Constraint damping in the Z4 formulation and harmonic gauge

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We show that by adding suitable lower-order terms to the Z4 formulation of the Einstein equations, all constraint violations except constant modes are damped. This makes the Z4 formulation a particularly simple example of a λ-system as suggested by Brodbeck et al. We also show that the Einstein equations in harmonic coordinates can be obtained from the Z4 formulation by a change of variables that leaves the implied constraint evolution system unchanged. Therefore the same method can be used to damp all constraints in the Einstein equations in harmonic gauge.

I. INTRODUCTION

Formulations of the Einstein equations as an initial-boundary value problem are required for the numerical simulation of astrophysical events, such as the inspiral and merger of a binary system of black holes. Simulations are still crucially limited by instabilities. Most of these instabilities arise already in the continuum system of partial differential equations, rather that at the stage of finite differencing.

There is now broad agreement that in order to avoid some of these instabilities, the formulation of the Einstein equations that one uses should give rise to well-posed problems. For the Cauchy problem (no boundaries in space) to be well-posed, strong hyperbolicity is a necessary and sufficient criterion. Symmetric hyperbolicity implies strong hyperbolicity and allows one to formulate a well-posed initial-boundary value problem. Symmetric hyperbolicity of the subsidiary constraint evolution system is likely to be a crucial ingredient in making the boundary conditions consistent with the constraints.

In a hyperbolic formulation, the error associated with constraint violation grows at a bounded rate, but this can be very fast in practice. It would be preferable if one could find a formulation of the Einstein equations in which the submanifold of solutions that also obey the constraints is an attractor. Clearly this requires a mechanism for breaking the time-reversal symmetry of general relativity away from the constraint surface. Mechanisms that have been suggested include dynamically adjusting the free parameters of the constraint addition [1], or adding derivatives of the constraints so that the system becomes mixed parabolic and hyperbolic [2].

Brodbeck et al [3] have suggested a general approach called the λ-system to solving a system of evolution equations and constraints such that the constraint surface is an attractor. This consists in adding one variable λ for each constraint, such that the time derivatives of the λ are the constraints, and the extended system is (or remains) symmetric hyperbolic. One then adds damping terms $\partial_t\lambda = \ldots - \kappa\lambda$ to the evolution equations for the variables λ. As they are lower order, they do not affect the hyperbolicity of the system. These terms should damp the λ, and therefore the constraints.

In [3] the Frittelli-Reula symmetric hyperbolic formulation of the Einstein equations was extended in this way, and it was shown analytically that when the system is linearised around Minkowski spacetime, the constraint surface is an attractor. In [4] the conformal field equations, reduced by two Killing vectors, were extended in the same way, and investigated numerically. This worked well for linear gravity, but not in strong field tests, where the constraint violations were reduced, but not to zero, and the error actually increased.

II. Z4 AS A SIMPLE λ-SYSTEM

Our starting point is the observation that another symmetric hyperbolic formulation of the Einstein equations, the Z4 formulation [5, 6, 7] is already a λ system, without the need to add extra variables. (This was already noted in [8] for the Z3 system, which is closely related to the Z4 system.) For simplicity we restrict our presentation to the vacuum case. Including matter would be straightforward.

The Z4 system is obtained in its 4-dimensional covariant form by replacing the vacuum Einstein equations $R_{ab} = 0$ by

$$R_{ab} + \nabla_a Z_b + \nabla_b Z_a = 0, \tag{1}$$

where R_{ab} is the Ricci tensor of the 4-dimensional spacetime metric g_{ab} and Z_a is an additional vector field. The main effect of this extension is to turn the 4 Einstein constraints into first-order evolution equations for the 4-vector Z_a. A solution of the Z4 equations is a solution of the Einstein equations if and only if $Z_a = 0$. (There is one exception: if the spacetime admits a Killing vector, a solution of the Z4 equations with Z_a equal to the Killing vector is also a solution of the Einstein equations. In the following, we assume for simplicity that the spacetime is generic and does not admit a Killing vector.)

We shall see that the variables Z_a are already variables of the λ type. All we need to do is to add the damping term. We do this in covariant notation by replacing the

Einstein equations by

$$R_{ab}+\nabla_a Z_b+\nabla_b Z_a-\kappa\left[t_a Z_b + t_b Z_a - (1+\rho)g_{ab}t^c Z_c\right] = 0, \quad (2)$$

or, with the trace reversed,

$$G_{ab}+\nabla_a Z_b+\nabla_b Z_a-g_{ab}\nabla^c Z_c-\kappa(t_a Z_b+t_b Z_a+\rho g_{ab}t^c Z_c) = 0, \quad (3)$$

where t^a is a non-vanishing timelike vector field and $\kappa \geq 0$ and ρ are real constants. (We shall later restrict to $\rho = 0$.) It is t^a that explicitly breaks time reversal invariance. A simple geometrical choice is $t^a = n^a$, the future pointing unit normal vector on the time slicing.

We carry out the usual 3+1 split of the metric as

$$ds^2 = -\alpha^2\, dt^2 + \gamma_{ij}(dx^i + \beta^i\, dt)(dx^j + \beta^j\, dt), \quad (4)$$

and split Z_a as $Z^a = X^a + n^a\theta$ where $\theta \equiv -n_a Z^a$. In adapted coordinates this gives $\theta = \alpha Z^0$ and $X_i = Z_i$. (Note that n^a as defined in [7] is past-pointing, while ours is future-pointing, so that the two definitions of θ are the same.) In the following we use the established, but slightly ambiguous, notation Z_i to denote X_i. We denote the Ricci tensor of γ_{ij} by R_{ij} and its covariant derivative by D_i. Spatial tensor indices i, j are moved with γ_{ij}. We also define the derivative operator $\partial_0 \equiv \alpha^{-1}(\partial_t - \mathcal{L}_\beta)$. In this notation, the 3+1 split of the Z4 equations is

$$\begin{aligned}
\partial_0\gamma_{ij} &= -2K_{ij}, && (5)\\
\partial_0 K_{ij} &= -\alpha^{-1}D_iD_j\alpha + R_{ij} - 2K_{ik}K^k{}_j + KK_{ij} \\
&\quad +D_iZ_j + D_jZ_i - 2\theta K_{ij} - \kappa(1+\rho)\gamma_{ij}\theta, && (6)\\
\partial_0\theta &= \frac{1}{2}H - \theta K + D_kZ^k - D_k(\ln\alpha)Z^k \\
&\quad -(2+\rho)\kappa\theta, && (7)\\
\partial_0 Z_i &= M_i + D_i\theta - D_i(\ln\alpha)\theta - 2K_i{}^kZ_k - \kappa Z_i. && (8)
\end{aligned}$$

In the last two equations H and M_i are shorthand for the Einstein constraints

$$\begin{aligned}
H &\equiv R - K_{ij}K^{ij} + K^2, && (9)\\
M_i &\equiv D^jK_{ji} - D_iK. && (10)
\end{aligned}$$

What happens when the initial data for γ_{ij} and K_{ij} do not obey the Einstein constraints? By substituting the definitions of H and M_i into the evolution equations (5)-(6), we find their formal time derivatives

$$\begin{aligned}
\partial_0 H &= -4M^iD_i(\ln\alpha) + 2KH - 2D^iM_i - 4\kappa(1+\rho)K\theta \\
&\quad +K_{ij}(\gamma^{ij}\gamma^{kl} - \gamma^{ik}\gamma^{jl})[4D_lZ_k - 4\theta K_{kl}], && (11)\\
\partial_0 M_i &= -\frac{1}{2}D_iH + (K-2\theta)M_i - D_i(\ln\alpha)H + 2R_{ij}Z^j \\
&\quad +D^j(D_jZ_i - D_iZ_j) + \alpha^{-1}D^j\alpha(D_iZ_j + D_jZ_i) \\
&\quad -2D_i(\ln\alpha)D^jZ_j - 2\alpha^{-1}K_{ij}D^j(\alpha\theta) \\
&\quad +2\left[K + \kappa(1+\rho)\right]\alpha^{-1}D_i(\alpha\theta). && (12)
\end{aligned}$$

In order for the constraint evolution system to close (in the usual sense, see below) it must be considered to consist of (11,12,7,8) for the constraint variables H, M_i, θ and Z_i. In particular, a solution of the evolution equations obeys $H = M_i = \theta = Z_i = 0$ at all times if and only if they all vanish at $t = 0$. The constraint system associated with the Z4 system is unusual in that θ and Z_i are genuine dynamic variables while, as in other formulations, H and M_i are only shorthands for combinations of the dynamic variables γ_{ij} and K_{ij} and their derivatives.

One can replace the 8 first-order evolution equations of the constraint system by a second-order wave equation for the 4-vector Z_a by taking the divergence of (3), and using the contracted Bianchi identity $\nabla^a G_{ab} = 0$. The result is

$$\Box Z_b + R_{ab}Z^a - \kappa\nabla^a\left(t_aZ_b + t_bZ_a + \rho g_{ab}t^cZ_c\right) = 0, \quad (13)$$

where $\Box$ is the covariant wave operator $\nabla_a\nabla^a$. Given Z_a, the Einstein constraints $G^{0\mu}$ can be read off from (3), or in 3+1 form H and M_i can be read off from (7)-(8). Note that for $Z_i = \theta = 0$ at some instant $t = 0$, the condition $\dot Z_i = \dot\theta = 0$ is equivalent to $H = M_i = 0$ at that instant. That means that all four constraints vanish at all times if and only if $Z_i = \theta = \dot Z_i = \dot\theta = 0$ at $t = 0$.

In either its first-order or second-order form, the constraint evolution system closes only in the sense that the right-hand side is proportional to the constraints, but not in the sense that it is autonomous: one cannot consistently evolve either (13) or (11,12,7,8) while considering the variables γ_{ij} and K_{ij} (or equivalently the spacetime metric g_{ab}) as fixed. Instead one should focus on the following question:

When one evolves initial data with a small constraint violation (set for example by finite differencing error) does the constraint violation grow or decay as the initial data are evolved? To address this, we perturb around a background solution $g^{(0)}_{ab}$ that obeys all 10 Einstein equations, and write

$$g_{ab} = g^{(0)}_{ab} + \epsilon\, g^{(1)}_{ab}, \quad (14)$$

where $R^{(0)}_{ab} = 0$ and $Z^{(0)}_a = 0$. To first order in ϵ the constraint violation then obeys a linear evolution equation with coefficients taken from the background Einstein spacetime, and admitting arbitrary data. In the Z4 formulation this is just a vector wave equation on the background spacetime, namely

$$\Box^{(0)}Z^{(1)}_b - \kappa\nabla^{a(0)}\left(t^{(0)}_aZ^{(1)}_b + t^{(0)}_bZ^{(1)}_a + \rho g^{(0)}_{ab}t^{c(0)}Z^{(1)}_c\right) = 0. \quad (15)$$

The growth of sufficiently small constraint violations is controlled by the damping; for larger constraint violations nonlinear lower-order terms also become important. We expect that with sufficiently large κ, and starting from sufficiently small initial constraint violations, the nonlinear terms never become important in practice.

[Note added in revision: Friedrich [9] has has independently carried out a partial non-linear analysis of (13) in the context of generalised harmonic coordinates and without damping. In (13) he considers the spacetime in the wave operator $\Box$ as given, but expresses R_{ab} in terms of Z_a using (2). The result is wave equation for Z_a with a quadratic lower-order term, but on a fixed spacetime. He shows that generic solutions blow up in finite time. This analysis accounts for some of the hidden nonlinearity of (13), but does not include all terms of $O(\epsilon^2)$. We have restricted ourselves to a consistent linear analysis.]

III. MODE ANALYSIS

The constraint evolution system of the Z4 system is simpler than in other formulations of the Einstein equations in that it has the form of a covariant wave equation. We can use this to carry out a mode analysis for the linearised constraint system (15) on an arbitrary Einstein background in the high-frequency, frozen coefficient limit in which the wavelength of $Z_a^{(1)}$ is much smaller than the curvature scale of the Einstein background. We can then locally approximate the background $g_{ab}^{(0)}$ as Minkowski space, and work in standard Minkowski coordinates. Assuming without loss of generality that $t_a t^a = -1$, we go to the frame in which $t^\mu = (1,0,0,0)$ and find (now dropping the expansion indices)

$$\Box Z_0 - \kappa\left[(2+\rho)\partial_t Z_0 - \partial^i Z_i\right] = 0, \tag{16}$$

$$\Box Z_i - \kappa\left(\partial_t Z_i + \rho \partial_i Z_0\right) = 0, \tag{17}$$

where $\Box$ is now the Minkowski wave operator $-\partial_t^2 + \partial_i\partial^i$. We make a plane-wave ansatz

$$Z_\mu(t,x^i) = e^{st+i\omega_i x^i}\,\hat Z_\mu, \tag{18}$$

with complex s and real ω_i. This gives rise to the eigenvalue problem

$$\begin{pmatrix} -s^2-\omega^2-\kappa(2+\rho)s & \kappa i\omega & 0 \\ -\kappa\rho i\omega & -s^2-\omega^2-\kappa s & 0 \\ 0 & 0 & -s^2-\omega^2-\kappa s \end{pmatrix} \begin{pmatrix} \hat Z_0 \\ \hat Z_n \\ \hat Z_A \end{pmatrix} = 0. \tag{19}$$

where Z_n is the component of Z_i in the direction of ω_i and Z_A stands for the projection of Z_i normal to ω_i. The 4 eigenvalues s for Z_A are

$$s = -\frac{\kappa}{2} \pm \sqrt{\left(\frac{\kappa}{2}\right)^2 - \omega^2} \tag{20}$$

(each of these occurring twice), independently of ρ. The other 4 eigenvalues s are in general complicated. However, for $\rho = 0$ they take the simple form

$$s = -\kappa \pm \sqrt{\kappa^2 - \omega^2} \tag{21}$$

(each of these occurring twice). Therefore with $\rho = 0$ all modes are damped for all $\omega_i \neq 0$. At high wave numbers, $\omega \gg \kappa$, the damping is by a constant factor, with

$$s \simeq -\kappa \pm i\omega, -\frac{\kappa}{2} \pm i\omega, \tag{22}$$

but at low wave numbers, $\omega \ll \kappa$, it is similar to a heat equation, with

$$s \simeq -\kappa, -\frac{\omega^2}{\kappa}, -2\kappa, -\frac{\omega^2}{2\kappa}. \tag{23}$$

Half of the modes are damped less with decreasing wavenumber, and not at all at zero wavenumber. The only other case in which the eigenvalues s are simple is $\rho = -1$. This also simplifies the field equations, but in this case the other 4 eigenvalues are $s = -\kappa \pm i\omega$ and $s = \pm i\omega$, and so there are undamped modes for all ω_i. It turns out that all but the constant modes are damped for any $\rho > -1$, but $\rho = 0$ is the most natural choice, and we assume this value in the following.

We have shown that the constraint manifold is an attractor for $\kappa > 0$ when the equations are linearised around Minkowski space, with the exception of constraint violations that are constant in space. The same is true in the high-frequency limit when linearising around any Einstein background. This analysis breaks down when the constraint violations are large and/or when their wavelength is large compared to the background curvature. In that case lower-order terms can potentially make the constraints grow against the explicit damping term. However, compared to the systems of [3, 4], $Z4$ has only a fraction of the number of variables and constraints, and so is less likely to be affected by undesirable growth arising from lower order terms.

IV. Z4 AND HARMONIC GAUGE

The hyperbolicity of any formulation of the Einstein equation depends on the choice of gauge (assuming that the formulation does not already fix the gauge). Formulations of the Einstein equations derived from the ADM formulation are typically not hyperbolic when the lapse α and shift β^i are given functions of the coordinates, but if they can be made hyperbolic at all, this is typically true for fixed shift and fixed densitised lapse Q, where α

and Q are related by

$$\alpha \equiv \gamma^{\sigma/2} Q, \tag{24}$$

for some constant parameter σ. The Z4 formulation with fixed densitised lapse and fixed shift is strongly hyperbolic for $\sigma > 0$. Surprisingly, it is not symmetric hyperbolic for any σ. (The most general energy that is conserved in the high-frequency, frozen coefficient approximation fails to be positive definite. Details will be given elsewhere.)

An evolved version of the densitised lapse (usually called the Bona-Massó lapse) is

$$\partial_0 \ln \alpha = -\sigma(K - m\theta) + \partial_0 \ln Q, \tag{25}$$

where m is another constant parameter. When $\theta = 0$, the ∂_0 derivative of (24) is just (25). As pointed out in [7], the Z4 formulation is symmetric hyperbolic with the lapse (25) and fixed shift for $\sigma = 1$ and $m = 2$. For any other $\sigma > 0$, it is strongly hyperbolic (with arbitrary m), but not symmetric hyperbolic. (The most general energy fails to be positive definite for $\sigma \neq 1$. Details will be given elsewhere.) $\sigma = 1$ is equivalent to harmonic slicing. This suggests that in some way Z4 is closely related to the Einstein equations in harmonic coordinates.

In fact, if in (1) we consider Z_μ as the shorthand

$$Z_\mu \equiv \frac{1}{2}\left(-H_\mu - g^{\alpha\beta} g_{\mu\alpha,\beta} + \frac{1}{2} g^{\alpha\beta} g_{\alpha\beta,\mu}\right). \tag{26}$$

where H_μ are given functions of the coordinates ("gauge source functions"), we obtain the Einstein equations in the generalised harmonic gauge. In harmonic gauge all ten $g_{\mu\nu}$, or equivalently γ_{ij}, α and β^i obey quasilinear second-order evolution equations whose principal part is just the wave operator on the spacetime with metric $g_{\mu\nu}$.

The constraints $Z_\mu = 0$ must still be obeyed to obtain a solution of the Einstein equations, but they are now the harmonic gauge constraints

$$\Box x^\mu = g^{\mu\nu}{}_{,\nu} + \frac{1}{2} g^{\mu\nu} g^{\alpha\beta} g_{\alpha\beta,\nu} = H^\mu, \tag{27}$$

where $\Box$ is the scalar wave operator [10]. In a 3+1 split, these constraints can be solved for $\dot\alpha$ and $\dot\beta^i$, and thus constrain the free data for the wave equations for α and β^i.

The substitution (26) does not change the constraint system itself at all, which is still the wave equation for Z_μ. The only difference is now that the Z_μ are no longer dynamical variables but are shorthands for the harmonic gauge constraints. We immediately obtain a prescription for damping all constraints (gauge and Einstein) in numerical free harmonic evolutions: we modify the Einstein equations in harmonic gauge to

$$R_{\mu\nu} + \nabla_\mu Z_\nu + \nabla_\nu Z_\mu - \kappa(t_\mu Z_\nu + t_\nu Z_\mu - g_{\mu\nu} t^\lambda Z_\lambda) = 0, \tag{28}$$

where Z_μ is the shorthand (26). (We have used Greek indices to indicate that these are not tensor equations but hold only in harmonic coordinates.) As the constraint system is identical to that of Z4, the same mode analysis applies, showing that all Fourier modes are damped except the constant in space modes.

V. Z4 AND NOR/BSSN

We obtain the NOR system [11] by the substitutions

$$\theta \to 0, \quad Z_i \to \frac{1}{2}\left(f_i - \gamma^{jk}\gamma_{ij,k} + \frac{\rho}{2}\gamma^{jk}\gamma_{jk,i}\right) \tag{29}$$

in terms of a new variable f_i. (ρ is the constant parameter of the same name in [11], but is different from the ρ introduced in (2), which is now set to zero.) The variable θ disappears, and Z_i is now the definition constraint of the new auxiliary variable f_i. (Note that $Z_i = -G_i$ in the notation of [11] and [12].) Essentially this change of variables was used in [6] to obtain the BSSN system from the Z4 system, where constraint damping was also pointed out. The constraints are $H = 0$, $M_i = 0$, and $Z_i = 0$. The constraint evolution system is obtained from that of the Z4 system by setting θ to zero. (The Z4 system with $\theta = 0$ but treating Z_i as dynamical variables is called the Z3 system, and was proposed in [8]. Its constraint evolution system is identical to that of the NOR system.)

If we repeat the mode analysis for the Z3 system, we find that the dispersion relation relation for the group of vectors transverse to ω_i, (M_A, Z_A) is given by

$$s = \frac{1}{2}\left(-\kappa \pm \sqrt{\kappa^2 - 4\omega^2}\right), \tag{30}$$

but for the group of scalars (H, M_n, Z_n) the modes are given by

$$s = -\kappa, \pm i\omega, \tag{31}$$

so there are two undamped modes. This means we can damp 5 of the 7 constraints of the NOR system (or the BSSN system) by a suitable modification given essentially by (28) with (29).

VI. CONCLUSIONS

We have given a λ system for general relativity that is optimal in the sense that the only variables in addition to the usual ADM variables γ_{ij} and K_{ij} are the four λ variables θ and Z_i that are associated with the four Einstein constraints H and M_i. All constraints except the constant in space modes are damped through lower order terms. This system is already symmetric hyperbolic (for harmonic slicing and fixed shift) without the need for other auxiliary variables. It is therefore a good testbed for investigating the usefulness of λ systems in general relativity in general.

Within the framework of any λ system, it is impossible to obtain damping of homogeneous constraint modes because the damping is through a friction term. However, it is likely that in the near future well-posed initial-boundary value problems will be constructed in which the constraint system implicitly obeys maximally dissipative boundary conditions. When these are of Dirichlet type, the homogeneous constraint modes should also be eliminated.

We have pointed out that the Z4 system is closely related to harmonic gauge, and that the two share the same constraint evolution system. Therefore constraint damping works in the same way in harmonic gauge evolutions, and all 8 constraints can be damped. Similarly, in the Z3 system and its counterpart the NOR/BSSN system, 5 of 7 constraints can be damped.

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