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The Kreuzer bi-homomorphism

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This is a brief review of simple current techniques and their application to string theory constructions, centered around Max Kreuzer's contribution to this field[1].

[1]Contribution to the Memorial Conference for Max Kreuzer, Vienna, 25-28 June 2011.

1 Introduction

The only paper[1] I co-authored with Max Kreuzer is rather atypical for his work, in that it is algebraic rather than geometrical. In the history of the subject that I am discussing, it plays an interesting rôle. The subject is that of simple currents in conformal field theory. The name "simple current" refers to a primary field of a (usually rational) two-dimensional conformal field theory that has the simplest possible fusion rules with any other field. The name was introduced by Shimon Yankielowicz and myself in a paper we wrote in 1989 [11]. It is a subject that has dominated my scientific career since then, even though on several occasions I wanted to move towards something else. However, each time someone drew my attention to something else that could still be done in this area. Max was one of those people. The paper we wrote closed the first stage of research, that of finding all simple current MIPFs (Modular Invariant Partition Functions), and formed the basis for later developments, in particular work on boundary and crosscap coefficients for open and unoriented strings. I will use this opportunity to review the developments in this area that took place before and after that paper.

2 MIPFs and Simple Currents

In 1989 there was a lot of interest in the problem of finding some or all modular invariant partition functions of a given two-dimensional conformal field theory (CFT). Modular invariance is a one-loop consistency condition for closed strings, and at that time the only string theory considered to be of interest was the $E_8 \times E_8$ heterotic string theory, a theory of closed strings. The requirement of modular invariance was initially applied to two-dimensional CFTs built out of free bosons and free fermions. For free bosons there is a beautiful and completely general solution due to Narain [2] in terms of Lorentzian even self-dual lattices. Although his original work concerned non-chiral theories with $N = 4$ supersymmetry, in [3] it was shown how to use Narain lattices to build chiral heterotic strings. A crucial ingredient in that work was the so-called bosonic string map, which allows mapping the heterotic string to a bosonic one, where a Narain lattice can be used. In the same year 1986, two groups [4] developed a method to satisfy the constraints of modular invariants using free fermions on the world sheet.

But all of this work from 1986 was limited to free two-dimensional CFTs. In 1987 Doron Gepner demonstrated [5] how one could use interacting CFTs, namely N=2 minimal models, to build string theories. A crucial ingredient in his construction was the same bosonic string map, this time used to map type-II strings to heterotic ones. This is needed because the only MIPFs that one can write down without further information are the diagonal one (and the charge conjugation invariant). Hence one must start with a left-right symmetric CFT, such as a type-II theory.

An N=2 minimal model is an example of a rational CFT (RCFT). These are conformal field theories with enough additional world sheet symmetries that the total number of unitary representations is finite. Without additional world-sheet symmetries, *i.e.* with only conformal symmetry, the only such theories are the Virasoro minimal models. Because a fermionic string theory requires world-sheet supersymmetry, the simplest building blocks one can use to construct them are $N = 1$ minimal models. If one wants to build theories with space-time supersymmetry, one must use $N = 2$ building blocks. The simplest ones are the minimal models, with central charge

$$c = \frac{3k}{k+2} \quad \text{for } k = 2 \ldots \infty \tag{1}$$

Since this is always less than the central charge 9 needed to build fermionic strings in four dimensions, one considers tensor products, whose central is the sum of those of the building blocks. There are 168 combinations of building blocks. Using diagonal MIPFs one then gets 168 different four-dimensional heterotic string theories.

But already at that time it was clear that diagonal MIPFs was not all there was. In general, the problem is to find a non-negative integer matrix M_{ij}, $i, j = 0, \ldots, P-1$ with $M_{00} = 1$, where P is the number of primary fields. The matrix M must commute with S and T, the matrix representation of the modular transformations $\tau \to -1/\tau$ and $\tau \to \tau + 1$ on the P Virasoro characters $\chi(\tau)$ of the CFT. In 1987 Cappelli, Itzykson and Zuber[6] classified all the MIPFs of $SU(2)$ WZW models, which admitted an ADE labelling (see also [9]). This naturally raised the question what possibilities might exist for other WZW models. Many papers appeared that found series or isolated examples of MIPFs far various CFTs, see *e.g.* [7, 8].

Another input for our work was a 1988 paper by Erik Verlinde [10], in which he expressed the coefficients appearing in the BPZ fusion rules in terms of the modular transformation matrix S. The fusion rules may be written formally as

$$[i] \cdot [j] = \sum_k N_{ij}{}^k [k] \tag{2}$$

which means that the non-negative integer $N_{ij}{}^k[k]$ specifies the number of ways the chiral algebra representations denoted $[i]$ and $[j]$ can couple to the chiral algebra representation $[k]$. Verlinde's result states that this can be expressed as

$$N_{ij}{}^k = \sum_m \frac{S_{im} S_{jm} S^*_{km}}{S_{0m}} \tag{3}$$

We realized that something special happens if the fusion rules are as simple as they can be. In general, the fusion of any two representations must yield *at least* one other representation. There is *precisely* one if either $[i]$, $[j]$ or $[k]$ are equal to the identity, $[0]$, but often there are special representations $[J]$ that have the property that

$$[J] \cdot [j] = [Jj] \quad \text{for all } j, \tag{4}$$

where "Jj" is merely a convenient notation for the unique fusion product of J and j. We called such representations (or the corresponding primary fields) "simple currents". The reason for the adjective is obvious, and the name "current" will be explained below. If a primary field $[J]$ has this property, this has an interesting consequence for its operator products. Since in any operator product with a field $[j]$ only $[Jj]$ and its descendants can appear, there is a well-defined monodromy phase when transporting $[J]$ on a contour around $[j]$ in the complex plane. On other words, the OPE has a well-defined branch cut, unlike a generic OPE. Note that all these manipulations are performed in a chiral half of the theory. This allows us to define a *monodromy charge*

$$Q_J(i) = h_i + h_J - h_{Ji} \tag{5}$$

This charge is conserved by all correlation functions, and hence defines a symmetry. The final step is then to perform an orbifold-like procedure and "mod out" that symmetry. In general this leads to a new theory with a new MIPF. If $h_J \in \mathbb{Z}$ this new MIPF defines an extension of the chiral algebra by the chiral field with label $[J]$, and one normally refers to such a field as a "current" in the extended algebra. Hence our name "simple current". If h_J is fractional this name is perhaps less suitable, although it is customary to talk about fermionic currents, supercurrents and parafermionic currents. After our paper had appeared, we learned that Fuchs and Gepner had already observed the appearance of fields with simple fusion in a study of four-point functions [14].

In a subsequent paper [12] we wrote the corresponding symmetry as a relation between matrix elements of S, and used that to check the conjectured modular invariance explicitly. This symmetry was discovered independently by Kenneth Intriligator [13], who called it "bonus symmetry". It relates elements of S_{ij} on simple current orbits

$$S_{Ki,j} = e^{2\pi Q_K(j)} S_{ij} \tag{6}$$

In [12] we also wrote down an explicit formula for the matrix M:

$$M_{J^p i, J^q j} = \delta_{ij} \sum_{\ell=1}^{N} \delta^{N_j}_{q,p+\ell} \delta^1 \left[Q_J(i) + \left(\frac{2p+\ell}{2N} \right) r \right]$$

where the "monodromy parameter" r is defined by

$$h_J = \frac{r(N-1)}{2N} \bmod 1$$

Note that r is defined modulo $2N$ if N is even. This formalism includes essentially all prior work on MIPFs as special cases, with the exception of the exceptional invariants. Strictly speaking this is a tautology, because it is precisely the definition of exceptional invariants, but the point is that in all cases studied thus far the exceptional invariants are rare. In the example of the ADE classification of $SU(2)$ only the three "E" invariants are exceptional.

The power of the simple current formalism lies in the fact that it can be applied to any CFT with simple currents, without detailed knowledge of the matrix S. In reality, the vast majority of the RCFT's we know are WZW-model based (for example coset CFTs), and one might argue that in those cases we already know S explicitly anyway.

However it is still a great convenience to be able to write down a huge number of MIPFs without invoking a formula for S. Nowhere is this demonstrated more clearly that in the application to Gepner models, which Shimon Yankielowicz and I attacked later in 1989 [15]. Gepner models have huge simple current groups: each minimal model factor has $4k+8$ simple currents. Not only were we able to vastly extend the total number of MIPFs, but this also lead to a significant simplification of the formalism and to new insights. The generalized GSO projection needed to get space-time supersymmetry was realized as a simple current extension of the chiral algebra, and it became obvious that the simple current formalism allowed us to make this extension only in the fermionic sector, and use a higher spin current (not appearing in the massless spectrum) in the bosonic sector. In this way a large number of spectra with $SO(10)$ unified gauge symmetry – instead of the less desirable E_6 gauge symmetry occurring in (2,2) models – could be constructed. We called these "$(1,2)$" models, where the "1" indicates that world-sheet in the left-moving, bosonic sector remained unbroken. We produced a huge "phonebook" of such spectra, which is available in pdf form (see [21]), but unfortunately not in a computer readable format. For (2,2) spectra similar lists were presented in [16]. Earlier results on subsets of MIPFs were presented in [17] and [18].

The method for building simple current MIPF used in [15] was still a bit primitive, though. We used the simple current matrix $M(J)$ defined in Eqn. (7) above, but we used the fact that we could also multiply such matrices. So we considered products

$$P(\tau,\bar{\tau}) = \chi(\tau)M(J_1)M(J_2)\ldots M(J_k)\bar{\chi}(\bar{\tau})$$

combined with a normalization condition for the identity. It was not clear how many such matrices one could multiply and still get something new, nor was it clear if anything was missed this way.

In 1990 it became clear that indeed something *was* missed this way. Up to then, integer spin simple currents were believed to yield extensions of the chiral algebra. Multiplying such MIPFs then only gives extensions as well. But it turned out that in certain cases automorphism MIPFs were also possible. An automorphism MIPF is characterized by a matrix M that acts as a permutation of the characters. A year later Beatriz Gato-Rivera and I started thinking about the classification problem of all simple current MIPFs [19]. We tried classifying all MIPFs with matrices M_{ij} that had vanishing entries if i and j are not linked by simple currents. We succeeded in part, achieving a full classification for simple current groups $(\mathbb{Z}_p)^k$, with p prime (the result is also valid if p is a product of prime factors, each occurring only once, since the entire problem then just factorizes, but it is not valid if there are factors $\mathbb{Z}_{p^n}$, $n>1$). We derived a formula for the total number of MIPFs

$$T(\rho,k,p) = \prod_{\ell=0}^{k}(1+p^{\ell}) \tag{7}$$

where ρ is the monodromy matrix of the currents

$$Q_i(J_j) = Q_j(J_i) \equiv R_{ij} = \frac{1}{p}\rho_{ij} \quad , \rho_{ij} \in \mathbb{Z}$$

Note that ρ_{ij} is defined modulo N_i as well as N_j. If N_i is even, there is an additional requirement that the diagonal elements ρ_{ii} must be equal to r_i , the monodromy parameter introduced in (7); this may require shifting ρ_{ii} by an amount N_i. The result (7) holds if all ρ_{ii} are even. Since this can always be achieved for N_i odd by shifting by N_i, this condition matters only for N_i even. Note that the total number of MIPFs does not depend on ρ at all. This was merely an observation, but it was not understood.

This is where Max entered the scene. It is now so long ago that I do not even remember the reason I met Max. Presumably he had been invited to NIKHEF to give a seminar. I also do not recall much of the work itself, except that it started with Max's suggestion to add discrete torsion to the orbifold procedure used in derive the simple current MIPF. This led to two important benefits: first of all we were able to extend the classification to all simple current groups, not just $(\mathbb{Z}_p)^k$, and secondly monodromy independence followed straightforwardly, because the orbifold procedure did not care about monodromies, except for the even integer subtleties mentioned above. This led to a general and complete formalism for simple current MIPFs which is still used today. Because it plays a pivotal rôle I quote the main result in its orginal form:

Suppose one has a conformal field theory with simple currents generating a center $\mathcal{C}$. Then the complete set of simple current invariants of that theory can be obtained by the following procedure

- Choose any subgroup $\mathcal{H}$ of $\mathcal{C}$
- Choose a basis of currents $J_1, \ldots, J_k$ that generate $\mathcal{H}$.
- Compute the current-current monodromies R_{ij} in that basis.
- Choose any properly quantized matrix X whose symmetric part is $\frac{1}{2}R$ mod 1 (in other words $X + X^T = R$).

The modular invariant partition function corresponding to this choice is then given by a matrix whose only non-zero elements are

$$M_{a,[\vec{\beta}]a} = \mathrm{Mult}(a) \prod_i \delta^1(Q_i(a) + X_{ij}\beta_j) \ , \tag{8}$$

where δ^1 is equal to 1 if its argument is an integer, and vanishes otherwise. The factor $\mathrm{Mult}(a)$ appears because a may be a fixed point of some currents. In that case the β-sum in (8) includes all terms involving a more than once, and $\mathrm{Mult}(a)$ is the number of times this happens. This is the generalization of (7) to more than one factor.

This paragraph was lifted directly from page 17 of the original paper. By "center" we meant the complete discrete group of simple currents; $[\vec{\beta}]a$ stands for $J_1^{\beta_1} \ldots J_k^{\beta_k} a$. Some subtleties regarding even integers can be found on the same page 17 in the original paper, but will not be repeated here.

The definition of "properly quantized" is that X must be a matrix of rational numbers satisfying $N_i X_{ij} \in \mathbb{Z}$, $X_{ij} N_j \in \mathbb{Z}$ (no summation implied). This matrix X is the object appearing in the title of this talk. Many years later, the authors of [20] defined an object they called Ξ, related to X as

$$\Xi(g,h) = \exp\left(2\pi \sum_{a,b=1}^{k} m_a X_{ab} n_b\right) \tag{9}$$

where g and h are elements of the simple current group written in the form $g = \prod_a (g_a)^{m_a}$ and $g = \prod_a (h_a)^{n_a}$. Obviously $\Xi(g,h)$ defines a bi-homomorphism on the simple current group, which the authors referred to as KSB, where "K" stands for Kreuzer and "B" for bi-homomorphism. We will see this quantity return later in this paper.

Recently we have applied this formalism to heterotic strings [39], in order to go a few steps beyond what was possible in 1989. In the course of doing that we re-obtained the results obtained in 1989 using matrix products. The results of 1989 were obtained by a deep, but randomized search, which was basically continued until the saturation point appeared to be reached (no new spectra appeared during some fixed time). We have not checked the entire list systematically (since it is not available electronically anymore this would be difficult to do), but very few cases, if any, are missing. This implies that all MIPFs that in principle cannot be obtained using matrix multiplication, but only using the complete formalism, can apparently still be obtained in different ways. This must be due to peculiar degeneracies in the Gepner models, which are only partly understood.

In the new complete scan of all $(2,2)$ symmetric as well as asymmetric MIPFs we found a total of 906 distinct Hodge number pairs. If we also consider the number of singlets and massless vector bosons the total number of distinct $(2,2)$ spectra is 2013. The number of Hodge pairs can be compared with similar numbers for free fermionic $(2,2)$ models, and numbers from Max Kreuzer's homepage [26]. For free fermions we obtained 26 distinct Hodge pairs in [23] (see also [24, 25] on related $\mathbb{Z}_2 \times \mathbb{Z}_2$ spectra). Kreuzer and Skarke [27] obtained 30108 distinct Hodge pairs in their work on reflexive polyhedra. Furthermore a list of 4370 Hodge number pairs are listed in [26] for Landau-Ginsburg models (after combining the "Untwisted", "Abelian" and "Discrete torsion" databases). This clearly illustrates the naive expectation that free theories are too special, that the geometric approach is more general, and the interacting CFTs are somewhere in between: almost the geometric mean, in fact. As might be expected, the list of 4370 LG Hodge number pairs includes all 906 Gepner Hodge number pairs, which in its turn includes the 26 free fermion Hodge number pairs.

3 Fixed point resolution

An important element in the story, although not directly related to Max's contribution, is fixed point resolution. This problem arises if the multiplicities $\text{Mult}(a)$ in the main formula above are larger than 1, which in its turn is a consequence of simple currents having fixed points: $Ja = a$. In MIPFs this merely leads to integer multiplicities larger than 1, but these multiplicities require further attention if one attempts to write down the modular transformations for the characters of the new, extended CFT. They lead to degeneracies: several characters of the new CFT are equal to one and the same character of the original CFT. Therefore modular transformations of the original CFT do not provide sufficient information to determine those of the extended CFT.

Based on early conjectures in a paper [28] with Shimon Yankielowicz I wrote in 1989, Jürgen Fuchs, Christoph Schweigert and I [29] were able to solve this problem in the middle of the nineties, for WZW based theories. The missing information is contained in modular transformation matrices S^J of a different algebraic object, which is often, but not always, itself a CFT. It is obtained as follows. The simple current J of a WZW model defines (with one exception, E_8 level 2) an automorphism of the extended Dynkin diagram underlying the WZW model. One can fold up the Dynkin diagram according to this automorhism, according to specific rules spelled out in [30]. This yields the extended Dynkin diagram of the "algebraic object" alluded to above. Usually this is an affine Lie algebra, but in some cases (the ones that were problematic in [28]) it is a twisted affine algebra, which is not a CFT, but still has the remarkable property that it has an associated modular group representation that can be used to define S^J.

The main result of [29] is as follows. Characters of the extended theory are labelled by some representation label a of the original CFT, plus a degeneracy label i distinguishing the individual representations of the extended CFT related to the same character in the original CFT. The modular transformation matrix $\tilde{S}$ of the extended CFT is given by:

$$\tilde{S}_{(a,i),(b,j)} = \frac{|\mathcal{H}|}{\sqrt{|\mathcal{U}_a|\,|\mathcal{S}_a|\,|\mathcal{U}_b|\,|\mathcal{S}_b|}} \sum_{J\in\mathcal{H}} \Psi_i^a(J)\, S^J_{a,b}\, \Psi_j^b(J)^* \tag{10}$$

Here $\mathcal{H}$ is the simple current subgroup used to define the MIPF, $\mathcal{S}_a$ is the stabilizer of a (the set of simple currents J with $J_a = a$), and $\mathcal{U}_a$ is a subgroup of $\mathcal{S}_a$, called the "untwisted stabilizer", and the Ψ^a's are characters of the group $\mathcal{U}_a$. The untwisted stabilizer is the subgroup on which a computable quantity called the twist $F(a, K, J)$ (which can be $+1$ or -1) is equal to 1. This twist is a factor that modifies (6) for fixed point resolution matrices S^J:

$$S^J_{Ki,j} = F_i(K, J)e^{2\pi i Q_K(j)} S^J_{ij} \tag{11}$$

and then the formal definition of the untwisted stabilizer is

$$\mathcal{U}_i = \{J \in \mathcal{S}_i \| F_i(K, J) = 1, \text{ for all } K \in \mathcal{S}_i\} \tag{12}$$

This formula holds for all simple current extensions of the chiral algebra. Since these only involve mutually local integer spin currents, the general simple current formalism of [1] is not needed to obtain them. However, we will see that all quantities introduced here will make a re-appearance later.

4 Boundary Conformal Field Theory

All previous results were for conformal field theory on closed surfaces. Work by the Tor Vergata group in Rome in the early nineties pointed out an interesting application to

CFT on surfaces with boundaries and crosscaps. I am grateful to Augusto Sagnotti for bringing this to my attention, and urging me to work on it.

In 1989, Cardy had written a breakthrough paper on CFT boundary conditions in an attempt to understand the Verlinde formula. The Rome group started applying that work to other cases, and to non-orientable surfaces. The "other cases" usually were simple current MIPFs, in particular those of the $SU(2)$ WZW model and the discrete series. They pointed out an interesting connection with fixed point resolution in [33]. Intuitively, the reason this matters is that if a field scatters of a boundary, one has to be able to decide into which of several degenerate fields it is reflected. It turns out that this choice is governed by the same fixed point resolution matrices S^J introduced above.

After the Rome group had worked out several series of examples, the natural question arose how this would work in general. Or more precisely: given a CFT and a simple current MIPF of the most general form, as given by the KS formalism, what is the set of boundary and crosscap coefficients that go with it.

This question bifurcated into several others. The first one was to determine which closed string states actually propagate in the closed channel between two boundaries (the transverse channel of the annulus). These states are called "Ishibashi states" [32]. In Cardy's work they are in one-to-one correspondence with the set of primary fields. This happens because he used the charge conjugation modular invariant, with $M_{mm^c} = 1$, and all other elements of M_{mn} vanishing (m^c is the charge conjugate of m). A plausible guess is that in non-trivial MIPFs the number of Ishibashi states for a given label i is equal to the value of M_{mm^c}, which now can in principle be any non-negative integer. This is indeed correct. Hence this number is given by the factor "Mult(a)" above. This implies that the Ishibashi states can be labelled by elements of the Stabilizer $\mathcal{S}_m$ of m. So we may introduce a set of labels (m, J), $J \in \mathcal{S}_m$, with

$$Q_L(m) + X(L, J) = 0 \bmod 1 \text{ for all } L \in \mathcal{H} \tag{13}$$

here $\mathcal{H}$ is the simple current subgroup that defines the MIPF, and X is the rational matrix related to the KSB. The MIPF is implicitly assumed to be multiplied by a charge conjugation MIPF.

The next, far less trivial question is which boundaries can be used. What we are looking for is the complete set of boundary states that respect all the symmetries of the original CFT, without taking into account the MIPF, even if the MIPF itself implies a chiral algebra extension. In a paper from 1996 [34], Pradisi, Sagnotti and Stanev had introduced the important concept of completeness of boundaries. This states that the set of boundary states forms a complete basis in the space of transverse channel states, and hence there should be as many boundary states as there are Ishibashi states. In simple examples one could fulfill that condition by associating one boundary label with every simple current $\mathcal{H}$-orbit (including orbits that are projected out in the MIPF). But is was far less obvious how to deal with fixed points. After a lot of experimentation, mainly done in several papers of the Rome group, and by Jürgen Fuchs and Christoph Schweigert, the following description was arrived at in [35]. The boundary states can be labelled by elements of a discrete group we called the "central stabilizer", which is defined as follows

$$\begin{aligned}\mathcal{C}_i &= \left\{J \in \mathcal{S}_i \| F_i^X(K,J)=1, \text{ for all } K \in \mathcal{S}_i\right\} \\ F_i^X(K,J) &= e^{2\pi i X(K,J)} F_i(K,J)^*\end{aligned}$$

Here the KSB makes its appearance.

Note that the rules defining Ishibashi states look rather different from those defining boundary states, and that it is far from obvious that the two set have the same size. This can be proved by writing down the boundary coefficients that relate the two basis, and showing that this matrix is invertible. The result is

$$B_{[a,\psi_a](m,J)} = \sqrt{\frac{|\mathcal{H}|}{|\mathcal{C}_a||\mathcal{S}_a|}}\psi_a^*(J)S^J_{am} \tag{14}$$

Here $\psi_a(J)$ is a character of the discrete group $\mathcal{C}_a$, which turns out to be the most convenient way to label the boundary states. Apart from invertibility there is a much more restrictive test the coefficients have to satisfy namely integrality and non-negativity of the annulus coefficients

$$A^i{}_{ab} = \sum_m \frac{S_{im}B_{am}B_{bm}}{S_{0m}} \ , \tag{15}$$

where a and b are a short-hand notation for the labels $[a,\psi_a]$. All this was checked explicitly in the thesis of my student Lennaert Huiszoon [36].

The next question to be answered was: which orientifolds are allowed for a given KS simple current MIPFs, and what is the expression for the crosscap coefficients that describe the reflection of Ishibashi states from a crosscap (a hole in a Riemann surfaces with opposite sides identified in an orientation-reversing way). As usual, the first answer was given by the Rome group, who showed that for the charge conjugation invariant it could be expressed in terms of the matrix element P_{i0} of the "P-matrix" $P=\sqrt{T}ST^2S\sqrt{T}$. They also found examples where several distinct orientation reversals were allowed, and worked out the result for D-invariants of $SU(2)$. Starting from there, with my students Huiszoon and Sousa we managed to derive the result for more general simple current MIPFs, and the final form was written down in [35].

$$\Gamma_{(m,J)} = \frac{1}{\sqrt{|\mathcal{H}|}}\sum_{L\in\mathcal{H}} \eta(K,L)P_{LK,m}\delta_{J,0} \tag{16}$$

where

$$\eta(K,L) = e^{i\pi(h_K-h_{KL})}\beta_K(L) \tag{17}$$

$$\beta_K(J)\beta_K(J') = \beta_K(JJ')e^{2\pi i X(J,J')};\ J,J' \in \mathcal{H}\ , \tag{18}$$

where once more we see the KS bi-homorphism appear. The simple current K is called the "Klein bottle current", and the equation for the coefficients β_K (which can be signs or

$\pm i$) may allow several solutions differing by signs. There are some further conditions, and some of the choices are equivalent, see section 2 of [37] for further details. These crosscap coefficients appear in the formulas for the Klein bottle and Moebius multiplicities, which are subject to severe integrality constraints, checked explicitly in [36]

$$\begin{aligned} K^i &= \sum_m \frac{S_{im}\Gamma_m\Gamma_m}{S_{0m}} \\ M^i{}_a &= \sum_m \frac{P_{im}B_{am}\Gamma_m}{S_{0m}} \end{aligned}$$

5 Applications

The formalism explained above can be put to practical use in the construction of exact perturbative string spectra in an area we still no little about: interacting two-dimensional CFT's. The vast majority of exact results has been obtained for free fermions or (twisted) free bosons (orbifolds). On the other side of the range of possibilities are geometric methods, such as Calabi-Yau manifold compactifications or F-theory. These cover undoubtely a larger piece of the landscape, but exact results can only be obtained for a limited number of quantities. Interacting CFTs provide a middle ground between the two. Because of the naive notion that these provide "rational points in continuous moduli spaces" one would have expected this to be a extremely large set, but in reality the class of accessible rational CFTs is rather limited. By "accessible" I mean here that the full set of modular data is available: the exact ground state weights and dimensions and the matrices S, T and S^J. Until recently, this was true only for tensor products of $N = 2$ minimal models. Recently, with my student Michele Maio [38], we succeeded in adding one more building block to this list, the $\mathbb{Z}_2$ permutation orbifolds of two identical, already known building blocks.

We have explored the RCFT orientifold and heterotic landscape in recent work [37, 39]. The general simple current formalism described in [1], and its open string analog [35] allowed us to survey a much larger part of the landscape than would otherwise be possible. The features we look for, chiral spectra containing an $SU(3) \times SU(2) \times U(1)$ gauge group and three families are not rare in the string theory landscape, but they are still statistically challenged by a factor about $10^{-5} \ldots 10^{-15}$ depending on the starting assumptions. Without this formalism, even scratching off this small layer from the surface would not have been possible.

Almost twenty years after [1] was written, Max lives on through his work.

Ackowledgements

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