A mathematical model for real-time control of the SILO4 leg

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ABSTRACT

Leg dynamics are often ignored in the real-time control of walking robots. The high gear reduction on actuators are the main reason. However, the use of gear reduction high enough to neglect leg dynamics yields additional non-desired effects. In order to make dynamic equations reflect the reality of the physical system, it is important to model the most important effects acting on it. In this paper we analyse the dynamics of the SILO4 leg, finding out the main sources of forces affecting the system. Then, we present a simple mathematical model that reflects the reality of the physical system which can be used by a real-time dynamic control system.

1 INTRODUCTION

Deriving an accurate mathematical model of a 3-dof mechanism like the SILO4 leg is a complex task and can incur unavoidable errors. In addition to the inherent complexity in deriving differential equations of a 3-dof leg motion, joint actuators, gears and coupling elements embedded in the mechanism provide inertial moments, friction, backlash and elasticity to the system. It is important to realize that the dynamic equations that generally describe the motion of a robot manipulator do not encompass all these effects acting on it (2). The modelling of elasticity between driving actuators and links in robot manipulators is extremely difficult and by itself a deep field of research (12) (3), and it is rarely included in the dynamic model of manipulators. As a general rule, limiting the closed loop natural frequency to half the resonant frequency avoids exciting unmodelled resonances (10).

Another source of forces that are usually not included in the dynamic model is friction. All mechanisms are, of course, affected by frictional forces; however in some manipulators in which significant gearing is typical, the forces due to friction can be up to 25 percent of the torque required to move the manipulator in common situations (2). Moreover, when these manipulators are legs of robots gearing is much higher, and friction effects can be greater than 50 percent of the total torque.
In order to make dynamic equations reflect the reality of the physical system, it is important to model, at least approximately, the most important effects acting on it. A trade off between an accurate model of the system and the viability of its real-time implementation for dynamic control has to be established. In this paper we analyse the dynamics of the SILO4 leg finding out the main sources of forces affecting the system. Then, we present a simple mathematical model that reflects the reality of the physical system which can be used by a real-time dynamic control system. (5) and (7) present the main features of the SILO4 walking robot.

2 DYNAMIC MODEL

Dynamics relates forces affecting a body with the motion induced on it. Thus, the dynamic model of a robot manipulator states the relationship between robot motion and the forces involved on it. Specifically, the dynamic model of a robot manipulator finds mathematical relationships among:

1. Robot location and its derivatives, velocity and acceleration.
2. Forces and torques applied in the robot joints or end-effector.
3. Dimensional parameters of the robot manipulator, such as link length, mass and inertia.

The dynamic model of a manipulator consists of the model of the mechanical part and the model of its actuators and transmission systems. The dynamic model of the mechanical part states the mathematical relationships between the manipulator motion and the forces and torques causing it. On the other hand, the dynamic model of actuators and transmission systems finds relationships between control signals and forces and torques required for motion (1).

The SILO4 leg can be studied from the dynamics point of view as a 3-dof manipulator, whose actuators are DC motors and with a foot as end-effector. We will derive the dynamic model of the actuators and the mechanical part separately in the following sections.

3 DYNAMIC MODEL OF THE SILO4 LEG

To derive the dynamic equations of the mechanical part of the SILO4 leg, the Lagrange-Euler formulation has been chosen (6). The direct application of the lagrangian dynamics formulation, together with the Denavit-Hartenberg link coordinate representation results in a convenient, compact and systematic algorithmic description of the SILO4 leg equations of motion. Although the real-time computation of the Newton-Euler formulation (6) is still more efficient than the Lagrange-Euler equations in open-loop control, the fact is that today’s processors are fast enough to compute efficiently the 4 x 4 homogeneous transformation matrices of the lagrangian formulation. The Lagrange-Euler formulation is a simple and secure method to derive the mathematical expressions. Later analysis of the dynamic model of the SILO4 leg will result in simplifications that enable the real-time computation of the final equations of motion. Table 1 lists all dynamic parameters of the SILO4 leg used for the derivation of the dynamic equations of motion. Accurate values of inertial moments and positions of centre of masses have been computed using Pro/ENGINEER® mechanical design software (11). Mass values were checked experimentally.
Table 1: Dynamic parameters of the SILO4 leg referred to Denavit-Hartenberg link coordinate representation.

<table>
<thead>
<tr>
<th>Link parameter</th>
<th>Link 1</th>
<th>Link 2</th>
<th>Link 3 + foot</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mass (Kg)</td>
<td>1.22</td>
<td>1.26</td>
<td>0.63</td>
</tr>
<tr>
<td>Length (mm)</td>
<td>60.0</td>
<td>238.4</td>
<td>238.5</td>
</tr>
<tr>
<td>Position of the c.o.m. (mm)</td>
<td>-12.2</td>
<td>-109.4</td>
<td>-84.5</td>
</tr>
<tr>
<td></td>
<td>101.0</td>
<td>11.4</td>
<td>-2.5</td>
</tr>
<tr>
<td></td>
<td>0.4</td>
<td>-0.8</td>
<td>3.9</td>
</tr>
<tr>
<td></td>
<td>18.2</td>
<td>0.6</td>
<td>0.3</td>
</tr>
<tr>
<td></td>
<td>1.7</td>
<td>1.8</td>
<td>-0.01</td>
</tr>
<tr>
<td>Inertia tensor (10⁻³ Kg m²)</td>
<td>0.002</td>
<td>-0.17</td>
<td>0.17</td>
</tr>
<tr>
<td></td>
<td>0.6</td>
<td>22.4</td>
<td>10.8</td>
</tr>
<tr>
<td></td>
<td>-0.03</td>
<td>0.01</td>
<td>0.0</td>
</tr>
<tr>
<td></td>
<td>18.4</td>
<td>22.5</td>
<td>10.8</td>
</tr>
</tbody>
</table>

The systematic derivation of the Lagrange-Euler equations yields a dynamic equation, which can be written in the form:

\[
\tau = D(\theta) \ddot{\theta} + H(\theta, \dot{\theta}) + G(\theta)
\]  
(1)

where \( D(\theta) \) is the 3 x 3 mass matrix of the leg, \( H(\theta, \dot{\theta}) \) is a 3 x 1 vector of centrifugal and Coriolis terms, and \( G(\theta) \) is a 3 x 1 vector of gravity terms. The matrices \( D, H \) and \( G \) for the SILO4 leg can be found in the Appendix B. Maple V software package has been used for symbolic simplification of the results (8).

4 DYNAMIC MODEL OF ACTUATORS

The actuators of the SILO4 leg are three low-inertia DC motors, located at each joint and connected through gear reduction to the load. Figure 1 shows the mechanical model of a DC torque motor connected through gear reduction to an inertial load. The torque applied to the rotor, \( \tau_m \), must balance both rotor and load inertias, which we can call equivalent inertia, \( J_{eq} \). Likewise it must balance damping effects due to motor and load friction, what we can call equivalent damping, \( B_{eq} \), that is:

\[
\tau_m = J_{eq} \ddot{\dot{\theta}}_m + B_{eq} \dot{\theta}_m
\]  
(2)

where

\[
J_{eq} = J_m + \frac{J}{N^2},
\]  
(3)

\[
B_{eq} = B_m + \frac{B}{N^2}.
\]
The torque balance can be also written in terms of load variables as:

\[ \tau = J_{\text{eff}} \ddot{\theta} + B_{\text{eff}} \dot{\theta} \]  \hspace{1cm} (4)

where

\[ J_{\text{eff}} = J + N^2 J_m, \]  \hspace{1cm} (5)

\[ B_{\text{eff}} = B + N^2 B_m. \]

The term \( J_{\text{eff}} \) can be called effective inertia seen at the output shaft, likewise \( B_{\text{eff}} \) can be called effective damping (2).

The three actuators of the SILO4 leg are connected through gear to the load. The first joint actuator is connected through planetary gear, however joints 2 and 3 have planetary and additional skew-axis gear (see Figure 2). Thus, the first joint-motor assembly will match the model of Figure 1, however joint-motor assemblies of joints 2 and 3 have two gear stages and thus will have a more complex model. If we want to achieve an accurate model of these actuators we should take into account that they are non-ideal actuators. Each gear stage has torque losses due to friction, what is usually represented in terms of efficiency. A mathematical model of this friction could improve the global mathematical model of the
actuator, but this is out of the scope of this paper. Thus, we model gear friction in terms of efficiency. This efficiency must be included in the three actuator models which here we present. Then let us name the rotor inertia and damping for the joint $i$, $J_{mi}$ and $B_{mi}$ respectively, and let us also name the inertia and damping of the elastic coupling element between the planetary gear and the skew-axis gear $J_{ei}$ and $B_{ei}$ respectively.

Let us assume that the three joint-motor assemblies are connected to zero load, but might need to balance some perturbation $\tau_{pi}$. The torque balance of expression (4), written in terms of load variables, for the three joint-motor assemblies of the leg is as follows:

$$\tau_1 - \tau_{p1} = \eta_{p1}(N_{p1}^2 J_{m1} \ddot{\theta}_1 + N_{p1}^2 B_{m1} \dot{\theta}_1 )$$

$$\tau_2 - \tau_{p2} = \eta_{p2}(N_{p2}^2 J_{m2} \ddot{\theta}_2 + N_{p2}^2 B_{m2} \dot{\theta}_2 )$$

$$\tau_3 - \tau_{p3} = \eta_{p3}(N_{p3}^2 J_{m3} \ddot{\theta}_3 + N_{p3}^2 B_{m3} \dot{\theta}_3 )$$

(6)

Where parameters $\eta_{pi}$ and $\eta_{si}$ denote gear efficiency. All actuator parameters are listed in table 3. Figure 3 shows a block diagram of the dynamic model of the $i$-th actuator of the SILO4 leg.

$$\tau_{pi} = \frac{1}{L_s + R} K_M (\tau_i - \tau_{pi}) + \frac{1}{J_{eq} s + B_{eq}} \tau_i$$

Table 2: Actuator parameters

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Actuator 1</th>
<th>Actuator 2</th>
<th>Actuator 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>$J_m$ (10$^6$ Kg m$^2$)</td>
<td>2.3</td>
<td>6.4</td>
<td>4.9</td>
</tr>
<tr>
<td>$B_m$ (10$^4$ Nm/rad/s)</td>
<td>1.77</td>
<td>9.14</td>
<td>3.0</td>
</tr>
<tr>
<td>$R$ (Ω)</td>
<td>10.5</td>
<td>2.0</td>
<td>5.5</td>
</tr>
<tr>
<td>$L$ (10$^3$ H)</td>
<td>0.94</td>
<td>0.27</td>
<td>0.85</td>
</tr>
<tr>
<td>$K_M$ (10$^3$ Nm/A)</td>
<td>46.81</td>
<td>42.88</td>
<td>41.05</td>
</tr>
<tr>
<td>$K_E$ (V/rad/s)</td>
<td>0.039</td>
<td>0.043</td>
<td>0.041</td>
</tr>
<tr>
<td>Planetary gear</td>
<td>$N_p$</td>
<td>246</td>
<td>14</td>
</tr>
<tr>
<td></td>
<td>$N_s$</td>
<td>20.5</td>
<td>20.5</td>
</tr>
<tr>
<td>Skew-axis gear</td>
<td>$B_e$</td>
<td>0.0</td>
<td>0.0</td>
</tr>
<tr>
<td></td>
<td>$J_e$ (10$^6$ Kg m$^2$)</td>
<td>6.5</td>
<td>6.5</td>
</tr>
</tbody>
</table>
5 MODEL ANALYSIS

Once that we have derived the dynamic equations of the mechanical part of the leg, we can compute the torques that the control system will need to balance in order to perform a desired trajectory at the foot. If we had a decoupled dynamic model of the leg, we could introduce these torques as perturbations to the actuator control system. Figure 4 shows this idea, where block diagrams are simplified for the sake of clarity. The real problem is that the dynamic model of the leg is a coupled non-linear system. In this section we analyse in detail the contribution of each term of the leg dynamic model to the torques required to balance during different trajectories of the foot. Such analysis will permit us consider the whole system as a decoupled linear one.

The first step in our analysis consists in separating the mass matrix $D$ of our leg dynamic model into two different matrices. Let us name $D_1$ the diagonal $3 \times 3$ matrix whose elements are the constant terms of $D$, that is:

$$
D_1 = \begin{bmatrix}
d_1 & 0 & 0 \\
0 & d_2 & 0 \\
0 & 0 & d_3
\end{bmatrix}
$$

and let us name $D_2$ the matrix that contains the rest of terms in $D$, that is:

Figure 4: Control system block diagram
\[
D_2 = D - D_1 = \begin{bmatrix}
D_{11} - d_1 & D_{12} & D_{13} \\
D_{12} & D_{22} - d_2 & D_{23} \\
D_{13} & D_{23} & D_{33} - d_3
\end{bmatrix}
\]  

(8)

Now the inverse dynamics equation can be expressed in the following form:

\[
\tau_p = D_1(\theta)\ddot{\theta} + D_2(\theta)\ddot{\theta} + H(\theta, \dot{\theta}) + G(\theta)
\]  

(9)

A comparison of the contributions of each matrix in the above expression to the torque in each joint has been realized for different foot trajectories. Figures 5 and 6 show the results of one of this experiments.

\[\text{Figure 5: Torque contribution of the dynamic model matrices: (a) mass matrix } D_1; \text{ (b) mass matrix } D_2; \text{ (c) centrifugal and Coriolis terms } H; \text{ (d) gravity terms } C.\]

It is important to note from Figure 5c the relative small effect that Centrifugal and Coriolis forces have on leg motion. Also from Figure 5d we can observe that the second joint of the leg is supporting all the gravitational effects. Thus, from Figures 5 and 6a we can conclude that the major contributions to the final torque correspond to a constant value \(d_1\) for the torque on joint 1 and the gravity term corresponding to the joint 2. This statement is reinforced when
we compare the output torque of the dynamic model of the leg and the torque the actuators employ to balance equivalent inertias and damping as it was explained in section 4. This is shown in Figure 6b, where we can see the dominance of the actuators equivalent inertial and damping effects in the total torque that the motors should hold. The most important conclusion is that the effect of SILO4 leg dynamics should never be neglected, which is a common simplification in a large number of walking robot dynamic models, usually named massless-leg robots. Therefore their effect on trajectory control must be taken into account. This effect is shown in Figure 6b for a given trajectory, and it appears in whatever trajectory we test.

![Figure 6: (a) Joint torques from the inverse dynamics of the SILO4 leg. (b) Joint torques and motor torque comparison for the three joints of the leg](image)

Figures 5 and 6 allow us to simplify the dynamic model of the mechanical part of the leg in the following way without losing accuracy:

\[
\begin{align*}
\tau_{p1} &= d_1 \dot{x}_1 \\
\tau_{p2} &= u \\
\tau_{p3} &= 0
\end{align*}
\]

where \(d_1\) is the first diagonal element of matrix \(D_1\), and \(u\) is a constant whose value can be found in the Appendix B.

We have finally found a simple linear decoupled perturbation effect of leg dynamics on the closed loop control of joint trajectories. Block diagrams of the control system of each joint are depicted in Figures 7 to 9, where the dynamic effects of the leg are modelled as perturbations given in equations (10) to (12). Each controller in these Figures can be a PID filter that can be easily tuned using classical control techniques to balance motor equivalent inertia, damping and perturbations (9). The complex foot trajectory control problem stated at the beginning of this section has turned to a simple decoupled and linear joint control scheme.
Many authors recommend not to take leg dynamics into account in the control of walking robots. The high gearing employed is often the reason to neglect the effect of leg dynamics on

6 MODEL ASSESSMENT AND CONCLUSIONS

Many authors recommend not to take leg dynamics into account in the control of walking robots. The high gearing employed is often the reason to neglect the effect of leg dynamics on
trajectory control. However, the use of gear reduction high enough to ignore leg dynamics implies a significant increase in backlash and elasticity of the transmission system (1). These non-desired additional effects are much more difficult to model than leg dynamics. The main conclusion of this paper is that it is not always the best option to consider robot legs as massless systems. Their effect on leg motion can be appreciable and more over, it can be used to improve the control system. In this paper we have derived a precise and accurate model of a robotic leg. The detailed analysis of this model led us to a very simplified and accurate model of the dynamic effect of the leg on motion control. It also permits the tuning of a PID controller, which we employ for the dynamic control of this leg, during trajectory following. The next step in the near future is the real-time control of the SILO4 robot using the leg dynamic model that we have presented here.

ACKNOWLEDGEMENT

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REFERENCES


APPENDIX A. SILO4 HOMOGENEOUS MATRICES

To derive the dynamic model of the SILO4 leg the Lagrange-Euler formulation has been used. The Denavit-Hartenberg homogeneous matrix representation has been used to describe the spatial displacement between neighbouring link co-ordinate frames to obtain the kinematic information. The relevant Denavit-Hartenberg parameters are given in Table A1. They are obtained from the kinematic parameters of the leg, which can be obtained from Figure A1. Finally, the Denavit-Hartenberg homogeneous matrices that contribute to the dynamic model are given in equation (A1) to (A3). Note that $S_i = \sin(\theta_i)$, $C_i = \cos(\theta_i)$.

Table A1: Denavit-Hartenberg link parameters of the SILO4 leg

<table>
<thead>
<tr>
<th>link</th>
<th>$a_i$</th>
<th>$d_i$</th>
<th>$\alpha_i$</th>
<th>$\theta_i$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$a_1$</td>
<td>0</td>
<td>$\pi/2$</td>
<td>$\theta_1$</td>
</tr>
<tr>
<td>2</td>
<td>$a_2$</td>
<td>0</td>
<td>0</td>
<td>$\theta_2$</td>
</tr>
<tr>
<td>3</td>
<td>$a_3$</td>
<td>0</td>
<td>0</td>
<td>$\theta_3$</td>
</tr>
</tbody>
</table>

$^0A_1 = \begin{bmatrix} C_1 & 0 & S_1 & a_1C_1 \\ S_1 & 0 & -C_1 & a_1S_1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ (A1)

$^1A_2 = \begin{bmatrix} C_2 & -S_2 & 0 & a_2C_2 \\ S_2 & C_2 & 0 & a_2S_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ (A2)

$^2A_3 = \begin{bmatrix} C_3 & -S_3 & 0 & a_3C_3 \\ S_3 & C_3 & 0 & a_3S_3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ (A3)

Figure A1: General view of the SILO4 leg
APPENDIX B. DYNAMIC MODEL OF THE SILO4 LEG

In this appendix we present the results obtained after applying the Lagrange-Euler formulation to derive the dynamic model of the SILO4 leg (See equation (1)). Numerical simplifications of the three matrices that match the dynamic model of the SILO4 leg have been performed and presented below.

B. 1 Mass matrix for the SILO4 leg (D)
The mass matrix is a 3 x 3 diagonal matrix containing inertia forces between two links of the leg. The general form of this matrix is

\[
D = \begin{bmatrix}
D_{11} & D_{12} & D_{13} \\
D_{12} & D_{22} & D_{23} \\
D_{13} & D_{23} & D_{33}
\end{bmatrix}
\] (B1)

The contribution of every term of each element of this matrix has been analysed for different foot trajectories, and finally non significant terms, whose contribution is less than \(10^{-4}\), have been omitted. Thus, after these mathematical simplifications, each element of the mass matrix has finally the following expression:

\[
D_{11} = a C_2 + b S_2 + c C_3 + d C_{23} + e \cos(\theta_3 + 2\theta_2) + e \sin(2\theta_2) + f \cos(2\theta_3 + 2\theta_2) + h \\
D_{12} = 0 \\
D_{13} = 0 \\
D_{22} = k C_3 + l \\
D_{23} = c C_3 + m \\
D_{33} = m
\] (B2)

The constant and diagonal matrix \(D_1\), whose elements are the constant terms of the diagonal of matrix \(D\) is of the form:

\[
D_1 = \begin{bmatrix}
d_1 & 0 & 0 \\
0 & d_2 & 0 \\
0 & 0 & d_3
\end{bmatrix}
\] (B3)

where \(d_1 = h\), \(d_2 = l\) and \(d_3 = m\).

Note that \(S_i = \sin(\theta_i)\), \(C_i = \cos(\theta_i)\), \(S_{ij} = \sin(\theta_i + \theta_j)\) and \(C_{ij} = \cos(\theta_i + \theta_j)\). The value of every constant can be found in table B1.

B. 2 Vector of centrifugal and Coriolis terms (H)
The vector of centrifugal and Coriolis terms is of the form:

\[
H = [H_1 \ H_2 \ H_3]^T
\] (B4)

where, after analysis and simplification, each element has the form:
\[
H_1 = h_{112} \dot{\theta}_1 \dot{\theta}_2 + h_{113} \dot{\theta}_1 \dot{\theta}_3 \\
H_2 = h_{211} \dot{\theta}_1^2 + h_{223} \dot{\theta}_2 \dot{\theta}_3 + h_{233} \dot{\theta}_3^2 \\
H_3 = h_{311} \dot{\theta}_1^2 + h_{322} \dot{\theta}_2^2
\] (B5)

\[
h_{112} = -a \ S_2 + n \ \sin(2\theta_2) - g \ \cos(2\theta_2) - d \ S_{23} - k \ \sin(2\theta_2 + \theta_3) + p \ \sin(2\theta_2 + 2\theta_3) \\
h_{113} = -c \ S_3 - d \ S_{23} - c \ \sin(2\theta_2 + \theta_3) + p \ \sin(2\theta_2 + 2\theta_3) \\
h_{211} = q \ S_2 + f \ \sin(2\theta_2) + r \ \cos(2\theta_2) + s \ S_{23} + c \ \sin(2\theta_2 + \theta_3) + g \ \sin(2\theta_2 + 2\theta_3) \\
h_{223} = -k \ S_3 \\
h_{233} = -c \ S_3 \\
h_{311} = t \ S_3 + s \ S_{23} + t \ \sin(2\theta_2 + \theta_3) + g \ \sin(2\theta_2 + 2\theta_3) \\
h_{322} = c \ S_3
\] (B6)

### B.3 Vector of gravity terms (G)

The vector of gravity terms is of the form:

\[
G = \begin{bmatrix} G_1 \\ G_2 \\ G_3 \end{bmatrix}
\] (B7)

where

\[
G_1 = 0 \\
G_2 = u \ C_2 + v \ S_2 + w \ C_{23} \\
G_3 = w \ C_{23} + x \ S_{23}
\] (B8)

### Table B1: Constant values in SI units for the dynamic model of the SILO4 leg

<table>
<thead>
<tr>
<th>a</th>
<th>0.0376</th>
<th>h</th>
<th>0.0532</th>
<th>r</th>
<th>0.00527</th>
</tr>
</thead>
<tbody>
<tr>
<td>b</td>
<td>-0.00173</td>
<td>k</td>
<td>0.0462</td>
<td>s</td>
<td>0.00581</td>
</tr>
<tr>
<td>c</td>
<td>0.0231</td>
<td>l</td>
<td>0.0856</td>
<td>t</td>
<td>0.0115</td>
</tr>
<tr>
<td>d</td>
<td>0.0116</td>
<td>m</td>
<td>0.0213</td>
<td>u</td>
<td>3.077</td>
</tr>
<tr>
<td>e</td>
<td>-0.00528</td>
<td>n</td>
<td>-0.0635</td>
<td>v</td>
<td>-0.142</td>
</tr>
<tr>
<td>f</td>
<td>0.0317</td>
<td>p</td>
<td>-0.0210</td>
<td>w</td>
<td>0.951</td>
</tr>
<tr>
<td>g</td>
<td>0.0105</td>
<td>q</td>
<td>0.0188</td>
<td>x</td>
<td>0.0152</td>
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</tbody>
</table>