ON THE SIZE OF DIVERGENCE SETS FOR THE
SCHRÖDINGER EQUATION WITH RADIAL DATA

JONATHAN BENNETT AND KEITH M. ROGERS

Abstract. We consider the Schrödinger equation $i\partial_t u + \Delta u = 0$ with initial data in $H^s(\mathbb{R}^n)$. A classical problem is to identify the exponents $s$ for which $u(\cdot, t)$ converges almost everywhere to the initial data as $t$ tends to zero. In one spatial dimension, Carleson proved that the convergence is guaranteed when $s = 1/4$, and Dahlberg and Kenig proved that divergence can occur on a set of nonzero Lebesgue measure when $s < 1/4$. In higher dimensions Prestini deduced the same conclusions when restricting attention to radial data. We refine this by proving that the Hausdorff dimension of the divergence sets can be at most $n - 1/2$ for radial data in $H^{1/4}(\mathbb{R}^n)$, and this is sharp.

1. Introduction

We consider the Schrödinger equation

$$i\partial_t u + \Delta u = 0$$

with initial data $u_0 \in H^s(\mathbb{R}^n)$, where

$$H^s(\mathbb{R}^n) = \{ G_s * f : f \in L^2(\mathbb{R}^n) \}.$$  

Here $G_s$ is the Bessel potential defined by $\hat{G}_s(\xi) = (1 + |\xi|^2)^{-s/2}$, where $\hat{\cdot}$ denotes the Fourier transform.

A classical problem is to identify the exponents $s$ for which

$$\lim_{k \to \infty} u(x, t_k) = u_0(x), \quad \text{a.e. } x \in \mathbb{R}^n,$$

whenever $u_0 \in H^s(\mathbb{R}^n)$ and $(t_k)$ is a sequence of times tending to zero. Carleson [5] proved the convergence with respect to Lebesgue measure in one spatial dimension for $s \geq 1/4$, and Dahlberg and Kenig [6] proved that there can be divergence on sets of nonzero Lebesgue measure when $s < 1/4$. In higher dimensions, Prestini [13] deduced the same conclusions when restricting attention to radial functions. For the best–known results with nonradial data, see [2, 10, 16, 21].

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\footnote{We take discrete times in order to avoid measurability issues.}
We consider a refinement of this question regarding the Hausdorff dimension of the sets on which the convergence fails. We denote by $\alpha_n(s)$ the supremum of
\[
\dim_H \{ x \in \mathbb{R}^n : u(x, t_k) \neq u_0(x) \text{ as } k \to \infty \}
\]
over all $u_0 \in H^s_{rad}(\mathbb{R}^n)$ and all sequences $(t_k)$ that converge to zero. Here, as usual, $\dim_H$ denotes the Hausdorff dimension, and $H^s_{rad}(\mathbb{R}^n)$ consists of the radial members of $H^s(\mathbb{R}^n)$.

Barceló, Carbery and the authors [1] refined the one dimensional result, so that
\[
\alpha_1(s) = \begin{cases} 
1, & s < 1/4 \\
1 - 2s, & 1/4 \leq s < 1/2 \\
0, & 1/2 \leq s.
\end{cases}
\]

In higher dimensions, upper bounds for $\alpha_n$ had previously been proven by Sjögren and Sjölin for nonradial data [15], and these bounds were lowered in certain ranges of $s$ in [1]. We prove the following sharp result for radial data.

**Theorem 1.1.**
\[
\alpha_n(s) = \begin{cases} 
n, & s < 1/4 \\
n - 2s, & 1/4 \leq s < 1/2 \\
0, & 1/2 \leq s.
\end{cases}
\]

Although we do not know the exact value at $s = 1/2$, we see that there are discontinuities when $s = 1/4$ and $s = 1/2$. Given that divergence can occur on a set of nonzero Lebesgue measure when $s < 1/4$, one might have expected divergence to occur on sets of full Hausdorff dimension when $s = 1/4$. We observe that this is not the case, and the sets of divergence can have Hausdorff dimension at most $n - 1/2$ at the critical exponent. This recalls the result of Carbery and Soria [3] regarding the dimension of the divergence sets of Fourier integrals (see also [4]).

2. Set-up

For initial data $u_0$ belonging to the Schwartz class, the solution can be written as
\[
u(x, t) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \hat{u}_0(\xi) e^{i(x \cdot \xi - t|\xi|^m)} d\xi,
\]
where $m = 2$. Our conclusions will hold for general $m > 1$.

For $u_0 \in H^s(\mathbb{R}^n)$, the integral in (2) does not in general exist in the sense of Lebesgue. In this broader setting we may define $u$ as the pointwise limit
\[
u(\cdot, t) = \lim_{N \to \infty} S^N_{t} u_0
\]
whenever the limit exists, where the operator $S^N_{t}$ is defined by
\[
S^N_{t} f(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \psi(N^{-1}|\xi|) \hat{f}(\xi) e^{i(x \cdot \xi - t|\xi|^m)} d\xi.
\]
Here, for convenience, we take $\psi$ to be the Gaussian $\psi(r) = e^{-r^2}$. By standard arguments, $\nu(\cdot, t)$ coincides with the traditional $L^2$-limit, almost everywhere with
We say that a positive Borel measure \( \mu \) is \( \alpha \)-dimensional if
\[
c_\alpha(\mu) := \sup_{x \in \mathbb{R}^n, r > 0} \frac{\mu(B(x, r))}{r^\alpha} < \infty, \quad 0 \leq \alpha \leq n.
\]
We denote by \( M_\alpha(A^n) \) the \( \alpha \)-dimensional probability measures which are supported in the unit annulus \( A^n = \{ x \in \mathbb{R}^n : 1/2 \leq |x| \leq 1 \} \).

It is pertinent to observe that \( u_0 \in H^s(\mathbb{R}^n) \) is integrable with respect to \( \mu \in M_\alpha(A^n) \) with \( \alpha > n - 2s \) due to the elementary inequality
\[
\|f\|_{L^1(d\mu)} \lesssim \sqrt{c_\alpha(\mu)} \|f\|_{H^s(\mathbb{R}^n)},
\]
which holds for these exponents. We prove this in Appendix A.

As usual in such contexts, upper bounds for \( \alpha \) follow from appropriate maximal estimates. Indeed, if
\[
\| \sup_{k \geq 1} \sup_{N \geq 1} |S_{t_k}^N f| \|_{L^1(d\mu)} \lesssim \sqrt{c_\alpha(\mu)} \|f\|_{H^s(\mathbb{R}^n)}, \quad \text{for all } \alpha > \alpha_0 \geq n - 2s,
\]
whenever \( \mu \in M_\alpha(\mathbb{R}^n) \), \( f \in H^s_{rad}(\mathbb{R}^n) \) and \( (t_k) \in \mathbb{R}^N \), then \( \alpha_0(s) \leq \alpha_0 \). This is proved by standard arguments including an application of Frostman’s lemma in Appendix B.

We remark that maximal operators of this type have also been bounded in \( L^p(\mathbb{B}^n) \) with \( p > 1 \), where \( \mathbb{B}^n \) is the unit ball in \( \mathbb{R}^n \) (see [8, 9, 14, 17]). In some sense, these improved integrability properties of the maximal operator are what permits the estimates with respect to fractal measures that we obtain here.

3. The proof of Theorem 1.1

By proving that there can be divergence on sets of nonzero Lebesgue measure, Prestini showed that \( \alpha_n(s) = n \) when \( s < 1/4 \). Hence it will be enough to prove Theorem 1.1 for \( s \geq 1/4 \).

We begin by dealing with the straightforward case \( s > 1/2 \).

**Proposition 3.1.** Let \( n \geq 2 \), \( s > 1/2 \). Then
\[
\| \sup_{k \geq 1} \sup_{N \geq 1} |S_{t_k}^N f| \|_{L^1(d\mu)} \lesssim \|f\|_{H^s(\mathbb{R}^n)}
\]
whenever \( \mu \in M_\alpha(\mathbb{R}^n) \), \( f \in H^s_{rad}(\mathbb{R}^n) \) and \( (t_k) \in \mathbb{R}^N \). This yields \( \alpha_n(s) = 0 \).

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\(^2\)The expression \( A \lesssim B \) denotes \( A \leq CB \), where the value of the positive constant \( C \), which may depend on \( m, n, s \) and \( \alpha \), but never on \( \mu, f, (t_k), N, x, \) or \( t \), will vary from line to line.
Proof. Writing \( f = G_s * g \), it will suffice to prove that if \( s > 1/2 \) then
\[
\left\| \sup_{A^{n^*}} \sup_{N \geq 1} \int_{\mathbb{R}^n} \frac{\psi(N^{-1}|\xi|) \hat{g}(\xi) e^{i(x \cdot \xi - ||\xi||^2)}}{(1 + ||\xi||^2)^{s/2}} \, d\mu(x) \right\|_{L^2(\mathbb{R}^n)} \lesssim \|g\|_{L^2(\mathbb{R}^n)}
\]
uniformly in \( g \in L^2_{\text{rad}}(\mathbb{R}^n) \) and \( \mu \in M^n(\mathbb{A}^n) \). Since \( \mu \) is a probability measure, this will of course follow from the pointwise inequality
\[
\left\| \int_{\mathbb{R}^n} \frac{\psi(N^{-1}|\xi|) \hat{g}(\xi) e^{i(x \cdot \xi - ||\xi||^2)}}{(1 + ||\xi||^2)^{s/2}} \, d\xi \right\|_{L^2(\mathbb{R}^n)} \lesssim \|g\|_{L^2(\mathbb{R}^n)}
\]
uniformly in \( N \geq 1, t \in \mathbb{R}, x \in \mathbb{A}^n \) and \( g \in L^2_{\text{rad}}(\mathbb{R}^n) \). Permitting the abuse of notation \( \hat{g}(r) = \hat{g}(\xi) \) when \(|\xi| = r\), and using polar coordinates, we have that
\[
\left\| \int_{\mathbb{R}^n} \frac{\psi(N^{-1}|\xi|) \hat{g}(\xi) e^{i(x \cdot \xi - ||\xi||^2)}}{(1 + ||\xi||^2)^{s/2}} \, d\xi \right\|_{L^2(\mathbb{R}^n)} = \int_0^\infty \left( \int_{S^{n-1}} e^{itr \omega} d\sigma(\omega) \right) e^{ir \gamma} r^{n-1} \hat{g}(r) e^{-itr^m} \left( \frac{J_{n-2}(r|x|)}{(r|x|)^{n-2}} \right) \, dr,
\]
where \( J_{n-2} \) denotes the Bessel function of order \((n-2)/2\) and \( C_n \) is a constant which depends on \( n \). Now it is well known that
\[
\|r|x|^1 J_{n-2}(r|x|)\| \lesssim C_n
\]
(see [22] or [7]), so that, by the Cauchy–Schwarz inequality and Plancherel’s theorem,
\[
\left\| \int_{\mathbb{R}^n} \frac{\psi(N^{-1}|\xi|) \hat{g}(\xi) e^{i(x \cdot \xi - ||\xi||^2)}}{(1 + ||\xi||^2)^{s/2}} \, d\xi \right\|_{L^2(\mathbb{R}^n)} \lesssim \frac{1}{|x|^{n-2}} \int_0^\infty \frac{r^{n-1} \hat{g}(r)}{(1 + r^2)^{s/2}} \, dr \lesssim \frac{1}{|x|^{n-2}} \left( \int_0^\infty \frac{r^{n-1} \hat{g}(r)^2 \, dr}{(1 + r^2)^{s/2}} \right)^{1/2} \left( \int_0^\infty \frac{dr}{(1 + r^2)^{s/2}} \right)^{1/2} \lesssim \frac{1}{|x|^{n-2}} \|g\|_2 \lesssim \|g\|_2
\]
uniformly in \( N \geq 1, t \in \mathbb{R}, x \in \mathbb{A}^n \) and \( g \in L^2_{\text{rad}}(\mathbb{R}^n) \), as required. We note that the second integral in the second line above converges as \( s > 1/2 \).

To complete the proof of Theorem 1.1, we will first prove that
\[
\alpha_n(s) \leq n - 2s, \quad 1/4 \leq s < 1/2,
\]
and then show that equality holds by means of an example. As remarked in the introduction and proven in Appendix B, the inequality (6) is a consequence of the maximal estimate in the following proposition. To prove the estimate, we employ the following lemma due to Carleson [5] when \( m = 2 \) and \( \gamma = 1/2 \), and due to Sjölin [19] in the stated generality.

Lemma 3.2. [19] Let \( m > 1, 1/2 \leq \gamma < 1 \), and let \( \eta \) be a Schwartz function. Then
\[
\left\| \int_{\mathbb{R}} \eta(N^{-1}|\xi|) e^{i(x \cdot \xi - ||\xi||^2)} \, d\xi \right\|_{L^2(\mathbb{R}^n)} \lesssim \frac{1}{|x|^{1-\gamma}}
\]
The proof of the maximal estimate follows an argument of Sjölin [17], who in turn followed Prestini [13], who in turn followed Carleson [5]. Sjölin used Pitt’s inequality to take advantage of the power weights which appear when applying Lemma 3.2. This yielded an $L^p$-estimate, improving the $L^1$-estimate of Prestini. Here we use the power weights to generate certain classical energies which lead to the $H^s \to L^1(d\mu)$ boundedness of the maximal function, where $\mu$ belongs to a class of fractal measures.\(^3\)

**Theorem 3.3.** Let $\frac{1}{2} \leq s < \frac{1}{2}$. Then for $\alpha > n - 2s$,

$$\left\| \sup_{k \geq 1} \sup_{N \geq 1} |S_k^N f| \right\|_{L^1(d\mu)} \lesssim c_\alpha(\mu) \left\| f \right\|_{H^s(\mathbb{R}^n)}$$

whenever $\mu \in M^\alpha(\mathbb{R}^n)$, $f \in H^s_{\text{rad}}(\mathbb{R}^n)$ and $(t_k) \in \mathbb{R}^N$. This yields $\alpha_n(s) \leq n - 2s$.

**Proof.** Writing $f = G_s \ast g$, it will suffice to prove that

$$\int \sup_{k \geq 1} \sup_{N \geq 1} |S_k^N G_s \ast g(x)| d\mu(x) \lesssim c_\alpha(\mu) \left\| g \right\|_{L^2(\mathbb{R}^n)},$$

whenever $\mu \in M^\alpha(\mathbb{R}^n)$ and $g \in L^2_{\text{rad}}(\mathbb{R}^n)$.

The operator $S_k^N$ maps radial functions to radial functions, so it will suffice to consider radial measures. We write $d\mu_0(v) = d\mu(x)$ when $|x| = v$, and recall the energy $I_{1-2s}(\mu_0)$ defined by

$$I_{1-2s}(\mu_0) = \int_0^1 \int_0^1 \frac{d\mu_0(v) d\mu_0(w)}{|v - w|^{1 - 2s}}.$$ 

Note that since $\text{supp}(\mu) \subseteq \mathbb{A}^n$, we have that $\text{supp}(\mu_0) \subseteq [1/2, 1]$. By decomposing dyadically around $w$, we have

$$\int_0^1 \int_0^1 \frac{d\mu_0(v) d\mu_0(w)}{|v - w|^{1 - 2s}} \lesssim \int_0^1 \sum_{j=0}^\infty c_{\alpha - (n-1)}(\mu_0) 2^{-j(a -(n-1))} 2^{j(1 - 2s)} d\mu_0(w)$$

$$\lesssim c_{\alpha - (n-1)}(\mu_0)$$

$$\lesssim c_\alpha(\mu)$$

for all $\mu \in M^\alpha(\mathbb{B}^n)$ with $\alpha > n - 2s$. In the final inequality, we use the fact that for each interval $I_r$ of length $r$, we can find a ball $B_{nr}$ of radius $nr$ satisfying $\mu_0(I_r) r^{n-1} \leq \mu(B_{nr})$.

Therefore, it will suffice to prove the somewhat sharper

$$\int \sup_{t \geq 1} \sup_{N \geq 1} |S_k^N G_s \ast g(v)| v^{\frac{1}{2} - \frac{\alpha}{2}} d\mu_0(v) \lesssim \sqrt{I_{1-2s}(\mu_0)} \left\| g \right\|_{L^2(\mathbb{R}^n)},$$

\(^3\)This result in fact “self-improves” to an $H^s \to L^2(d\mu)$ estimate. See Appendix C of [1].
where we have again permitted the abuse of notation $S_t^N G_s \ast g(v) = S_t^N G_s \ast g(x)$. As in the proof of Proposition 3.1, we have

$$S_t^N G_s \ast g(v) = C_N \frac{1}{v^{2s-1}} \int_0^\infty \psi(N^{-1} r) \hat{g}(r) (rv)^{1/2} J_{n-2} (rv) e^{-itrv} \frac{r^{n-1}}{(1 + r^2)^{s/2}},$$

where $J_{(n-2)/2}$ denotes the Bessel function. By linearising the maximal function, (7) is implied by the estimate

$$\left| \int_0^1 S_{t(v)}^N G_s \ast g(v) e^{-\frac{x}{2} h(v)} d\mu_0(v) \right| \lesssim \sqrt{I_{1-2s}(\mu_0)} \|g\|_{L^2(\mathbb{R}^N)} \|h\|_{L^\infty(d\mu_0)},$$

uniformly in the measurable functions $t : [0, 1] \to \mathbb{R}$, $N : [0, 1] \to [1, \infty)$ and $h \in L^\infty(d\mu_0)$. By duality it will suffice to prove that

$$\|Th\|_{L^2(0, \infty)} \lesssim \sqrt{I_{1-2s}(\mu_0)} \|h\|_{L^\infty(d\mu_0)},$$

where

$$Th(r) = \int_0^1 \psi(N(v)^{-1} r) (rv)^{1/2} J_{n-2} (rv) e^{i(t+v|r|)h(v)} d\mu_0(v).$$

As in [13], we recall that

$$|y^{1/2} J_{n-2} (y) - (a_1 e^{iy} + a_2 e^{-iy})| \leq C y^{-1}, \quad y > 1,$$

and

$$|y^{1/2} J_{n-2} (y) - (a_1 e^{iy} + a_2 e^{-iy})| \leq C, \quad 0 < y \leq 1,$$

(see [22] or [7]), so we can write

$$Th(r) = a_1 T_1 h(r) + a_2 T_2 h(-r) + T_2 h(r),$$

where

$$T_1 h(r) = \int_0^1 \psi(N(v)^{-1} |r|) e^{(tv+t|v|)h(v)} d\mu_0(v),$$

and $T_2 h$ is a remainder term satisfying

$$|T_2 h(r)| \leq (1 + r^2)^{-s/2} \int_0^1 |h(v)| \frac{1}{\max(1, rv)} d\mu_0(v).$$

Thus, by the triangle inequality, it will suffice to prove that

$$\|T_1 h\|_{L^2(\mathbb{R})} \lesssim I_{1-2s}(\mu_0) \|h\|_{L^\infty(d\mu_0)},$$

$$\|T_2 h\|_{L^2(0, \infty)} \lesssim I_{1-2s}(\mu_0) \|h\|_{L^\infty(d\mu_0)},$$

and

$$\|T_2 h\|_{L^2(1, \infty)} \lesssim I_{1-2s}(\mu_0) \|h\|_{L^\infty(d\mu_0)}.$$

In order to prove (10), we note that

$$\|T_1 h\|_{L^2(\mathbb{R})} \leq \|\overline{T_1} h\|_{L^2(\mathbb{R})} + \|T_2 h\|_{L^2(0, \infty)},$$

where

$$\overline{T_1} h(r) = \frac{1}{|r|^s} \int_0^1 \psi(N(v)^{-1} |r|) e^{(tv+t|v|)h(v)} d\mu_0(v),$$

and

$$\|T_2 h\|_{L^2(0, \infty)} \leq \|T_2 h\|_{L^2(1, \infty)},$$

and

$$\|T_2 h\|_{L^2(1, \infty)} \leq \|T_2 h\|_{L^2(0, \infty)}.$$
and by Fubini’s theorem,
\[ \| T_1 h \|_2^2 = \int_\mathbb{R} T_1 h(r) \overline{T_1 h(r)} dr = \int \int I(v, w) h(v) \overline{h(w)} d\mu_0(v) d\mu_0(w), \]
where
\[ I(v, w) = \int_\mathbb{R} \psi(N(v)^{-1}|r|) \psi(N(w)^{-1}|r|) e^{i(r(v-w)+(\ell(v)-\ell(w))|v|^m)} dr \frac{1}{r^{2s}}. \]
By Lemma 3.2, we have \(|I(v, w)| \lesssim |v - w|^{1-2s}\), so that when \(r > 1\), we have
\[ \| T_1 h \|_{L^2(\mathbb{R})}^2 \lesssim \int_0^1 \int_0^1 |h(v)||h(w)| d\mu_0(v) d\mu_0(w) \]
which is the desired inequality.

In order to prove (11), we note that when \(0 \leq r \leq 1\),
\[ |T_2 h(r)| \leq \int_0^1 |h(v)| d\mu_0(v), \]
and so
\[ \| T_2 h \|_{L^2[0,1]}^2 \lesssim \int_0^1 \int_0^1 |h(v)||h(w)| d\mu_0(v) d\mu_0(w) dr \]
\[ \lesssim \int_0^1 \int_0^1 |h(v)||h(w)| d\mu_0(v) d\mu_0(w) \]
as required.

Finally, in order to prove (12) we follow an argument of Sjölin [17, 18, 20]. We note that when \(r > 1\), we have
\[ |T_2 h| \lesssim T_3 h + T_3 h, \]
where
\[ T_3 h(r) = \frac{1}{r^{1/s}} \int_0^{1/r} |h(v)| d\mu_0(v) \quad \text{and} \quad T_4 h(r) = \frac{1}{r^{1+2s}} \int_0^{1/r} |h(v)| \frac{d\mu_0(v)}{v}. \]
Now, by Fubini’s theorem,
\[ \| T_3 h \|_{L^2(1,\infty)}^2 = \int_1^\infty \frac{1}{r^{2s}} \int_0^{1/r} |h(v)| d\mu_0(v) \int_0^{1/r} |h(w)| d\mu_0(w) dr \]
\[ = \int_0^1 \int_0^1 |h(v)||h(w)| \left( \int_1^{\max\{|v|,|w|\}} \frac{1}{r^{2s}} dr \right) d\mu_0(v) d\mu_0(w). \]
Supposing that \(v > w\), we have
\[ \int_1^{\max\{|v|,|w|\}} \frac{1}{r^{2s}} dr \lesssim \frac{1}{v^{1-2s}} \leq \frac{1}{|v - w|^{1-2s}}. \]
Similarly, this is true when $v < w$, so that, by substituting into (15),

\begin{equation}
\|T_3h\|_{L^2(1, \infty)} \lesssim \int_0^1 \int_0^1 \frac{|h(v)||h(w)|}{|v - w|^{1-2s}} \, d\mu_0(v) \, d\mu_0(w) \\
\lesssim I_{1-2s}(\mu_0) \|h\|_{L^2}^2.
\end{equation}

Turning to the operator $T_4$, by Fubini’s theorem we have

\begin{equation}
\|T_4h\|_{L^2(1, \infty)} = \int_1^{\infty} \frac{1}{r^{2+2s}} \int_1^1 |h(v)| \frac{d\mu_0(v)}{v} \int_1^1 |h(w)| \frac{d\mu_0(w)}{w} \, dr \\
= \int_0^1 \int_0^1 |h(v)||h(w)| \left( \frac{1}{vw} \int_{\min\{v,w\}}^{\infty} \frac{1}{r^{2+2s}} \, dr \right) \, d\mu_0(v) \, d\mu_0(w).
\end{equation}

Supposing again that $v > w$, we have

\[
\frac{1}{vw} \int_{\min\{v,w\}}^{\infty} \frac{1}{r^{2+2s}} \, dr \lesssim \frac{w^{1+2s}}{v} \leq \frac{w^{2s}}{v} \leq \frac{1}{v^{1-2s}} \leq \frac{1}{|v - w|^{1-2s}}.
\]

Similarly, this is true when $v < w$, so that, by substituting into (15),

\begin{equation}
\|T_4h\|_{L^2(1, \infty)} \lesssim \int_0^1 \int_0^1 \frac{|h(v)||h(w)|}{|v - w|^{1-2s}} \, d\mu_0(v) \, d\mu_0(w) \\
\lesssim I_{1-2s}(\mu_0) \|h\|_{L^2}^2.
\end{equation}

Combining this with (14), we obtain (12) by the triangle inequality. This completes
the proof of (9) and hence the theorem. \hfill \Box

It remains to prove that $\alpha_n(s) \geq n-2s$ in the range $1/4 \leq s < 1/2$. For $\alpha$ satisfying $n - 1 < \alpha < n - 2s$, consider the set $E$ defined by

\[ E = \{ \xi : 1/2 \leq |\xi| \leq 3/2, \quad |\xi| \in E_0 \}, \]

where $E_0$ is the generalized Cantor set with $\text{dim}_H(E_0) = \alpha - (n-1)$. By countable
stability, the Hausdorff dimension of $E$ is the same as a segment of arbitrarily
small width. In such segments, the curvature of the circle is negligible, and so the
Hausdorff dimension of $E$ is the same as that of the cross product of the Cantor
set of dimension $\alpha - (n-1)$ with a small interval of dimension $n-1$, and so
$\text{dim}_H(E) = \alpha$.

As $\alpha < n - 2s$, we can fix a $\gamma$ so that $s < \gamma < n-\alpha$, and define $f$ by

\[ f(x) = \chi_{B_2}(x) d(x, E)^{-\gamma}, \]

where $B_2$ is the ball in $\mathbb{R}^n$ with centre 0 and radius 2, and $d(x, E)$ denotes the usual Euclidean distance from $x$ to $E$. It is clear that $f$ is radial, Hardt and Mou
[12, Lemma 3.6] proved that $f \in L^2(\mathbb{R}^n)$, and Žubrinič [23, Theorem 2] proved that $G_\gamma f$ is singular on $E$. This renders the pointwise convergence question moot for
these exponents, yielding $\alpha_n(s) \geq n - 2s$, and so we are done.
Appendix A. Proof of (4)

Writing \( f = G_s \ast g \), we are required to prove that, for \( 0 < s \leq n/2 \),
\[
\| G_s \ast g \|_{L^1(\mu)} \leq \sqrt{c_\alpha(\mu)} \| g \|_{L^2(\mathbb{R}^n)}, \quad \alpha > n - 2s.
\]
It is well–known (see [11]) that \( G_s \geq 0 \), so by Fubini’s theorem and the Cauchy–Schwarz inequality,
\[
\| G_s \ast g \|_{L^1(\mu)} \leq \int \int G_s(x - y) \, d\mu(x) \, |g(y)| \, dy
\leq \| G_s \ast \mu \|_{L^2(\mathbb{R}^n)} \| g \|_{L^2(\mathbb{R}^n)}.
\]
Thus it remains to prove that
\[
\| G_s \ast \mu \|_{L^2(\mathbb{R}^n)}^2 \lesssim c_\alpha(\mu), \quad \alpha > n - 2s.
\]
By Plancherel’s theorem,
\[
\| G_s \ast \mu \|_{L^2(\mathbb{R}^n)}^2 = \| \hat{G}_s \hat{\mu} \|_{L^2(\mathbb{R}^n)}^2,
\]
from which it is clear that we can restrict attention to the range \( 0 < s < n/2 \), as the result for \( s = n/2 \) will follow as a consequence.

Now, it is also well–known (see [11]) that on a compact set, the Bessel potential \( G_s \) satisfies
\[
\frac{C_1}{|x|^{n-s}} \leq G_s(x) \leq \frac{C_1}{|x|^{n-s}},
\]
so by Plancherel’s theorem again,
\[
\| \hat{G}_s \hat{\mu} \|_{L^2(\mathbb{R}^n)}^2 = \int \hat{\mu}(\xi) \, \overline{\hat{G}_{2s}(\xi)} \, d\xi
\leq \int \mu \ast G_{2s}(y) \, d\mu(y)
\leq \int \int \frac{d\mu(x) \, d\mu(y)}{|x - y|^{n-2s}},
\]
which is nothing more than the \((n - 2s)\)–energy. By an appropriate dyadic decomposition,
\[
\int \int \frac{d\mu(x) \, d\mu(y)}{|x - y|^{n-2s}} \lesssim \sum_{j=0}^{\infty} c_\alpha(\mu) 2^{-j\alpha} 2^{j(n-2s)} \, d\mu(y)
\lesssim c_\alpha(\mu), \quad \alpha > n - 2s
\]
for \( \mu \in M^\alpha(\mathbb{R}^n) \), which completes the proof.

Appendix B

Here we prove that the maximal estimate (5) yields an upper bound on the Hausdorff dimension of the divergence sets. Consider a Schwartz function \( h \) satisfying
\[ \|u_0 - h\|_{H^s(\mathbb{R}^n)} < \varepsilon, \text{ and note that} \]
\[ |S_k^N u_0 - u_0| \leq |S_k^N u_0 - S_k^N h| + |S_k^N h - h| + |h - u_0|. \]

We have,
\[
\mu\{ x : \limsup_{k \to \infty} \limsup_{N \to \infty} |S_k^N u_0 - u_0| > \lambda \} \leq \mu\{ x : \sup_{k \geq 1} \sup_{N \geq 1} |S_k^N (u_0 - h)| > \lambda/3 \}
+ \mu\{ x : \lim_{k \to \infty} \lim_{N \to \infty} |S_k^N h - h| > \lambda/3 \} + \mu\{ x : |h - u_0| > \lambda/3 \}.
\]

Now, if \( t_k \to 0 \), the second term on the right hand side of the inequality is zero, so by the maximal inequalities (4) and (5),
\[
\mu\{ x : \lim_{k \to \infty} \lim_{N \to \infty} |S_k^N u_0 - u_0| > \lambda \} \lesssim \lambda^{-1} \sqrt{c_{\alpha}(\mu)} \|u_0 - h\|_{H^s(\mathbb{R}^n)} \lesssim \lambda^{-1} \sqrt{c_{\alpha}(\mu)} \varepsilon.
\]

Letting \( \varepsilon \) tend to zero, then \( \lambda \) tend to zero, we see that
\[
\mu\{ x : u(x, t_k) \not\to u_0(x) \text{ as } k \to \infty \} = 0
\]
whenever \( \mu \in M^\alpha(H^n) \) with \( \alpha > \alpha_0 \). By Frostman’s lemma (see [11]),
\[
H^\alpha\{ x \in H^n : u(x, t_k) \not\to u(x, 0) \text{ as } k \to \infty \} = 0,
\]
where \( H^\alpha \) denotes the \( \alpha \)-Hausdorff measure. By scaling and the countable additivity of Hausdorff measure, this implies that
\[
H^\alpha\{ x \in \mathbb{R}^n : u(x, t_k) \not\to u(x, 0) \text{ as } k \to \infty \} = 0,
\]
Thus, since this holds for every \( \alpha > \alpha_0 \), we have that
\[
\dim_H \{ x \in \mathbb{R}^n : u(x, t_k) \not\to u_0 \text{ as } k \to \infty \} \leq \alpha_0
\]
whenever \( u_0 \in H^{\alpha}_\text{rad}(\mathbb{R}^n) \) and \( t_k \to 0 \), and we are done.

This is a natural continuation of the results obtained in [1]. It is a pleasure to recognise the influence that the numerous conversations with Juan Antonio Barceló and Tony Carbery have had on this work.

**References**


School of Mathematics, The Watson Building, University of Birmingham, Edgbaston, Birmingham, B15 2TT, UK

E-mail address: J.Bennett@bham.ac.uk

Instituto de Ciencias Matematicas CSIC-UAM-UC3M-UCM, 28049 Madrid, Spain

E-mail address: keith.rogers@icmat.es