Dynamics of the semiclassical Einstein equations

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An investigation is done on the behavior of the Einstein equation for the case of a conformally invariant field in a conformally flat spacetime when higher-order derivative terms with logarithmic dependence on the scalar curvature are introduced. It is seen that in the quantum case flat spacetime is always stable to conformally flat perturbations.

General relativity with higher-derivative terms could constitute a suitable route to quantize gravity. The gravitational Lagrangian with $R^2$ terms preserves many of the good properties of simple general relativity and may lead to a framework in which quantum gravity would become renormalizable. However, it was soon realized that this kind of theory confronts two fundamental problems. The first difficulty was essentially that in general relativity with higher-derivative terms one ultimately has either unitarity or renormalizability, but never both. Renormalization-group techniques have been used by Julve and Tonin and Salam and Strathdee to move the ghost mass off to infinity, where it should become innocuous. Resummation based on the $1/N$ approximation was carried out by Tomboulis who showed that the real poles vanish in the $1/N$ expansion when gravity is coupled to $N$ massless fermions, and a new pole complex pair of poles in the physical sheet thus violating the analyticity of the $S$ matrix.

The second difficulty was raised by Horowitz and Wald who studied the dynamics of Einstein equations modified by higher-order derivative terms and showed that flat spacetime then is either catastrophically unstable to conformally flat perturbations that grow exponentially on the Planck time scale, or oscillates at the Planck frequency resulting in radiation with higher energy than the most energetic cosmic-ray events by some seven orders of magnitude or so.

It has been recently suggested, nevertheless, that in a theory with higher-order derivative terms modulated by a logarithmic dependence on $R$, the graviton propagator ought to have no tachyon or unphysical-state poles, or complex-conjugate poles on the physical sheet, while the theory appears to be renormalizable and analytical. These kinds of theories could restore also a physical behavior for the semiclassical theory of back reaction. The purpose of this Brief Report is to investigate further this subject starting with a theory of general relativity with higher-order derivative terms containing a logarithmic dependence on $R$.

The stress-energy operator for a conformally invariant quantum field in a conformally flat spacetime has the general form

$$ T_{\mu \nu} = :T_{\mu \nu}: + (K_1 H_{\mu \nu} + K_2 I_{\mu \nu}) I , $$

where $:T_{\mu \nu}:$ denotes the normal-ordered operator, $I$ denotes the identity operator, $K_1$ and $K_2$ are two numerical constants that need not be specified now, and the explicit form for $H_{\mu \nu}$ and $I_{\mu \nu}$ depends on the theory being considered.

The semiclassical Einstein equation for conformally flat spacetime then takes the form

$$ G_{\mu \nu} = 8 \pi (K_1 H_{\mu \nu} + K_2 I_{\mu \nu}) , $$

in which we will assume nonzero, finite values for $K_2$. In order to study the stability of small perturbations for the Minkowski solution, we consider the conformally flat metric $g_{\mu \nu} = \Omega^2 g_{\mu \nu}$, where $g_{\mu \nu}$ is a flat metric, and take $\Omega = 1 + \gamma$, keeping only terms first order in $\gamma$.

We consider then the quantically corrected action integral for the conformally coupled gravity field suggested in Ref. 10, which in reduced form is

$$ I = - (16\pi)^{-1} \int d^4 x \left( -g \right)^{1/2} [R - R^2 R_0^{-1} (1 - \ln R R_0^{-1})] , $$

where $R_0$ is the maximum value of the curvature allowed by the theory. $R \ll R_0$ [$R_0$ could be of the order of (Planck mass)$^{-2}$], and $R^2$ can be generalized in the form

$$ R^2 = \omega (R_{\mu \nu} R_{\mu \nu} - \frac{1}{2} R^2) + \frac{1}{2} \zeta R^2 . $$

$\omega$ and $\zeta$ are coefficients whose values need not be specified now. The generalized Einstein tensor derived from (3) is

$$ G_{\mu \nu} = R_{\mu \nu} - \frac{1}{2} R g_{\mu \nu} + \frac{1}{2} \delta_{[\mu} R_{\nu]} + [\Delta R_{\mu \nu} (2 \ln R R_0^{-1} - 1) - \frac{1}{2} \delta_{[\mu} R^2 (2 \ln R R_0^{-1} - 1) + \frac{1}{2} \delta_{[\mu} R^2 (2 \ln R R_0^{-1} - 1) + (\delta_{[\mu} g^{\alpha \beta} - \delta_{[\mu} p^{\alpha \beta} R_{\alpha \beta} (2 \ln R R_0^{-1} - 1)] . $$

The terms $\{I_{\mu \nu}\}$ and $\{H_{\mu \nu}\}$ (which contain no quadratic or higher-order terms in the curvature) entering into the linearized Einstein equation then become

$$ \{I_{\mu \nu}\} = 2 [\nabla_{\mu} \nabla_{\nu} R - (\nabla^2 R) g_{\mu \nu}] (2 \ln R R_0^{-1} - 1) , $$

$$ \{H_{\mu \nu}\} = 0 . $$

Now, since $R < R_0$, $\ln R R_0^{-1}$ can be expanded in an infinite series, so that the linearized Einstein equation in terms of $\gamma$ becomes

$$ - \partial_{\mu} \partial_{\nu} \gamma + (\Box \gamma) \eta_{\mu \nu} - 48 \pi [R_0]_{\text{ren}}^{-1} \times \left[ 1 + 2 \sum_{n=1}^{\infty} n^{-1} \right] = 0 , $$

where we allow the renormalized value of $R_0$ to take any finite value and $\Box = \partial^\alpha \partial_\alpha$. Equation (7) can be rewritten in

$$ - \partial_{\mu} \partial_{\nu} \gamma + (\Box \gamma) \eta_{\mu \nu} - 48 \pi [R_0]_{\text{ren}}^{-1} \times \left[ 1 + 2 \sum_{n=1}^{\infty} n^{-1} \right] = 0 , $$

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the form
\[(\partial_{\mu}\partial^{\mu} - \eta_{\mu\nu}\square)F = 0\, ,\] (8)
in which
\[F = \gamma - 48\pi [R_0]_{\text{ren}}^{-1}\Box\gamma\left(1 + 2\sum_{n=1}^{\infty} n^{-1}\right).\] (9)

Equation (8) leads finally to the Klein-Gordon homogeneous equation
\[\Box\psi - [R_0]_{\text{ren}}\left[48\pi\left(1 + 2\sum_{n=1}^{\infty} n^{-1}\right)\right]^{-1}\psi = 0\, ,\] (10)

which is the same as that obtained by Horowitz and Wald,\(^9\) unless the term in \(\psi\) is now modulated by a factor \((1 + 2\sum_{n=1}^{\infty} n^{-1})^{-1}\). Since this factor tends to zero, the nontrivial perturbation will be described by a massless Klein-Gordon equation. Therefore, a general spatially homogeneous solution will have the form
\[\gamma = \text{const}\, .\] (11)

Hence, if at \(t=0\) one were to perturb Minkowski spacetime slightly in the direction of this solution, that perturbation would remain unchanged at all further time. Thus, introducing a logarithmic factor depending on \(R R_0^{-1}\) (with \(R < R_0\)), flat spacetime is completely stable with respect to initial small perturbations. This result is valid for any finite value of \(R_0\). The case where \(R_0 = \infty\) must correspond to the classical limit of the theory for which singularities are unavoidable, and the effects of an initial small perturbation could grow at a catastrophic rate or oscillate unphysically at the Planck frequency.

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