Finite-size scaling of the left-current correlator with non-degenerate quark masses

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Abstract: We study the volume dependence of the left-current correlator with non-degenerate quark masses to next-to-leading order in the chiral expansion. We consider three possible regimes: all quark masses are in the $\epsilon$-regime, all are in the $p$-regime and a mixed-regime where the lightest quark masses satisfy $m_\pi \Sigma V \leq 1$ while the heavier $m_\pi \Sigma V \gg 1$. These results can be used to match lattice QCD and the Chiral Effective Theory in a large but finite box in which the Compton wavelength of the lightest pions is of the order of the box size. We consider both the full and partially-quenched results.

Keywords: Lattice QCD, Chiral Lagrangians.
1. Introduction

The first principles determination of the low-energy couplings of the Chiral Effective Theory, that describes the meson interactions at low momenta, is one of the milestones of lattice QCD. This matching can only be carried out reliably close to the chiral limit, and this is often a limitation for lattice simulations, because the computational cost of lattice simulations increases very significantly with decreasing quark masses and increasing volumes.

Even though there has been important algorithmic progress in recent years, it seems quite difficult to reach the range of the \( u \) and \( d \) quark masses, at least within the \( p \)-regime, i.e. satisfying the condition \( M_\pi L \gg 1 \).

The \( \epsilon \)-regime \cite{1,2} has been advocated \cite{3} as an alternative to perform the matching, that a priori could be more economic in the sense that the quark masses can be taken to zero without increasing the box-size proportionally, since in this regime \( M_\pi L \leq 1 \). Finite-size effects are large in this situation, but they are calculable within the Chiral Effective Theory in terms of the infinite-volume low-energy constants \cite{4}. Even though the truly chiral regime requires that the volume is scaled to infinity eventually, not just the quark mass to zero, the scaling with the volume at zero quark mass is more universal in the sense...
that it involves less low-energy couplings, since most of the operators that appear at higher orders in Chiral Perturbation Theory (ChPT) include explicit powers of the quark mass.

In particular it can be shown that in the $\epsilon$-regime only the leading order couplings $F$ and $\Sigma$ appear in two-point functions\(^1\) at next-to-leading order (NLO) of ChPT.

A number of correlation functions have been computed to NLO in the $\epsilon$-regime of ChPT. The two-point functions of scalar, pseudoscalar, vector and axial-vector correlators were presented in [4]. In [3, 5] the same correlators were computed in quenched ChPT and also in the full theory, but in a fixed topological sector [12]. Three-point functions relevant for determining the weak low-energy couplings were first presented in [8] both for the full and quenched theories. The $\epsilon$-regimes has also been recently applied to the study of baryon properties [9]. These results have been used in various simulations to extract low-energy couplings mostly in the quenched approximation [3], but more recently also in unquenched simulations [10].

In many of these computations quark masses were taken degenerate. The purpose of this paper is to present the results for the left-current correlator for non-degenerate quark masses. The methods developed here can be readily applied to other correlation functions.

More concretely, we will consider the situation with $N_v$ heavier quarks with common mass $m_v$ and $N_s$ light quarks with common mass $m_s$. At this point one could imagine having three different situations:

- $m_v \Sigma V \gg 1$: all quarks are in the $p$-regime.
- $m_v \Sigma V \ll 1$: all quarks are in the $\epsilon$-regime.
- $m_v \Sigma V \leq 1$, $m_s \Sigma V \gg 1$: some quarks are in the $\epsilon$ and some in the $p$ regime. It appears natural to identify $m_v$ with the light quarks $u, d$ and $m_s$ with the $s$ quark mass in realistic simulations.

The explicit $N_v$ and $N_s$ dependences will be shown in such a way that the partial quenching of the $v$ or $s$ quarks can be easily done via the replica method [11]. Considering the partial quenching of the lighter quarks is interesting in the context of mixed-actions [19], where the valence and sea quarks are treated in different regularizations, for example with overlap valence quarks and Wilson sea quarks.

The structure of the paper is as follows. In section 2, we present the results for the current correlator in the full theory, in the $p$, $\epsilon$ and mixed regimes. In 3 we present the results for the partially-quenched theory, where the $v$ quarks are quenched, also in the three regimes. Although the $\epsilon$ and $p$ regime results could have been obtained from earlier literature, we include them for completeness. The mixed-regime on the other hand involved a new method to separate the perturbative and non-perturbative modes, that will be discussed in detail. In section 4 we present our conclusions and outlook.

The results in this paper rely heavily on previous similar computations in [3]. We refer the reader to those papers for further details of some intermediate steps.

\(^1\)Up to contact terms.
2. Full theory results

We start by considering $SU(N_s + N_v)$ Yang-Mills theory with $N_s$ flavours with masses $m_s$ and $N_v$ flavours with masses $m_v$. The quark part of the Euclidean continuum Lagrangian reads

$$L_E = \sum_{r=1}^{N_v} \bar{\psi}_r (\gamma_\mu D_\mu + m_v) \psi_r + \sum_{r=N_v+1}^{N} \bar{\psi}_r (\gamma_\mu D_\mu + m_s) \psi_r,$$

(2.1)

where $r$ is a flavour index; the Dirac matrices $\gamma_\mu$ are assumed normalised such that $\gamma_\mu^\dagger = \gamma_\mu$, $\{\gamma_\mu, \gamma_\nu\} = 2 \delta_{\mu\nu}$; $D_\mu$ is the covariant derivative and $N \equiv N_v + N_s$. We will consider external sources $J$ that have non-zero elements only in the $SU(N_v)$ flavour subgroup.

At large distances, the physics of QCD can be reproduced by chiral perturbation theory. The leading order chiral Lagrangian reads

$$L_{\text{ChPT}} = \frac{F^2}{4} \text{Tr} \left[ \partial_\mu U \partial_\mu U^\dagger \right] - \frac{\Sigma}{2} \text{Tr} \left[ e^{i\theta} MU + U^\dagger M e^{-i\theta} \right],$$

(2.2)

where $U \in SU(N)$. The mass matrix is diagonal with eigenvalues $(m_v, \ldots, m_v, m_s, \ldots, m_s)$ and $\theta$ is the vacuum angle. Apart from $\theta$, this Lagrangian contains two parameters, the pseudoscalar decay constant $F$ and the chiral condensate $\Sigma$. At NLO in the momentum expansion, additional operators appear in the chiral Lagrangian, with the associated low-energy constants $L_1, L_2, \ldots$ [13]. For a general $N$, the number of independent couplings is $11+2$ [14], but for $N = 2$ and $N = 3$, not all of them are independent and smaller subsets of $7+3$ and $10+2$ couplings respectively are commonly used in this situation [13]. These couplings do not depend on the quark masses, but do depend on $N$.

One of the simplest correlation functions that can be used to measure $F$ and is also sensitive to the NLO couplings $L_4, L_5, L_6$ and $L_8$ is the left-current two-point function. The numerical advantages of such correlator have been discussed in [15].

In QCD, the left-handed flavour current can be formally defined as

$$J_\mu^a \equiv \bar{\psi} T^a \gamma_\mu P_- \psi,$$

(2.3)

where $T^a$ is a traceless generator of the subgroup $SU(N_v)$, and all colour, flavour, and spinor indices are assumed contracted. $P_- = (1 - \gamma_5)/2$ is the left projector. Note that $J_\mu^a$ defined this way is formally purely imaginary.

The two-point correlation function between the left-handed currents, averaged over the spatial volume, now reads:

$$\text{Tr} [T^a T^b] C(x_0) \equiv \int d^3x \left\langle J^a_0(x) J^b_0(0) \right\rangle.$$

(2.4)

On the ChPT side, the operator corresponding to eq. (2.3) becomes, at leading order in the momentum expansion,

$$J_\mu^a = \frac{F^2}{2} \text{Tr} \left[ T^a U \partial_\mu U^\dagger \right].$$

(2.5)

We use this “unphysical” convention since it removes a number of unnecessary overall minus signs from the ChPT predictions.
The two-point correlation function $C(x_0)$ is defined (apart from contact terms) by
\[
\mathrm{Tr}[T^a T^b] C(x_0) = \int\! d^3 x \left\langle \mathcal{J}^a_0(x) \mathcal{J}^b_0(0) \right\rangle. \tag{2.6}
\]

\section*{2.1 The $p$-regime}
In the $p$-regime, we express the outcome as a power series in $M^2/F^2$, where $M^2 \equiv 2m\Sigma/F^2$ is the pseudoscalar mass. The power-counting rules for the $p$-regime are
\[
M \sim p \sim L^{-1},
\]
where $p$ is assumed a small quantity, $p \ll 4\pi F$. The temporal extent $T$ can in principle be small or large, as long as $T \gtrsim 1/(4\pi F)$. It follows from these assignments that the Goldstone field $\xi$, defined through $U = \exp(2i\xi/F)$, behaves effectively as a small quantity, and can be expanded in. Here we have also set $\theta = 0$, as is usually done in the $p$-regime.

Inserting the Taylor-series of $U$ into eq. (2.2), the propagator in the quark basis becomes
\[
\left\langle \xi_{ca}(x) \xi_{db}(y) \right\rangle = \frac{1}{2} \left[ \begin{matrix} \delta_{cb}\delta_{da}G(x - y; M_{ab}^2) - \delta_{ca}\delta_{db}E(x - y; M_{ab}^2, M_{cc}^2) \end{matrix} \right], \tag{2.8}
\]
where
\[
G(x; M_{ab}^2) \equiv \frac{1}{V} \sum_{n \in \mathbb{Z}^4} \frac{e^{ip \cdot x}}{p^2 + M_{ab}^2}, \quad p \equiv (p_0, \mathbf{p}) \equiv 2\pi \left( \frac{n_0}{T}, \frac{\mathbf{n}}{L} \right). \tag{2.9}
\]

$V \equiv TL^3$ is the volume and $M_{ab}^2 = \Sigma(m_a + m_b)$ is the mass of a meson constructed out of an $a$ and $b$-flavour quark, which in practice can be either $s$ or $v$. On the other hand the singlet contribution is
\[
E(x; M_{ss}^2, M_{cc}^2) \equiv \frac{1}{V} \sum_{n \in \mathbb{Z}^4} \frac{e^{ip \cdot x}}{(p^2 + M_{ss}^2)(p^2 + M_{cc}^2)F(p)}, \tag{2.10}
\]
with
\[
F(p) \equiv \left[ \frac{N_s}{p^2 + M_{ss}^2} + \frac{N_v}{p^2 + M_{vv}^2} \right], \tag{2.11}
\]
if $N_s + N_v \neq 0$.

The result for the left-current correlation function in the $p$-regime, after spatial integration over the source positions and up to contact terms, is:
\[
C^p(x_0) = F^2/2 \left\{ (1 + \Delta_F) M_{vv}^2 P_v(x_0) - \frac{N_v}{F^2} \frac{dG(0; M_{vv}^2)}{dT} - \frac{N_s}{F^2} \frac{dG(0; M_{ss}^2)}{dT} \right. \\
+ \Delta_M \frac{d}{dM_{vv}^2} \left[ M_{uu}^2 P_u(x_0) \right] \right\}, \tag{2.12}
\]
where
\[
\Delta_F = -\frac{N_s}{F^2} G(0; M_{ss}^2) - \frac{N_v}{F^2} G(0; M_{vv}^2) + \frac{8}{F^2} \left[ N_s M_{ss}^2 L_4 + N_v M_{vv}^2 (N_v L_4 + L_5) \right], \tag{2.13}
\]
\[
\Delta_M = \frac{E(0; M_{uu}^2, M_{vv}^2)}{F^2} - \frac{8}{F^2} \left[ (N_s M_{ss}^2 + N_v M_{vv}^2) (L_4 - 2L_6) + M_{vv}^2 (L_5 - 2L_8) \right]. \tag{2.14}
\]
The temporal dependence (for $|x_0| \leq T$) is contained in the function

$$P_v(x_0) \equiv \int d^3x G(x; M_{vv}^2) = \frac{1}{T} \sum_{p_0} \frac{e^{ip_0x_0}}{p_0^2 + M_{vv}^2} = \frac{\cosh[M_{vv}(T/2 - |x_0|)]}{2M_{vv} \sinh[M_{vv}T/2]}.$$  \hfill (2.15)

Up to the second term, which is a constant finite-volume effect, the NLO propagator has the same temporal dependence as the LO, if the decay constant and pseudoscalar mass squared are scaled by a relative correction given by $\Delta F$ and $\Delta M$ respectively:

$$F_{NLO}^2 = F^2(1 + \Delta F), \quad M_{NLO}^2 = M_{vv}^2(1 + \Delta M).$$ \hfill (2.16)

These results agree with those obtained by Gasser and Leutwyler in infinite volume \cite{libeskind:1996} and finite volume \cite{gasser:2001} for $N_v = 2$ and $N_s = 1$. Finite volume corrections to $F$ and $\Sigma$ have been also obtained to two-loops \cite{gasser:2001}.

The finite volume corrections can be isolated by

$$G(0; M^2) \equiv G_\infty(M^2) + G_V(M^2) \quad E(0; M^2, M'^2) \equiv E_\infty(M^2, M'^2) + E_V(M^2, M'^2),$$ \hfill (2.17)

where $G_\infty, E_\infty$ are the infinite-volume closed propagators, where instead of momentum sums in eqs. (2.9) and (2.10) there are integrals. The (finite) functions $G_V(M^2)$ and $E_V(M^2, M'^2)$ incorporate the volume dependence \cite{gasser:2001}.

The limit in which $m_s \rightarrow m_v$, we of course recover the degenerate mass result of \cite{gasser:2001}.

2.2 The $\epsilon$-regime

We consider now the case where all quark masses satisfy $m_{s/v} \Sigma V \leq 1$. The results for the correlator obtained in a $\theta$-vacuum in \cite{gasser:1987} and in a fixed-topology in \cite{gasser:2001} are valid for non-degenerate quark masses:

$$C^{\epsilon}(x_0) = \frac{F^2}{2T} \left[ 1 + \frac{N}{F^2} \left( \frac{\beta_1}{\sqrt{V}} - \frac{T^2k_{00}}{V} \right) + \frac{2T^2}{F^2\epsilon} \mu_s^{(N_s,N_v)}(M) h_1(\hat{x}_0) \right],$$ \hfill (2.18)

where $\hat{x}_0 = x_0/T$. The only non-trivial mass dependence is in the function $\mu_s^{(N_s,N_v)}(M)$:

$$\mu_s^{(N_s,N_v)}(M) \equiv \int_{U(N)} dU \frac{\mu_v}{2N_v} \text{Tr} \left[ P_vU + U^\dagger P_v \right] \left( \text{det} U \right)^{\nu} \exp \left( \frac{\Sigma V}{2} \text{Tr} \left[ M U + U^\dagger M \right] \right),$$ \hfill (2.19)

where $P_v$ is the projector onto the sector of masses $m_v$, and $\mu_v \equiv m_v \Sigma V$.

The constants $\beta_1$ and $k_{00}$ are related to the (dimensionally regularised) value of

$$\bar{G}(x, M^2) \equiv \frac{1}{V} \sum_{n \in \mathbb{Z}^4} \left( 1 - \delta_n^{(4)} \right) \frac{e^{ip\cdot x}}{p^2 + M^2},$$ \hfill (2.20)

\footnote{The UV divergences of these quantities for $d \approx 4$ cancel against those \cite{gasser:1987} in the $L_i$’s as expected.}

\footnote{In ref. \cite{gasser:2001}, the function $G_V$ was denoted by $g_1$.}
by
\[ \bar{G}(0,0) = -\frac{\beta_1}{\sqrt{V}}, \quad T\frac{d}{dT}\bar{G}(0,0) = \frac{T^2k_{00}}{V}. \] (2.21)

Introducing \( \rho \equiv T/L \) and
\[
\hat{\alpha}_p(t_0, t_i) = \int_0^1 dt t^{p-1} \left[ S\left( \frac{t_0^2}{t} \right) S^3\left( \frac{t_i^2}{t} \right) - 1 \right],
\] (2.22)

where \( S(x) \) is an elliptic theta-function, \( S(x) = \sum_{n=-\infty}^{\infty} \exp(-\pi x n^2) = \omega_3(0, \exp(-\pi x)) \), a numerical evaluation of these coefficients is allowed by (see, e.g., refs. [16, 4])

\[
\beta_1 = \frac{1}{4\pi} \left[ 2 - \hat{\alpha}_{-1}\left( \rho^\frac{3}{4}, \rho^{-\frac{1}{4}} \right) - \hat{\alpha}_{-1}\left( \rho^{-\frac{3}{4}}, \rho^{\frac{1}{4}} \right) \right],
\] (2.23)

\[
k_{00} = \frac{1}{12} - \frac{1}{4} \sum_{n \neq 0} \frac{1}{\sinh^2(\pi \rho |n|)}. \] (2.24)

The function \( h_1(\tau) \) appearing in eq. (2.18) reads (for \( |\tau| \leq 1 \))
\[
h_1(\tau) = \frac{1}{2} \left[ \left| \tau - \frac{1}{2} \right|^2 - \frac{1}{12} \right]. \] (2.25)

The integral of eq. (2.19) for non-degenerate quark masses can be written in terms of a functional derivative:
\[
\mu_{\sigma}^{(N_s, N_v)}(M) = \frac{m_v}{N_v} \frac{1}{Z^{(N_s, N_v)}_{\nu}(M_J)} \frac{\partial Z^{(N_s, N_v)}_{\nu}(M_J)}{\partial J} \bigg|_{J=0}, \] (2.26)

where
\[
Z^{(N_s, N_v)}_{\nu}(M_J) \equiv \int_{U(N)} (\det U)^\nu \exp\left( \frac{\Sigma V}{2} \text{Tr} \left[ M_J U + U^\dagger M_J \right] \right), \] (2.27)

where \( M_J \) is the block-diagonal matrix:
\[
M_J = \begin{pmatrix} (m_v + J)I_v & 0 \\ 0 & m_s I_s \end{pmatrix}, \] (2.28)

and \( I_v \) and \( I_s \) are the identity matrices in the \( v \) and \( s \) subgroups respectively.

The functional \( Z^{(N_s, N_v)}_{\nu} \) is known in terms of modified Bessel functions [21] [22]
\[
Z^{(N_s, N_v)}_{\nu}(M_J) = C_{\nu} \frac{\det \left[ \mu_i^{j-1} I_{v+j-1}(\mu_i) \right]_{i,j=1,...,N}}{\prod_{j=1,...,N} (\mu_j^2 - \mu_i^2)}, \] (2.29)

where \( I_n(x) \) is the modified Bessel function and \( \mu_i^2 \) are the eigenvalues of the matrix \( M_J^4 M_J \) multiplied by \( (\Sigma V)^2 \).
In our case we just have to consider two distinct eigenvalues $m_s^2$ and $(m_v + J)^2$. As a useful example we consider the case with $N_v = 2$ and $N_s = 1$, corresponding to the 2 + 1 flavour QCD:

$$Z^{(1,2)}_{\nu}(M) = \frac{1}{2} I_{\nu}(\mu_s) \left( I_{\nu}(\mu_s) - I_{\nu+1}(\mu_v) I_{\nu-1}(\mu_v) \right) + \frac{I_{\nu}(\mu_v)}{\mu_s^2 - \mu_v^2} \left[ \mu_s I_{\nu+1}(\mu_v) I_{\nu}(\mu_s) - \mu_v I_{\nu}(\mu_v) I_{\nu+1}(\mu_s) \right],$$

while

$$\mu \sigma^{(1,2)}_{\nu}(M) = \frac{\mu_v}{2 Z^{(1,2)}_{\nu}(M)} \frac{d}{d\mu_v} Z^{(2,1)}_{\nu}(M).$$

Another interesting case is that of $N_v = 2$ and $N_s = 0$, corresponding to 2 flavour QCD. In this case we have

$$Z^{(0,2)}_{\nu}(M) = \frac{1}{2} I_{\nu}(\mu_v) \left( I_{\nu}(\mu_v) - I_{\nu+1}(\mu_v) I_{\nu-1}(\mu_v) \right),$$

while

$$\mu \sigma^{(0,2)}_{\nu}(M) = \frac{I_{\nu+1}(\mu_v) I_{\nu-1}(\mu_v)}{I_{\nu}(\mu_v) \left( I_{\nu+1}(\mu_v) I_{\nu-1}(\mu_v) \right) .$$

### 2.3 The mixed-regime

Now we turn to the most complicated case of the mixed regime. In this case, some quarks are in the $\epsilon$ and some in the $p$ regime and therefore a different factorization of zero and non-zero modes is needed. As in the previous sections we start by considering the full theory case with the $v$ and $s$ quarks are both unquenched.

The power-counting for this regime is

$$m_v \sim \epsilon^4 \quad m_s \sim p^2 \sim L^{-2} \sim \epsilon^2.$$

The inspection of the $p$-regime propagator of eq. (2.8) shows that the modes that become massless in the $m_v \to 0$ limit are those corresponding to the generators of $SU(N_v)$. Therefore a factorization that would treat the zero-momentum modes of these fields non-perturbatively is

$$U = \begin{pmatrix} U_0 & 0 \\ 0 & I_s \end{pmatrix} \exp \left( \frac{2i\xi}{F} \right),$$

where $I_s$ is the identity matrix in the $s$ sector and $U_0 \in SU(N_v)$. The perturbative fields $\xi$ satisfy

$$\int d^4x \ Tr \left[ T^a \xi \right] = 0,$$

where $T^a$ is a generator of the subgroup $SU(N_v)$.
It is convenient to include the $\theta$ dependence as

$$e^{i\theta} U = \begin{pmatrix} e^{i\frac{\xi}{N_v} U_0} & 0 \\ 0 & I_s \end{pmatrix} \exp \left( \frac{2i\xi}{F} \right) = \begin{pmatrix} U_0 & 0 \\ 0 & I_s \end{pmatrix} \exp \left( \frac{2i\xi}{F} \right),$$

(2.37)

If the topology is fixed so that $\theta$ is integrated over, the path integral at LO in the $\epsilon$ expansion is

$$Z_\nu \simeq \int d\xi J(\xi) e^{-\int d^4x (\text{Tr}[\partial_\mu \xi(x)\partial_\mu \xi(x)] + M_\nu^2 \text{Tr}[P_\nu \xi^2])}
\int_{U(N_v)} dU_0 \det(U_0)^2 e^{\frac{1}{2} \text{Tr}[P_\nu (M U_0 + U_0^\dagger M)]},$$

(2.38)

and the integration over the zero and non-zero modes factorizes.

The term $J(\xi)$ is the Jacobian that comes about from the change in the measure when moving from $[dU]$ to $[dU_0][d\xi]$, which will contribute at NLO as in the $\epsilon$-regime. We describe the computation of this measure term in appendix A.

The integration over the $\xi$ variables is done in perturbation theory. In order to write the $\xi$ propagator, we need to distinguish the indices in the $v$ and $s$ sector, we denote the former by latin letters $a, b, \ldots$ and the latter by greek ones $\alpha, \beta, \ldots$. The propagator for the $\xi$ fields is:

$$\langle \xi_{ca}(x) \xi_{db}(y) \rangle = \frac{1}{2} \left[ \delta_{cb} \delta_{da} \bar{G}(x - y; 0) - \delta_{ca} \delta_{db} \left( \bar{E}(x - y; 0, 0) - \frac{N_a}{N_s V M_{ss}^2} \right) \right],$$

(2.39)

$$\langle \xi_{ca}(x) \xi_{db}(y) \rangle = \frac{1}{2} \delta_{cb} \delta_{da} \bar{G}(x - y; \frac{M_{2a}^2}{2})$$

(2.40)

$$\langle \xi_{ca}(x) \xi_{db}(y) \rangle = -\frac{1}{2} \delta_{ca} \delta_{db} \frac{1}{N} \bar{G}(x - y; \frac{N_v}{N} M_{ss}^2)$$

(2.41)

$$\langle \xi_{\gamma a}(x) \xi_{\delta b}(y) \rangle = \frac{1}{2} \left[ \delta_{\gamma \delta} \delta_{ab} \bar{G}(x - y; M_{ss}^2) - \delta_{\gamma a} \delta_{\delta b} \bar{E}(x - y; M_{ss}^2) \right],$$

(2.42)

where $\bar{G}(x, M^2)$ is defined in eq. (2.20) and

$$\bar{E}(x; M_{aa}^2, M_{cc}^2) \equiv \frac{1}{V} \sum_{n \in \mathbb{Z}^4} \left( 1 - \delta_{n,0}^{(4)} \right) \frac{e^{ip \cdot x}}{(p^2 + M_{aa}^2)(p^2 + M_{cc}^2) F(p)},$$

(2.43)

with

$$F(p) \equiv \left[ \frac{N_s}{p^2 + M_{ss}^2} + \frac{N_v}{p^2} \right].$$

(2.44)

It is easy to check in eqs. (2.39)-(2.42) that the $\xi$ fields in the SU($N_v$) subgroup contain no zero modes as expected from eq. (2.36), while the others do contain zero modes.

The computation of the left-current correlator at NLO, that is at relative order $\epsilon^2$, gives a result which has the same structure as in the $\epsilon$-regime

$$C^{\text{mixed}}(x_0) = \frac{F^2}{2T} \left[ 1 - \frac{1}{F^2} \left( N_v G(0, 0) + N_s \bar{G} \left( 0, \frac{M_{ss}^2}{2} \right) - 8 L_4 N_s M_{ss}^2 + \frac{T_2^2}{V} \left( N_v k_{00} + N_s k_{00}^s \right) \right) \right]$$

$$+ \frac{2T^2}{F^2 V} \mu \sigma_{\nu}^{(0, N_v)}(M) h_1(\bar{x}_0) \quad \text{.}$$

(2.45)
where
\[
\frac{T^2 k_{00}^s}{V} \equiv T \frac{d}{dT} G \left( 0, \frac{M_{ss}^2}{2} \right).
\] (2.46)

and
\[
\mu \sigma_v^{(0, N_v)}(M) \equiv \int_{U(N_v)} dU \frac{\mu_v}{2N_v} \text{Tr} \left[ (U + U^\dagger)^{\nu} \right] \left( \det U \right)^{\nu} \exp \left( \frac{\mu_v}{2} \text{Tr} \left[ U + U^\dagger \right] \right). \] (2.47)

2.4 Decoupling of the s quarks

It is useful to rewrite the result of eq. (2.45) in a way which is almost identical to the result in the $\epsilon$-regime for a full theory with $N_v$ degenerate flavours but with a modified $F$:
\[
C^{\text{mixed}}(x_0) = \frac{\tilde{F}^2}{2T} \left[ 1 + \frac{N_v}{F^2} \left( \frac{\beta_1}{\sqrt{V}} - \frac{T^2}{V} k_{00}^s \right) + \frac{2T^2}{F^2 V} \mu \sigma_v^{(0, N_v)}(M) h_1(\hat{x}_0) \right],
\]
\[
- \frac{N_s}{2T} \left( G_V \left( 0, \frac{M_{ss}^2}{2} \right) + \frac{T^2}{V} k_{00}^s \right),
\] (2.48)

where
\[
\tilde{F}^2 = F^2 \left[ 1 - \frac{N_s}{F^2} \left( G_\infty \left( 0, \frac{M_{ss}^2}{2} \right) - 8L_4 M_{ss}^2 \right) \right].
\] (2.49)

The only difference between this expression and that of the full theory with $N_v$ degenerate quarks are the finite volume effects in the second line, that are exponentially suppressed in $M_{ss} L$.

It is easy to understand these results: the $s$ quarks in the mixed-regime contribute as decoupling particles, because the mixed regime probes much lower energy scales than $M_{ss}$, since the $v$ quarks are much lighter and the size of the box is also much larger than the Compton wavelength of the heavy pions:
\[
M_{vv}^2 \leq L^{-2} \leq M_{ss}^2 \ll (4\pi F)^2.
\] (2.50)

In this situation one can integrate out the $m_s$ quark within the effective theory \[13, 24, 25\]. According to general symmetry arguments we expect that the theory in this limit can be matched to a theory with $SU(N_v)$ flavour symmetry. The effects of the heavy particles can be absorbed in the low-energy couplings of the resulting effective theory. Since all the $v$ quarks are degenerate in mass, the result for the correlator should be identical to that of eq. (2.18) with $m_s = m_v$ and $N_s + N_v \to N_v$ which is precisely what we have found, apart from exponentially suppressed finite volume effects. In fact the result for the renormalized coupling, $\tilde{F}^2$, with $N_s = 1$ coincides with that obtained in \[13\] where the matching of the $SU(3)$ flavour and the $SU(2)$ flavour effective theories for large strange quark mass was first considered.

Another observation is that also within the $p$-regime we can consider a separation of scales $L^{-2} \leq M_{vv}^2 \ll M_{ss}^2$. A similar factorization would then be possible for correlators involving only $v$ quarks as external legs, and up to exponentially suppressed terms in $M_{ss} L$. 

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Figure 1: Dependence of the effective couplings $\tilde{F}^2$ (left) and $\tilde{\Sigma}$ (right) in a theory with $N_v = 2$, and with $N_s = 1$ quarks integrated out, as functions of $M_{ss}^2 / F^2$ for $F = 90 \text{MeV}$. The two lines correspond to the extreme values of $L_4, L_6$ obtained in the phenomenological determinations reviewed in [7].

The result can be written as the correlator in the $p$-regime for $N_v$ degenerate quarks with mass $m_v$ with modified couplings $\tilde{F}$ as in eq. (2.49) and $\tilde{\Sigma}$:

$$\tilde{\Sigma} = \Sigma \left( 1 - \frac{N_s}{F^2} G_{\infty} \left( 0, \frac{M_{ss}^2}{2} \right) + \frac{E_{\text{sub}}(0,0,0)}{F^2} + \frac{16}{F^2} N_s M_{ss}^2 L_6 \right),$$

(2.51)

where

$$E_{\text{sub}}(x, M_{vv}^2, M_{ss}^2) \equiv E_{\infty}(x, M_{vv}^2, M_{ss}^2) - \frac{1}{N_v} G_{\infty}(x, M_{vv}^2),$$

(2.52)

which also coincides with the result of [13].

In figures [**] we show the $\tilde{F}$ and $\tilde{\Sigma}$ as functions of $M_{ss}^2 / F^2$.

The reason that $\tilde{\Sigma}$ does not appear in the mixed-regime of eq. (2.47) is because $\Sigma$ appears there only at NLO and therefore any correction to it, would be of higher order.

It is important to stress however that the decoupling only works in a finite volume up to exponentially suppressed corrections in $M_{ss} L$, since there is no way to predict these finite volume corrections within the effective theory after the $s$ quarks are integrated out.

3. Partially-quenched theory

We will now consider a partially-quenched theory in which there are $N_v$ quenched valence quarks of mass $m_v$ and $N_s$ sea quarks of mass $m_s$. Note that we consider the generators appearing in the left-currents belong to the valence subgroup.
In order to obtain the partially-quenched results \[18\], it is simpler to use the replica method of \[11\]. In this method one enlarges the valence sector to \(N_r\) degenerate flavours of mass \(m_v\), where only \(N_v\) of these appear in the external sources. The full symmetry group for zero quark masses is therefore \(SU(N_r + N_s)\) and the path integral of this theory at the quark level is

\[
Z[J] = \int [dA_\mu] \det (\mathcal{D} + m_v + J)^{N_v} \det (\mathcal{D} + m_v)_{N_r} e^{-S[A_\mu]}. \tag{3.1}
\]

Taking the limit \(N_r \to 0\) in this expression, one obtains the supersymmetric formulation of the theory \[5\]. The replica method therefore dictates that one should construct the chiral effective theory for the \(N = N_r + N_s\) flavours, do perturbative calculations keeping the explicit dependence on \(N_r\) and take the limit \(N_r \to 0\) at the end. We know consider the three regimes in this context.

### 3.1 \(p\)-regime

In the \(p\)-regime, as long as \(N_s \neq 0\), the replica limit can be taken and coincides with the result obtained using the supersymmetric method \[18\]. The \(\xi\) propagator is that of eq. (2.8) with \(N_v \to N_r\).

The result for the left-current correlator is that of eq. (2.12) with \(N_v \to N_r = 0\) and

\[
\Delta_P^{pq} = -\frac{N_s}{F^2} G(0; M_{ss}^2) + \frac{8}{F^2} \left( N_s M_{ss}^2 L_4 + M_{vv}^2 L_5 \right), \tag{3.2}
\]

\[
\Delta_P^{pq} = \left[ \frac{E(0; M_{vv}^2, M_{ss}^2)}{F^2} \right]_{N_v=0} - \frac{8}{F^2} \left( N_s M_{ss}^2 (L_4 - 2L_6) + M_{vv}^2 (L_5 - 2L_8) \right). \tag{3.3}
\]

In the limit \(m_s \to m_v\), the full theory result for \(N_s\) degenerate flavours is recovered. Results for the meson masses and decay constants at NNLO have been recently obtained \[26\].

It is important to realize that in the partially-quenched theory, the full set of \(O(p^4)\) couplings need to be used. The reduction of independent couplings in the full theory with \(N = 2\) or \(N = 3\) only takes place in the unquenched limit, i.e. \(m_v \to m_s\). In particular this implies that for \(N_s = 2\), the partially-quenched predictions involve more couplings that those that are physical in the unquenched limit. Obviously these couplings cannot be determined from phenomenology (not even in principle) and need to be determined on the lattice.

An interesting observation is that the partially-quenched correction to the meson mass \(\Delta_M^{pq}\) has no logarithm in the sea-quark mass, just in the valence quark. If valence quarks masses could be simulated in the light regime, for example using Ginsparg-Wilson fermion regularizations \[20\], and only the sea quark masses would be kept unphysically large, the \(m_s\) dependence of the meson mass would be strictly linear at this order of the chiral expansion. In the case of the decay constant the logarithm remains but with a smaller coefficient. These features are shown in figure 2 where we show the dependence of the meson mass and decay constant with the sea-quark mass for a value of the valence quark mass of 5 MeV for \(N_s = 2\). This is compared with the \(m_s = m_v\) dependence in the full theory case, for \(N_s + N_v = 2\).
In order to recover the fully-quenched case $N_s \to 0$, it is necessary to keep the singlet meson in the theory. When the singlet with a mass $m_0^2$ is kept in the theory the singlet part of the propagator in eq. (2.10) is modified to

$$E^q(x; M_{aa}^2, M_{cc}^2) \equiv \frac{1}{V} \sum_{n \in \mathbb{Z}^4} \frac{e^{ip \cdot x}}{(p^2 + M_{aa}^2)(p^2 + M_{cc}^2)F^q(p)},$$

(3.4)

with

$$F^q(p) = 1 + \left(\frac{\alpha p^2 + m_0^2}{2N_c}\right) \left[\frac{N_s}{p^2 + M_{ss}^2} + \frac{N_r}{p^2 + M_{vv}^2}\right],$$

(3.5)

which is well-defined for $N_s = N_r = 0$. Note that as long as either $N_s$ or $N_r$ are different from zero, the limit $m_0^2 \to \infty$ can be safely taken. The results for the two-point function in this limit agree with those obtained in [8].

### 3.2 $\epsilon$-regime

The results for the partially-quenched theory when all quarks are in the $\epsilon$-regime are

$$C^{\epsilon,Pq}(x_0) = \frac{F^2}{2T^2} \left[1 + \frac{N_s}{F^2} \left(\frac{\beta_1}{\sqrt{V}} - \frac{T^2k_{00}}{V}\right) + \frac{2T^2}{F^2V} \mu \sigma^{Pq(N_s,N_v)}(M) h_1(\hat{x}_0)\right],$$

(3.6)
where
\[
\mu\sigma^Pq(N_v,N_s)(M) \equiv \lim_{N_v \to 0} \int_{U(N_s+N_v)} dU \frac{\mu v}{2N_v} \text{Tr} \left[ P_v U + U^\dagger P_v \right] (\det U)^\nu \\
\times \exp \left( \frac{\Sigma V}{2} \text{Tr} \left[ MU + U^\dagger M \right] \right),
\]
(3.7)
where \( P_v \) is the projector on the \( SU(N_v) \) subgroup of \( SU(N_v) \). This limit has been defined in [22] as
\[
\mu\sigma^Pq(N_v,N_s)(M) \equiv \int_{\text{Gl}(N_s+N_v\mid N_v)} dU \frac{\mu v}{2N_v} \text{Tr} \left[ P_v U + U^\dagger P_v \right] (\det U)^\nu \\
\times \exp \left( \frac{\Sigma V}{2} \text{Tr} \left[ MU + U^\dagger M \right] \right),
\]
(3.8)
where \( U \) is an element of the maximal Riemannian manifold, \( \text{Gl}(N_s+N_v\mid N_v) \). This integral can be obtained as a functional derivative of the functional
\[
Z^Pq(N_v,N_s)(M) = \frac{\det \left[ \mu^j_{i+1} \mathcal{I}_{\nu+1-j}(\mu_i) \right]}{\prod_{j>i=1,...,N_v}(\mu^2_j - \mu^2_i) \prod_{j>i=N_v+1,...,N}(\mu^2_j - \mu^2_i)},
\]
(3.9)
with
\[
\mathcal{I}_{\nu}(x_i) = \left\{ \begin{array}{ll}
(-1)^\nu K_{\nu}(x_i) & i = 1,\ldots,N_v \\
I_{\nu}(x_i) & i = N_v+1,\ldots,2N_v+N_s
\end{array} \right.
\]
(3.10)
\( \mu_i = \mu_v, i = 1,\ldots,N_v; \mu_s = (m_v + J)\Sigma V, i = N_v + 1,\ldots,2N_v \) and \( \mu_i = \mu_s \equiv m_s\Sigma V, i = 2N_v + 1,\ldots,2N_v+N_s \).

\( Z^Pq(N_v,N_s)(M) \) is the same for any value of \( N_v \). This is easy to check for small values of \( N_v \) and \( N_s \). Essentially the choice of \( N_v \) is determined by the dimension of the external source, \( N_v \) in eq. (3.3). In order to obtain the function \( \mu\sigma^Pq(N_v,N_s)(M) \) it is enough to have an external source coupled to one of the valence quarks, since due to the \( SU(N_v) \) invariance, the contribution of each \( v \) quark to eq. (3.8) is the same. We can therefore choose the simplest case, \( N_v = 1 \). Any other choice would give the same result.

Let us consider two simple examples.

**Example 1.** \( N_v = 1, N_s = 1 \)

It is easy to check that the partition functional is that of a one flavour theory:
\[
Z^Pq(1,1)(M) = I_{\nu}(\mu_s),
\]
(3.11)
while the function \( \mu\sigma^Pq(1,1)(M) \) is:
\[
\mu\sigma^Pq(1,1)(M) = \mu_0 \Sigma^q(\mu_v) + \frac{2\mu^2_v}{\mu^2_v - \mu^2_s} K_{\nu}(\mu_s) \left[ \mu_s I_{\nu+1}(\mu_v) I_{\nu}(\mu_v) - \mu I_{\nu+1}(\mu_v) I_{\nu}(\mu_s) \right]
\]
(3.12)
where \( \Sigma^q(\mu) \) is the quenched quark condensate in the \( \epsilon \)-regime at LO [23]:
\[
\Sigma^q(\mu) \equiv \mu \left[ I_{\nu}(\mu_v) K_{\nu}(\mu) + I_{\nu+1}(\mu) K_{\nu-1}(\mu) \right] + \frac{\mu}{\mu_s}.
\]
(3.13)
It is easy to check from this expression that the quenched limit is obtained as \( \mu_s \to \infty \):

\[
\lim_{\mu_s \to \infty} \mu \sigma^{Pq(1,1)}(\mu) = \mu_\nu \Sigma^q(\mu_\nu), 
\tag{3.14}
\]

and the full theory with just one flavour \( N_\nu = 1 \) is obtained when the valence and sea masses are the same, that is in the limit \( \mu_s \to \mu_v \):

\[
\lim_{\mu_s \to \mu_v} \mu \sigma^{Pq(1,1)}(\mu) = \mu \sigma^{(0,1)}(\mu) = \frac{I_\nu^p(\mu_v)}{I_\nu^v(\mu_v)}, 
\tag{3.15}
\]

Note that this last result is an explicit non-perturbative check of the Sharpe and Shoresh statement that the full theory with \( N_\nu \) flavours can be smoothly obtained from the partially-quenched approximation with \( N_s \) sea and \( N_v \) valence quarks, within the effective theory.

**Example 2.** \( N_\nu = 1, \; N_s = 2 \)

The functional for this case if the one corresponding to a two-flavour theory:

\[
Z^{Pq(2,1)}_\nu(M) = \frac{1}{2} \left[ I_\nu(\mu_s)^2 - I_{\nu+1}(\mu_s)I_{\nu-1}(\mu_s) \right], 
\tag{3.16}
\]

while the function \( \mu \sigma^{Pq(2,1)}_\nu(M) \):

\[
\mu \sigma^{Pq(2,1)}_\nu(M) = \mu_\nu \Sigma^q(\mu_\nu) - \frac{2 \mu_\nu^2}{\mu_\nu^2 - \mu_s^2} + \frac{2 \mu_s^3 \mu_s}{(\mu_\nu^2 - \mu_s^2)^2} \frac{G_\nu(\mu_s, \mu_v)}{Z^{Pq(2,1)}_\nu(\mu_s)}, 
\tag{3.17}
\]

where

\[
G_\nu(\mu_s, \mu_v) \equiv I_\nu(\mu_s)I_{\nu+1}(\mu_s)(I_{\nu+1}(\mu_v)K_\nu(\mu_v) - I_\nu(\mu_v)K_{\nu+1}(\mu_v)) \\
+ \frac{\mu_\nu}{\mu_s} I_\nu(\mu_s)^2 K_{\nu+1}(\mu_v)I_{\nu+1}(\mu_v) - \frac{\mu_\nu}{\mu_v} I_{\nu+1}(\mu_s)^2 K_\nu(\mu_v)I_\nu(\mu_v). 
\tag{3.18}
\]

As in the previous example one can explicitly check that in the limit \( \mu_s \to \infty \) one recovers the quenched limit, while in the limit \( \mu_s \to \mu_v \) one recovers the full theory with \( N = 2 \) degenerate quarks.

The partially-quenched result then interpolates between the quenched and the full theory with \( N_\nu \) flavours. This is shown in left figure of figure 3 where the function \( (\mu \sigma^{Pq(2,1)}_\nu(M) - |\nu|)/\mu_\nu \) is shown for the partially quenched case as a function of \( \mu_s \) and compared with full \( N = 2 \) (\( \mu_s = \mu_v \)) and quenched results. On the right figure the \( \mu_v \) dependence of the condensate for two topological sectors is shown and compared with the quenched and full theory, setting \( \mu_s = 1 \).

### 3.3 Mixed-regime

Both in the \( p \) and \( \epsilon \) regimes we could obtain the partially-quenched result from the full one with \( N_s + N_v \) quarks by taking the limit \( N_r \to 0 \) at the end of the calculation. While for the current correlator the limit seems to be well-defined and indeed is the right answer, in other correlation functions such as the pseudoscalar correlator, the limit does not exist. It is easy to see this simply by looking at the \( \xi \) propagators of eq. \( (2.33) – (2.42) \). In the partially-quenched theory the \( p \)-regime propagator is the same but with \( N_v \to N_r \). If we
try to take the $N_r \to 0$ limit, the zero-momentum mode contributions in the first and third equations explode. This is exactly the same effect that happens in the quenched case if the singlet field is not kept in the effective theory \[18\].

In the partially-quenched case, the $U(N_s + N_r)$ singlet can be integrated out and this is true no matter whether we are in the $p, \epsilon$ or mixed regimes. However, what plays the role here of the non-decoupling singlet is the traceless generator of the flavour group $SU(N_s + N_r)$, that is a singlet under the $SU(N_r)$ subgroup, whose normalized generator is

$$T_\eta = \sqrt{\frac{N_r N_s}{2(N_s + N_r)}} \text{diag} \left\{ \frac{1}{N_r}, \ldots, \frac{1}{N_r}, \frac{1}{N_s}, \ldots, -\frac{1}{N_s} \right\}.$$ \hspace{1cm} (3.19)

In the partially-quenched approximation not only the $SU(N_r)$ generators become massless in the limit $m_v \to 0$, also the pion associated with the $\eta$ field gets massless in the limit $N_r \to 0$. In fact the propagator for this field can be easily derived from eq. \[2.43\]

$$\langle \eta(x) \eta(y) \rangle = \frac{1}{2} G \left( x - y; \frac{N_r}{N} M_{ss}^2 \right),$$ \hspace{1cm} (3.20)

and therefore the $\eta$ field becomes massless if $N_r \to 0$, and its zero-mode contribution diverges. Note that in the full case this is however a massive mode, since the mass goes with the largest massgap.

The way out of this problem is to modify the factorization in such a way that the zero-momentum mode of the $\eta$ field is also treated non-perturbatively. So instead of eq. \[2.34\]
we have
\[ U = \left( e^{i \frac{\bar{\eta}}{N_s} \tau} U_0 \right) \exp \left( \frac{2i\xi}{F} \right), \] (3.21)
and now the \( \xi \) field do not contain the zero-modes of the \( SU(N_r) \) generators nor the \( \eta \) one. As in the quenched theory \([8]\), the LO Lagrangian would not factorize into \( \xi \) and \( U_0 \) fields in this case, however it does after the integration over \( \theta \) is performed, that is in a fixed topological sector. In appendix B we derive the path integral and show that at the LO in the \( \epsilon \) expansion, the integration over the zero and non-zero modes factorize:
\[ Z^{\text{LO}}_\nu = \int_{U(N_r)} dU_0 \exp \left[ \frac{2i}{F} \text{Tr}[U_0 + U_0^\dagger] \right] \int d\xi e^{- \int d^4x \left( \text{Tr}[\partial_x (\xi(x) \partial_x (\xi(x)) + M_{\xi}^2 \text{Tr}[P, \xi^2]) \right]. \] (3.22)
The new \( \xi \) propagator is:
\[ \langle \xi_{ca}(x) \xi_{db}(y) \rangle = \frac{1}{2} \left[ \delta_{cb} \delta_{da} G(x-y; 0) - \delta_{ca} \delta_{db} E^{pq}(x-y; 0, 0) \right], \] (3.23)
\[ \langle \xi_{ca}(x) \xi_{db}(y) \rangle = \frac{1}{2} \delta_{cb} \delta_{da} G \left( x - y; \frac{M_{ss}^2}{2} \right), \] (3.24)
\[ \langle \xi_{ca}(x) \xi_{\beta\beta}(y) \rangle = - \frac{1}{2} \delta_{ca} \delta_{\beta\beta} N \bar{G} \left( x - y; \frac{N_r}{N} M_{ss}^2 \right) \] (3.25)
\[ \langle \xi_{\gamma\alpha}(x) \xi_{\beta\beta}(y) \rangle = \frac{1}{2} \left[ \delta_{\gamma\alpha} \delta_{\beta\beta} G(x-y; M_{ss}^2) - \right. \right.
\[ \left. \left. \delta_{\gamma\alpha} \delta_{\beta\beta} \left( E^{pq}(x-y; M_{ss}^2, M_{ss}^2) + \frac{1}{N_s V M_{ss}^2} \right) \right] \right], \] (3.26)
where the latin indices refer to the valence and the greek to the sea, and
\[ E^{pq}(x; M_{ss}^2, M_{ss}^2) \equiv \frac{1}{V} \sum_{n \in \mathbb{Z}^4} \left( 1 - \delta^{(4)}_{n,0} \right) \frac{e^{i p \cdot x}}{(p^2 + M_{ss}^2)^2 + M_{ss}^2} \] (3.27)
with
\[ F^{pq}(p) = \left[ \frac{N_s}{p^2 + M_{ss}^2} + \frac{N_r}{p^2} \right]. \] (3.28)
It is easy to check that the propagator is now well-defined in the limit \( N_r \to 0 \). The singlet part of the propagator of the \( v \) modes in eq. (3.23) has a double pole structure as in the quenched case, but instead of the singlet mass, what appears in the numerator is the heavy mass gap, \( M_{ss}^2 \). This double pole is a non-decoupling effect that only appears because the theory is partially-quenched.

With this parametrization it is easy to check that the left-current propagator is
\[ C^{\text{mixed},pq}(x_0) = \frac{F^2}{2T} \left[ 1 - \frac{N_s}{F^2} \left( G \left( 0, \frac{M_{ss}^2}{2} \right) - 8 L_4 M_{ss}^2 + \frac{T^2}{V} k_0^5 \right) \right] \]
\[ + \frac{2T^2}{F^2 V} \mu \sigma^{pq(0,N_r)}(M) h_1(\bar{x}_0) \right], \] (3.29)
where \( \mu \sigma^{pq(0,N_r)}(M) \) is also the fully quenched result. No double-pole appears for the same reason that it did not appear in the quenched case: this observable is not sensitive to them at NLO.
The decoupling of the $s$ quarks is not possible in the partially-quenched case, because the $\eta$ field remains light. However, we expect that we should be able to integrate out the scale associated to $M_{ss}$ and match the result to a quenched effective theory. Provided $M_{ss} \ll 4\pi F$, this integration can be done perturbatively. The quenched Chiral Lagrangian contains additional couplings besides $F$ and $\Sigma$: $m_0^2$ and $\alpha$ in the standard notation (a mass of the $\eta'$ field and a kinetic term). The tree-level matching of $m_0^2$ and $\alpha$ can be easily read from the propagator of eq. (2.8). The expected $p$-regime propagator in a quenched theory with $N_v$ valence quarks would be of the form

$$\langle \xi_{ca}(x) \xi_{db}(y) \rangle = \frac{1}{2} \left[ \delta_{ib} \delta_{da} \tilde{G}(x-y; M_{ab}^2) - \delta_{ca} \delta_{db} \tilde{E}(x-y; M_{aa}^2, M_{cc}^2) \right],$$  

with $\tilde{G} = G$, and

$$\tilde{E}(x; M_{aa}^2, M_{cc}^2) = \frac{1}{V} \sum_{n \in \mathbb{Z}^4} \frac{e^{ip \cdot x}}{p^2 + M_{aa}^2} \frac{(\frac{\alpha p^2 + m_0^2}{2N_c})^2}{(p^2 + M_{cc}^2)}. \quad (3.31)$$

Identifying $\tilde{E}$ with $E$ in eq. (2.10) with $N_v \to N_r = 0$ we find

$$\frac{\alpha}{2N_c} = \frac{1}{N_s}, \quad \frac{m_0^2}{2N_c} = \frac{M_{ss}^2}{N_s}. \quad (3.32)$$

The quenched $\tilde{F}$ is still the same as that of eq. (2.43), but the matching of the quenched $\tilde{\Sigma}$ gives instead

$$\tilde{\Sigma} \left( 1 + \frac{E_{\infty}(0,0,0)}{F^2} \right) = \Sigma \left( 1 - \frac{N_s}{F^2} G_{\infty}(0, \frac{M_{ss}^2}{2}) + \frac{E_{\infty}(0,0,0)}{F^2} + \frac{16}{F^2} N_s M_{ss}^2 L_6 \right). \quad (3.33)$$

$\tilde{\Sigma}$ gets renormalized in the $m_v \to 0$ limit in the quenched theory, as is well-known. Therefore at one-loop the renormalized coupling is

$$\tilde{\Sigma}_r = \tilde{\Sigma} \left( 1 + \frac{E_{\infty}^{UV}(0,0,0)}{F^2} \right), \quad (3.34)$$

while the logs in $E_{\infty}$ would cancel on the two sides of the eq. (3.33). The curve $\tilde{\Sigma}_r/\Sigma$ as a function of $M_{ss}^2/F^2$ is also shown for $N_s = 1, 2$ in figure 4.

4. Conclusions

We have computed the current correlator at next-to-leading order in finite volume Chiral Perturbation Theory for non-degenerate quark masses in several interesting regimes. The case when all the quarks are in the $p$-regime could have been easily obtained from earlier literature [13, 18]. However we have also considered the case when all quarks are in the $\epsilon$-regime both in the full and partially-quenched theories. In the latter case, we explicitly checked the Sharpe-Shoresh statement, concerning the recovery of QCD from partially-quenched approximations, within the effective theory.
The main result of this paper is that in the mixed-regime in which some of the quarks masses are in the $\epsilon$ and some in the $p$. This regime requires a new zero versus non-zero mode factorization, that we introduced and worked out in detail. A further modification was required to treat the partially-quenched theory in this case. Our results in the mixed-regime show that the quarks in the $p$-regime behave essentially as decoupling particles, so that the correlator (up to some exponentially suppressed finite volume corrections) is that of the $\epsilon$-regime for a theory with a reduced number of flavours (i.e. those in the $\epsilon$-regime), but with corrected low-energy couplings by the heavier quarks (i.e. those in the $p$-regime).

These results can be useful for matching lattice QCD and Chiral Perturbation Theory in finite volumes, when the volume is not sufficiently large compared with the Compton wavelength of the lighter pions.

Clearly the methods developed here can be used for the computation of other correlation functions [27].

Acknowledgments

We wish to thank C. Haefeli, M. Laine, S. Necco and C. Pena for useful discussions on the topics of this work and a critical reading of the manuscript. We also thank S. Aoki, S. Hashimoto and T. Degrand for informing us of their interest in this calculation. F.B. acknowledges the financial support of the FPU grant AP2005-5201. This work was also partially supported by the Spanish CICYT (Project No. FPA2004-00996 and FPA2005-01678), by the Generalitat Valenciana (Project No. GRUPOS03-13) and the Integrated Grant HA2005-0066.
A. Calculation of the Jacobian

The parametrization with some or all of the zero modes factorized, that we encounter in the mixed and \( \epsilon \) regimes, has a non trivial Jacobian with respect to the \( SU(N_f) \) Haar measure. Here we first review the calculation of the Jacobian for the \( \epsilon \)-regime parametrization at NLO, as first obtained in ref. [4]. The method of [4] is easily extendable to the parametrization of the mixed regime, eqs. (2.35) and (3.21), so that we briefly mention which are the differences and give the results for the mixed regime too. In particular we show that the contribution coming from the \( \xi \) fields factorizes.

A metric can be defined through:

\[
\langle dU^\dagger dU \rangle = g_{ab} dU_a dU_b \quad (A.1)
\]

so that the volume element \( d\mu \) is obtained as:

\[
d\mu = \sqrt{\det g} \bigwedge U_i \quad (A.2)
\]

The parenthesis \( (...) \) means here and in the following that both an integration and a trace are executed \( (\int_V \frac{d^4x}{4!} \text{Tr}[\ldots]) \).

With the physical fields in the game, our metric matrix \( g \) will be of the form

\[
g = \begin{bmatrix} g_1 & g_2 \\ g_3 & g_4 \end{bmatrix}
\]

and we will see that \( g_1 \) contains elements of order 1, \( g_2 = g_3^T \) contain elements of order \( \epsilon \), and \( g_4 = 1 + O(\epsilon^2) \).

At next to leading order we are lead by the seesaw formula to consider the matrix

\[
g = \begin{bmatrix} g_1 & 0 \\ 0 & g_4 - g_2 g_3^{-1} g_2 \end{bmatrix}
\]

that has the same eigenvalues and eigenvectors of \( g \) to \( \epsilon^2 \) order.

We will use the parametrization:

\[
U = U_0 U_\xi = e^{i \phi} e^{2 i \xi} \quad (A.3)
\]

and the expansions:

\[
\phi = \phi^a T^a \quad \xi(x) = \xi_{m_1}^n f_{m_1}(x) T^{m_2} \quad .
\]

Since \( \xi \) only contain nonzero modes, the \( f_{m_i} \) satisfy:

\[
\int dx f_{m_i}(x) = 0 \quad . \quad (A.4)
\]

Adding the constant function to the \( f_{m_i} \) we obtain a complete set:

\[
\sum_{m_i} f_{m_i}(x) f_{m_i}(y) = \delta(x - y) - 1 \quad . \quad (A.5)
\]
The completeness relations for generators $T^a$ such that $Tr[T^a T^b] = \frac{1}{2} \delta^{ab}$ read:

\[
\sum_{a=1}^{N^2-1} \Tr[T^a A T^a B] = -\frac{1}{2N} \Tr[AB] + \frac{1}{2} \Tr[A] \Tr[B] \tag{A.6}
\]

\[
\sum_{a=1}^{N^2-1} \Tr[T^a A] \Tr[T^a B] = -\frac{1}{2N} \Tr[AB] + \frac{1}{2} \Tr[AB] \tag{A.7}
\]

and are valid if $A, B$ are hermitian matrices.

Inserting (A.3) in (A.4) we obtain:

\[
\langle U^\dagger dU \rangle = \langle U^\dagger dU \xi_0 dU_0 \xi_0^\dagger \xi_0 \rangle + \langle U^\dagger dU \xi_0 dU_0 \xi_0 \rangle = \langle U^\dagger dU \rangle \tag{A.8}
\]

and note that blocks like $U^\dagger dU$ are easy to calculate since they are elements of the algebra.

The block containing the $\xi$ fields gives:

\[
U^\dagger \xi dU \xi \simeq \frac{2i d\xi}{\mathcal{F}} + \frac{2}{\mathcal{F}^2} (\xi d\xi - d\xi \xi) + \frac{4i}{\mathcal{F}^2} (2\xi d\xi - \xi^2 d\xi - d\xi \xi^2) \tag{A.9}
\]

and consequently:

\[
dU^\dagger \xi dU \xi = (U^\dagger dU \xi)^\dagger = -U^\dagger \xi dU \xi \tag{A.10}
\]

To calculate the block containing the zero modes we define the $M(y)$ matrix by:

\[
M(y) \equiv e^{\frac{2i y}{\mathcal{F}}} e^{-\frac{2i y}{\mathcal{F}} (\phi + d\phi)} \tag{A.11}
\]

and we see that $U_0 dU_0^\dagger = M(1) - 1$. $M$ is a solution for the Cauchy problem:

\[
\frac{\partial M(y)}{\partial y} \simeq \frac{2i}{\mathcal{F}} d\phi^a \left[ e^{-\frac{2i y}{\mathcal{F}} F^c \phi^c} \right]^{ab} T^b \quad M(0) = 1, \tag{A.12}
\]

if $F^a$ are the generators of the adjoint irrep ($[T^a, T^b] = i f^{abc} T^c$, $[F^a]^{bc} = -i f^{abc}$). Finally:

\[
U_0 dU_0^\dagger = - \left[ 1 - e^{-\frac{2i y}{\bar{\phi}}} \right] d\phi^a T^b \equiv - A^{ab} d\phi^a T^b = - dU_0 U_0^\dagger \tag{A.13}
\]

where $\bar{\phi} = F^a \phi^a$. Last equality is a consequence of the reality of the structure constants.

Exploiting the orthonormality of the $f_{m_1}$, for $g_4$ one obtains:

\[
(g_4)_{m_{n_2} n_1}^{m_{n_2} n_1} = \langle dU^\dagger \xi_1 U^\dagger \xi U dU^\dagger \xi_2 \rangle = 2 \left( \delta_{m_1 n_1}^{m_2 n_2} + \frac{4}{3 \mathcal{F}^2} \int \frac{dx}{V} f_{m_1} f_{n_1} \Tr[T^{m_2} \xi T^{n_2} \xi - \xi^2 T^{m_2} T^{n_2}] \right). \tag{A.14}
\]

At NLO $\det(1 + a) \sim 1 + \Tr[a]$ if the entries of $a$ are small. To take the trace of (A.14) one uses the completeness relations. The perturbative correction to the determinant is:

\[
\frac{2}{\mathcal{F}^2} (\delta(0) - 1)(-\frac{2N}{3 \mathcal{F}^2}) \langle \xi^2 \rangle \tag{A.15}
\]

The addend proportional to $\delta(0)$ would be there even in the p regime but it is zero in dimensional regularization. And this is the explanation why we do not need to consider a measure term in the p regime.
The other correction at NLO comes from \( Tr[g_2^T g_1^{-1} g_2] \) and after straightforward calculations one sees that this amounts to
\[
Tr[g_2^T g_1^{-1} g_2] = \frac{2}{F^2} \frac{2N}{F^2} (\xi^2) .
\] (A.15)
Combining these results we can calculate the measure:
\[
d\mu \simeq d\mu(\phi)_{\text{Haar}} d\xi J(\xi) \simeq d\mu(\phi)_{\text{Haar}} d\xi \sqrt{\frac{2}{F}} \left( 1 - \frac{2N}{3F^2} (\xi^2) \right) .
\] (A.16)
The same procedure can be applied to the mixed regime parametrizations. Of course the relations (A.4) and (A.5) need to be modified properly. We obtained:
\[
J(\xi) \simeq \frac{\sqrt{2}}{F} \left( 1 - \frac{4}{3F^2 \sqrt{V}} \int dz \sum_{m \in SU(N_s)} Tr[(T^m)^2] \right.
\]
\[
\left. - \frac{2}{F^2 V^2} \int dz \int dw \sum_{b, m \in SU(N_s)} Tr[T^b(\xi T^m - T^m \xi)](x)Tr[T^b(\xi T^m - T^m \xi)](y) \right) .
\] (B.1)
in the factorization for full theory calculations,
\[
J(\xi) \simeq \frac{\sqrt{2}}{F} \left( 1 - \frac{4}{3F^2 \sqrt{V}} \int dz \sum_{m \in SU(N_s) \cup T^n} Tr[(T^m)^2] \right.
\]
\[
\left. - \frac{2}{F^2 V^2} \int dz \int dw \sum_{b, m \in SU(N_s) \cup T^n} Tr[T^b(\xi T^m - T^m \xi)](x)Tr[T^b(\xi T^m - T^m \xi)](y) \right) .
\] (B.2)
in the factorization for PQ calculations.

**B. Mixed-regime path integral**

In this appendix we discuss in detail the path integral in the mixed-regime. Starting with the factorization in eq. (2.34), the path integral in the absence of sources can be written as
\[
Z_\nu = \int d\xi J(\xi) \int_{SU(N_s)} dU_0 \int d\theta e^{i\nu \theta} \exp(-S) ,
\] (B.1)
where \( S \) can be organized as an expansion in \( \epsilon \) according to the power-counting of eq. (2.34). The leading order is:
\[
S^{(0)} = -\mu_s \cos \left( \frac{\theta - \eta}{N_s} \right) - \frac{\Sigma V}{2} Tr \left[ p_\nu \left( M \epsilon^{in/N_s} U_0 + U_0^\dagger \epsilon^{-in/N_s} M \right) \right] + \int d^4 x \left[ \partial_\mu \xi(x) \partial_\mu \xi(x) \right] + M_{ss}^2 \cos \left( \frac{\theta - \eta}{N_s} \right) Tr \left[ p_\nu \xi^2 \right] ,
\] (B.2)
where the first term if \( O(\epsilon^{-2}) \) and the others are all of \( O(1) \).

Passing to the variables \( \hat{\theta} \equiv \theta - \eta \) and \( \hat{U}_0 \equiv e^{in/N_s} U_0 \):
\[
Z_\nu = \int d\xi J(\xi) e^{-\int d^4 x [\partial_\nu \xi(x) \partial_\mu \xi(x)]} \int_{U(N_s)} d\hat{U}_0 \det(\hat{U}_0)^\nu \epsilon^{-\frac{\Sigma V}{2} \hat{p}_\nu \left( M \hat{U}_0 + \hat{U}_0^\dagger M \right) \hat{p}_\nu \xi^2} ,
\] (B.3)
where
\[ A(\xi) \equiv -M_{ss}^2 \int d^4x \text{Tr} \left[ P_s \xi^2 \right] + \mathcal{O}(\epsilon^2), \]  
(B.4)

and \( B(\xi, \bar{U}_0) \) contains the \( \mathcal{O}(\epsilon^2) \) terms of the original action that do not depend on \( \bar{\vartheta} \).

The integral over \( \bar{\vartheta} \) is therefore of the form:
\[ \int d\bar{\vartheta} e^{F(\bar{\vartheta})}, \]  
(B.5)

where the real part of \( F(\bar{\vartheta}) \) has a maximum at \( \bar{\vartheta} = 0 \) and can be expanded in \( \epsilon \) via a saddle point approximation. Expanding \( F(\bar{\vartheta}) \) around this maximum:
\[ F(\bar{\vartheta}) = \left( \mu_s + A(\xi) \right) \left[ 1 - \frac{1}{2} \left( \frac{\bar{\vartheta}}{N_s} \right)^2 + \mathcal{O} \left( \frac{1}{N_s^4} \right) \right] + i\nu \bar{\vartheta}, \]  
(B.6)

the integral can be rewritten as
\[ \int d\bar{\vartheta} e^{F(\bar{\vartheta})} = \epsilon^{\mu_s + A(\xi)} \int d\bar{\vartheta} e^{-\frac{1}{2} \left( \frac{\bar{\vartheta}}{N_s} \right)^2 (\mu_s + A(\xi))} e^{i\nu \bar{\vartheta}} (1 + (\mu_s + A(\xi)) \mathcal{O}(\bar{\vartheta}^4)) \]
\[ \simeq \epsilon^{\mu_s + A(\xi)} \sqrt{2\pi N_s} e^{-\frac{\nu^2 N_s^2}{2(\mu_s + A(\xi))}} \left( 1 + \mathcal{O} \left( \frac{1}{(\mu_s + A(\xi))} \right) \right) \]
\[ \simeq C_\nu \epsilon^{A(\xi)} \left( 1 - \frac{A(\xi)}{2\mu_s} + \mathcal{O}(\epsilon^4) \right). \]  
(B.7)

Therefore at the order we need to go, the path integral can be written as:
\[ Z_\nu \simeq C_\nu \int d\xi \ J(\xi) e^{-\int d^4x \left( \text{Tr}[\partial_\mu \xi(x) \partial^\mu \xi(x)] + M_{ss}^2 \text{Tr}[P_s \xi^2] \right)} \]
\[ \int_{U(N_v)} d\bar{U}_0 \det(\bar{U}_0) e^{\frac{1}{2} \text{Tr} \left[ P_s (MC_0 + U_0^\dagger M) \right]} \left( 1 - \frac{A(\xi)}{2\mu_s} + B(\xi, \bar{U}_0) \right). \]  
(B.8)

The LO path-integral factorizes into zero and non-zero mode contributions and the computation of correlation functions is then like in the \( \epsilon \)-regime.

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