tfrac{1}{2}\epsilon_{hkl}\delta_{ij} - \tfrac{1}{2}\epsilon_{ikl}\delta_{hj})\Gamma^{kl} - 3\epsilon_{hi}{}^{k}\Gamma_{\hat{k}\hat{j}} \\
&\quad + m'(2\delta_{hi}\Gamma_{\hat{j}} - \delta_{ij}\Gamma_{\hat{h}} + \delta_{hj}\Gamma_{\hat{i}}) \ , && \text{(A.11)}\\
M_{h\hat{i}\hat{j}} &= 3\delta_{hi}\Gamma_{0\hat{j}} - 3\delta_{hj}\Gamma_{0\hat{i}} + 3\epsilon_{hi}{}^{k}\Gamma_{k\hat{j}} - 3\epsilon_{hj}{}^{k}\Gamma_{k\hat{i}} + 2m'(\epsilon_{hij}\Gamma_0 + \delta_{hj}\Gamma_i - \delta_{hi}\Gamma_j) \ , && \text{(A.12)}\\
M_{\hat{h}0j} &= -6\Gamma_{\hat{h}\hat{j}} + 2m'\epsilon_{hjk}\Gamma^{\hat{k}} \ , && \text{(A.13)}\\
M_{\hat{h}0\hat{j}} &= 3\Gamma_{j\hat{h}} - 3\delta_{hj}\delta^{kl}\Gamma_{k\hat{l}} - m'(2\delta_{hj}\Gamma_0 + \epsilon_{hjk}\Gamma^{k}) \ , && \text{(A.14)}\\
M_{\hat{h}ij} &= 6\epsilon_{ij}{}^{k}\Gamma_{\hat{k}\hat{h}} + 2m'(\delta_{hj}\Gamma_{\hat{i}} - \delta_{hi}\Gamma_{\hat{j}}) \ , && \text{(A.15)}\\
M_{\hat{h}i\hat{j}} &= 3\delta_{hj}\Gamma_{0\hat{i}} - 3\delta_{ij}\Gamma_{0\hat{h}} - 3\delta_{hi}\epsilon_{jkl}\Gamma^{k\hat{l}} - 3\epsilon_{ij}{}^{l}\Gamma_{h\hat{l}} \\
&\quad + m'(-\epsilon_{hij}\Gamma_0 - 2\delta_{hj}\Gamma_i + \delta_{ij}\Gamma_h - \delta_{hi}\Gamma_j) \ , && \text{(A.16)}\\
M_{\hat{h}\hat{i}\hat{j}} &= 6\delta_{hj}\Gamma_{0i} - 6\delta_{hi}\Gamma_{0j} - 6\epsilon_{ij}{}^{k}\Gamma_{kh} + 4m'(\delta_{hj}\Gamma_{\hat{i}} - \delta_{hi}\Gamma_{\hat{j}}) \ . && \text{(A.17)}
\end{aligned}
$$

Not all the generators included in (A.3)–(A.17) are linearly independent, however. After all, they are built up from Dirac matrices $\{\Gamma_{ab}, \Gamma_a\}$, that is, from generators of SO(8), so at most 28 can be linearly independent.

In fact, only 21 linearly independent generators are contained in (A.3)–(A.17), as we will now show. Some redundant generators are straightforward to detect, since the Bianchi identities for the Weyl tensor, $\nabla_{[a}C_{bc]de} = 0$ and $C_{[bca]d} = 0$ place the restrictions

$$M_{[abc]} = 0 \ . \tag{A.18}$$

Further manipulations show that only the generators (A.11) and (A.16) are relevant, the rest being linear combinations of them. The generators (A.4), (A.6), (A.9), (A.13), (A.15) and (A.17) are obtained from (A.11):

$$\begin{aligned}
M_{00\hat{j}} &= \tfrac{1}{5}\delta^{kl}(M_{kl\hat{j}} + M_{kj\hat{l}} + M_{jk\hat{l}}) \ , && \text{(A.19)}\\
M_{0i\hat{j}} &= \tfrac{1}{5}\epsilon_{[i|}{}^{kl}(4M_{k|j]\hat{l}} - M_{|j]k\hat{l}}) - \tfrac{1}{5}\epsilon_{ij}{}^{k}\delta^{lm}M_{lm\hat{k}} \ , && \text{(A.20)}\\
M_{i0\hat{j}} &= -\tfrac{1}{5}\epsilon_{[i|}{}^{kl}(M_{k|j]\hat{l}} - 4M_{|j]k\hat{l}}) - \tfrac{1}{5}\epsilon_{ij}{}^{k}\delta^{lm}M_{lm\hat{k}} \ , && \text{(A.21)}\\
M_{\hat{j}0i} &= -\epsilon_{[i|}{}^{kl}(M_{k|j]\hat{l}} - M_{|j]k\hat{l}}) \ , && \text{(A.22)}\\
M_{\hat{h}ij} &= M_{ji\hat{h}} - M_{ij\hat{h}} \ , && \text{(A.23)}\\
M_{\hat{h}\hat{i}\hat{j}} &= \tfrac{1}{5}(M_{hi\hat{j}} - M_{hj\hat{i}} + M_{ih\hat{j}} - M_{jh\hat{i}}) - \tfrac{4}{5}\delta^{kl}(\delta_{hi}M_{kl\hat{j}} - \delta_{hj}M_{kl\hat{i}}) \ , && \text{(A.24)}
\end{aligned}$$

while (A.3), (A.5), (A.7), (A.8), (A.10), (A.12) and (A.14) are linear combinations of (A.16):

$$\begin{aligned}
M_{00j} &= \tfrac{1}{3}\delta^{kl}(M_{\hat{k}j\hat{l}} - M_{\hat{j}k\hat{l}}) \ , && \text{(A.25)}\\
M_{0hj} &= -\tfrac{1}{3}\epsilon_{h}{}^{kl}(M_{\hat{k}l\hat{j}} + 3M_{\hat{j}k\hat{l}}) + \tfrac{1}{3}\epsilon_{j}{}^{kl}\left(M_{\hat{k}l\hat{h}} + 3M_{\hat{h}k\hat{l}}\right) \ , && \text{(A.26)}\\
M_{0\hat{h}\hat{j}} &= \epsilon_{h}{}^{kl}(M_{\hat{k}l\hat{j}} + 2M_{\hat{j}k\hat{l}}) - \epsilon_{j}{}^{kl}(M_{\hat{k}l\hat{h}} + 2M_{\hat{h}k\hat{l}}) \ , && \text{(A.27)}\\
M_{h0j} &= -\tfrac{1}{6}\epsilon_{h}{}^{kl}(2M_{\hat{k}l\hat{j}} + 5M_{\hat{j}k\hat{l}}) + \tfrac{1}{6}\epsilon_{j}{}^{kl}M_{\hat{h}k\hat{l}} \ , && \text{(A.28)}\\
M_{hij} &= \tfrac{1}{2}\delta^{kl}\left(\delta_{hi}(M_{\hat{k}l\hat{j}} - 2M_{\hat{j}k\hat{l}}) - \delta_{hj}(M_{\hat{k}l\hat{i}} - 2M_{\hat{i}k\hat{l}})\right) \\
&\quad + \tfrac{7}{3}(M_{\hat{h}i\hat{j}} - M_{\hat{h}j\hat{i}}) + M_{\hat{i}j\hat{h}} - M_{\hat{j}i\hat{h}} - \tfrac{2}{3}\epsilon_{h}{}^{kl}\epsilon_{ij}{}^{m}(M_{\hat{k}l\hat{m}} + 4M_{\hat{m}k\hat{l}}) \ , && \text{(A.29)}\\
M_{h\hat{i}\hat{j}} &= M_{\hat{i}h\hat{j}} - M_{\hat{j}h\hat{i}} \ , && \text{(A.30)}\\
M_{\hat{h}0\hat{j}} &= \epsilon_{h}{}^{kl}(M_{\hat{k}l\hat{j}} + M_{\hat{j}k\hat{l}}) - \epsilon_{j}{}^{kl}M_{\hat{h}k\hat{l}} \ , && \text{(A.31)}
\end{aligned}$$

Moreover, both (A.11) and (A.16) contain redundant generators. The following combinations obtained from (A.11):

$$\begin{aligned}
\mathcal{C}_{0i} &= \tfrac{1}{6}\delta^{kl}M_{ik\hat{l}} \ , && \text{(A.32)}\\
\mathcal{C}_{ij} &= -\tfrac{1}{30}\epsilon_{[i|}{}^{kl}(M_{k|j]\hat{l}} - 9M_{|j]k\hat{l}}) - \tfrac{1}{30}\epsilon_{ij}{}^{k}\delta^{lm}M_{lm\hat{k}} \ , && \text{(A.33)}\\
M_{ij} &= \tfrac{1}{6}M_{\hat{j}0i} = -\tfrac{1}{6}\epsilon_{[i|}{}^{kl}(M_{k|j]\hat{l}} - M_{|j]k\hat{l}}) && \text{(A.34)}
\end{aligned}$$

(the expressions of which in terms of Dirac matrices are the first two equations in (3.9) and the first equation in (3.10), respectively) are linearly independent. Thus (A.11) [and so (A.4), (A.6), (A.9), (A.13), (A.15) and (A.17)] can be uniquely written in terms of them:

$$\begin{aligned}
M_{hi\hat{j}} &= 2\delta_{hi}\mathcal{C}_{0j} + \delta_{ij}\mathcal{C}_{0h} + \delta_{hj}\mathcal{C}_{0i} + (2\epsilon_{j}{}^{kl}\delta_{hi} - \tfrac{1}{2}\epsilon_{h}{}^{kl}\delta_{ij} - \tfrac{1}{2}\epsilon_{i}{}^{kl}\delta_{hj})\mathcal{C}_{kl} \\
&\quad -3\delta_{hi}\epsilon_{j}{}^{kl}M_{kl} - 3\epsilon_{hi}{}^{k}M_{kj} \ . && \text{(A.35)}
\end{aligned}$$

Similarly, the following combinations contained in (A.16):

$$\mathcal{C}_{i\hat{j}} = \tfrac{1}{3}\epsilon_i{}^{kl} M_{\hat{j}k\hat{l}} - \tfrac{1}{6}\epsilon_j{}^{kl}(M_{\hat{k}l\hat{i}} + M_{\hat{i}k\hat{l}}) \ , \tag{A.36}$$

$$M_i = \tfrac{1}{12}\delta^{kl}(M_{\hat{k}l\hat{i}} - 2M_{\hat{i}k\hat{l}}) \ , \tag{A.37}$$

$$M = -\tfrac{1}{6}\epsilon^{hij} M_{\hat{h}i\hat{j}} \tag{A.38}$$

(which can be written in terms of Dirac matrices as in the final equation of (3.9) and the last two equations of (3.10), respectively) are linearly independent. Hence (A.16) [and so (A.3), (A.5), (A.7), (A.8), (A.10), (A.12) and (A.14)] can be uniquely written in terms of them:

$$\begin{aligned} M_{\hat{h}i\hat{j}} = {} & 6\delta_{hi}\epsilon_j{}^{kl}\mathcal{C}_{k\hat{l}} - 2\epsilon_{ij}{}^{k}(\mathcal{C}_{k\hat{h}} - 2\mathcal{C}_{h\hat{k}}) \\ & +6\delta_{hj}M_i + 3\delta_{ih}M_j - 3\delta_{ij}M_h - \epsilon_{hij}M \ . \end{aligned} \tag{A.39}$$

In summary, the linearly independent generators associated to the second order integrability condition (3.8) are the 21 linearly independent generators (3.9) and (3.10), namely $\{\mathcal{C}_{0i}, \mathcal{C}_{ij}, \mathcal{C}_{i\hat{j}}, M_{ij}, M_i, M\}$ (notice that $\mathcal{C}_{i\hat{j}}$ contains 8 generators, since it is traceless), which close into an algebra whenever m^2 takes the value required by the equations of motion, $m^2 = \frac{9}{20}$. Since the only condition for the generators to close the algebra is placed on m^2, they will close regardless of the orientation (*i.e.*, of the sign of m). In fact, they generate the 21-dimensional algebra of $SO(7)$, for both orientations.

Note that, by further choosing linear combinations of (3.9), the 14 generators $\{\mathcal{C}_{0i}, \mathcal{C}_{ij}, \mathcal{C}_{i\hat{j}}\}$ of G_2 may be re-expressed in symmetric form

$$\begin{aligned} &\Gamma_{1\hat{1}} - \Gamma_{2\hat{2}}, \qquad \Gamma_{1\hat{1}} - \Gamma_{3\hat{3}}, \\ &\Gamma_{0\hat{i}} + \Gamma_{j\hat{k}}, \qquad \Gamma_{0\hat{i}} + \Gamma_{\hat{j}k}, \qquad (i,j,k = 123, 231, 312) \\ &\Gamma_{0i} + \Gamma_{\hat{j}\hat{k}}, \qquad \Gamma_{0i} - \Gamma_{jk}, \qquad (i,j,k = 123, 231, 312). \end{aligned} \tag{A.40}$$

The 7 additional generators $\{M_{ij}, M_i, M\}$ of (3.10) extending (A.40) to SO(7) may also be simplified in appropriate linear combinations. One possible set of generators is given by:

$$\begin{aligned} &\Gamma_{1\hat{1}} \pm i\Gamma_0, \\ &\Gamma_{0\hat{i}} \mp i\Gamma_i, \\ &\Gamma_{\hat{j}\hat{k}} \mp i\Gamma_{\hat{i}}, \qquad (i,j,k = 123, 231, 312). \end{aligned} \tag{A.41}$$

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