Classical and quantum Lovelock cosmology

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In this work we consider a $D$-dimensional ($D \geq 3$) homogeneous and isotropic minisuperspace model provided with a Lovelock Lagrangian. We introduce a procedure by which we associate to the system a unique classical solution. For very weak conditions on the Lovelock coefficients and at least for positive Lorentzian energy densities we show that this solution is uniquely determined by the rather natural physical requirement that the Euclidean solution is flat when we add no matter to the system. In these cases, the given classical solution can be interpreted in terms of perturbations to the corresponding Einstein solution. We also discuss the associated Wheeler-DeWitt equation and apply our results to the model with a cosmological constant and a massive scalar field model.

I. INTRODUCTION

The study of gravity in more than four dimensions has received a great deal of attention in recent years. This interest has been mainly motivated by the success of Kaluza-Klein and string theories which use multidimensional frameworks.

In dealing with multidimensional gravity it seems necessary to introduce higher-order curvature corrective terms in addition to the usual Hilbert-Einstein term. It appears that these terms should produce the dimensionally extended Euler densities of order larger than unity because such densities are the natural topological generalization of the Hilbert-Einstein Lagrangian. On the other hand, Zwiebach and Zumino have shown that, if the low-energy limit of the supergravity obtained from string theory is to respect unitarity, the corrective terms have to be set in groups giving rise to these Euler densities, in such a way that they would lead to ghost-free nontrivial interactions. In fact, Lovelock demonstrated that the most general classical gravitational Lagrangian with associated dynamical equations $\mathcal{G}^{\mu}=0$ preserving that (i) $\mathcal{G}^{\mu}$ is symmetric, (ii) $\mathcal{G}^{\mu}=\mathcal{G}^{\mu}(g_{ab},g_{abc},g_{ab,cd})$, and (iii) $\mathcal{G}^{\mu}=0$ is formed up by the dimensionally extended Euler densities.

Appealing as they may be, gravity theories with a Lovelock Lagrangian must confront some serious problems; the main difficulty is the resulting multivalued expression for the metric derivatives in terms of the associated momenta. As a consequence, the Hamiltonian is multivalued and the system accepts more than one classical solution, which implies a breakdown of the classical predictability.

In order to circumvent this problem, Henneaux, Teitelboim, and Zanelli proposed using an effective Hamiltonian which corresponds to a particular combination of the different branches of the original heuristic Hamiltonian. Their procedure was based, however, on formal manipulations in the path integral. On the other hand, the classical solution associated with the effective Hamiltonian generally differs from those corresponding to the original Hamiltonian. Thus, one should not expect this classical solution to approach the Einstein solution even in the cases where Hilbert-Einstein gravity is thought to be a sufficiently accurate approximation.

Here, we adopt an alternative view which is much closer to the classical description and is interpretable in terms of perturbation theory. If the higher-derivative terms are considered as corrections to the unique Einstein solution, then, locally, the relation between momenta and derivatives is invertible in the vicinity of the Einstein solution. This local inversion is defined by given branches of algebraic functions, which can be analytically continued to the whole complex plane, except for some singularities and cuts, giving rise to monovalued functions whose range restricts the possible variation of the metric derivatives. The unique remaining classical solution is the one with derivatives on the corresponding range of the inversion branch. That solution can be then interpreted as a perturbed Einstein solution.

All other solutions have no physical meaning. They are generated due to the singular character of the perturbations. In general, most of them are complex, and those that are real appear to violate nature physical requirements. Thus, Boulware and Deser have shown that, among the two classical solutions for a spherically symmetric minisuperspace model for a Lagrangian containing quadratic curvature terms and no matter, only the flat solution is stable. More directly related with the content of this paper is the work done by Whitt who, by considering a pure Lovelock gravity, was able to show that, for no cosmological constant, the unique spherically symmetric black-hole solution respecting the physical requirements of positiveness for black-hole mass and effective gravitational constant, while preventing naked singularities, is asymptotically Schwarzschild. Whitt has also shown that this solution is only possible whenever the polynomial associated to the Lovelock Lagrangian $[P(x)=\sum_{m}L_{m}x^{m}$, Eq. (2.15)] is strictly increasing for all $x\geq 0$.

In this paper we shall study a $D$-dimensional ($D \geq 3$) isotropic, homogeneous minisuperspace model, with $ds^{2}=-N^{2}dt^{2}+a^{2}(t)\Omega_{D-1}^{2}$, where $N$ denotes the lapse.
function, \( a(t) \) is the scale factor, and \( d\Omega^2_{D-1} \) is the element on the unit \((D-1)\)-sphere, and assume a Lovelock Lagrangian
\[
L = \sum_{m=1}^{\infty} a_m L_m ,
\]
(1.1)
where the \( a_m \) are real constants (we initially consider a model with no cosmological term, \( a_0 \)) and the \( L_m \) are the \( m \)-th order dimensionally extended Euler densities
\[
L_m = \epsilon_{a_1 \ldots a_D} R^{a_1 a_2} \ldots R^{a_{2m-1} a_{2m}} e^{a_{2m+1}} \ldots e^{a_D} .
\]
(1.2)
Here, \( \epsilon_{a_1 \ldots a_D} \), \( a = 1, \ldots, D \) is the \( D \)-tetrad associated to the \( D \)-dimensional manifold, \( R^a_b \) is the corresponding curvature two-form, and \( \epsilon_{a_1 \ldots a_D} \) is the Levi-Civita tensor.\(^{14}\)
If \( D < 2m \), \( L_m \) vanishes on the considered manifold and (1.1) becomes a finite sum. For \( D = 2m \) the term \( L_{D/2} \) is a topological invariant having no effect in the dynamical equations. The physically relevant Lagrangian reduces to
\[
L = \sum_{m=1}^{M} a_m L_m , M = \begin{cases} \frac{D-2}{2} & \text{for } D \text{ even}, \\ \frac{D-1}{2} & \text{for } D \text{ odd}. \end{cases}
\]
(1.3)
The matter content of the theory is usually described by the Lagrangian \( L_{\text{mat}} \). Our system will verify then the Hamiltonian constraint
\[
\frac{\partial (L + L_{\text{mat}})}{\partial \dot{N}} = H = 0 .
\]
(1.4)
Formally, at least, by substituting in \( H \) the canonical moments for the corresponding differential operators, we get the Wheeler-DeWitt equation governing the quantum evolution of the wave functions \( \Psi \) of the system, \( H \Psi = 0 \).
For our minisuperspace model, the momentum associated to the scale factor \( a \) is quantized by \( p_a = -i(\partial / \partial a) \)

\[
S = \int_\mathcal{M} \sum_{m=1}^{M} \frac{L_m}{(D-1)!!(D-2m)} \epsilon_{a_1 \ldots a_D} R^{a_1 a_2} \ldots R^{a_{2m-1} a_{2m}} e^{a_{2m+1}} \ldots e^{a_D} ,
\]
(2.1)
where \( \mathcal{M} \) denotes the \( D \)-dimensional manifold and \( M \) is given by (1.3). In (2.1) the \( L_m \)'s are real constants. We assume that the first term in the above expansion corresponds to the Hilbert-Einstein action, so that
\[
L_1 = \frac{(D-1)(D-2)}{16\pi G_D} ,
\]
(2.2)
\( G_D \) being the \( D \)-dimensional gravitational constant.
If the manifold \( \mathcal{M} \) has a boundary, we should add a surface term to the action (2.1).\(^{15,16}\) In what follows, we shall use \( i \) and \( j \) for spatial indices: \( \{ 2, \ldots, D \} \).
We choose a minisuperspace with metric \( g = \eta_{ab} e^a e^b \), where
\[
e^1 = N dt ,
\]
(2.3)
\[
e^i = a(t) \left| \prod_{j < i} \sin \theta_j \right| d\theta_i ;
\]
(2.4)
\( N \) denotes the lapse function and \( a \) is the scale factor.
Hence
\[
R^{11} = 0 ,
\]
(2.5)
\[ R^{ij} = \frac{\ddot a}{aN^2} e^i e^j, \]  
(2.6)  
\[ R^{ij} = \frac{1}{a^2} \left[ 1 + \frac{\dot a^2}{N^2} \right] e^i e^j. \]  
(2.7)

Inserting (2.5)–(2.7) in the action we obtain

\[
S = \int dt \sum_{m=1}^{M} L_m a^{D-1-2m} \left[ 1 + \frac{\dot a^2}{N^2} \right]^{m-1} - 2m \sum_{n=0}^{m-1} \left[ \frac{1}{2n+1} \frac{\dot a^2}{N^2} \right]^{m+n+1} + \text{boundary terms},
\]  
(2.8)

in which \( d\Omega_{D-1} \) is the area element on the unit \((D-1)\)-sphere.

Integrating by parts the first term in square brackets in (2.8), after expanding the binomial, we get

\[
S = \int dt \sum_{m=1}^{M} L_m a^{D-1-2m} \left[ 1 + \frac{\dot a^2}{N^2} \right]^{m-1} - 2m \sum_{n=0}^{m-1} \left[ \frac{1}{2n+1} \frac{\dot a^2}{N^2} \right]^{m+n+1} + \text{boundary terms},
\]  
(2.9)

where the boundary terms resulting from integration are canceled by the surface terms, constructed for this minisuperspace in Appendix A [Eq. (A14)].

Equation (2.9) can be rewritten

\[
S = -\int dt \sum_{m=1}^{M} L_m a^{D-1-2m} \left[ \frac{1}{2p(2p-1)} \frac{\dot a^2}{N^2} \right]^p.
\]  
(2.10)

Using

\[
\left( \frac{\dot a^2}{N^2} \right)^p \frac{1}{2p(2p-1)} = \int_0^{a/N} dx \int_0^x dy y^{2p-2}
\]  
(2.11)

for \( p \geq 1 \), and Newton’s formula, we obtain, finally,

\[
S = -\int dt \sum_{m=1}^{M} L_m a^{D-1-2m} \left[ -1 + 2m \int_0^{a/N} dx \int_0^x dy (1+y^2)^{m-1} \right].
\]  
(2.12)

Alternatively, if we use the expression

\[
\frac{1}{2n+1} \left( \frac{\dot a}{N} \right)^{2n+1} = \int_0^{a/N} dx x^{2n}
\]  
(2.13)

for \( n \geq 0 \) in (2.9), we get

\[
S = \int dt \sum_{m=1}^{M} L_m a^{D-1-2m} \left[ 1 + \frac{\dot a^2}{N^2} \right]^{m-1} - 2m \int_0^{a/N} dx \int_0^x dy (1+x^2)^{m-1},
\]  
(2.14)

where we have integrated over the unit \((D-1)\)-sphere area \( V_{D-1} \). Equation (2.14) can also be obtained by integrating by parts (2.12).

Following Whitt, we associated a polynomial

\[
P(x) = \sum_{m=1}^{M} L_m x^m
\]  
(2.15)

to our gravitational theory, with derivative

\[
Q(x) = \sum_{m=1}^{M} L_m mx^{m-1}.
\]  
(2.16)

Inserting polynomials \( P \) and \( Q \), actions (2.14) and (2.12) can be reexpressed as

\[
S = \int dt \sum_{m=1}^{M} L_m a^{D-1} P \left[ 1 + \frac{\dot a^2}{N^2} \right] - \frac{2a^2}{a^2} \int_0^{a/N} dx Q \left[ \frac{1+x^2}{a^2} \right],
\]  
(2.17)
\[
S = \int dt \sum_{m=1}^{M} L_m a^{D-1} P \left[ \frac{1}{a^2} \right] - \frac{1}{a^2} \int_0^{a/N} dx \int_0^x dy Q \left[ \frac{1+y^2}{a^2} \right],
\]  
(2.18)
respectively.

Thus, polynomials \( P \) fix the gravitational action and contain all relevant physics.\(^{13} \)

Denoting \( S = \int dt L \), we can define the momentum \( p = \partial L / \partial \dot{a} \). Then, from (2.18) it follows

\[
p = -2V_{D-1}a^{D-1} \int_0^{\dot{a}/N} dy \left( \frac{1+\dot{y}^2/a^2}{a^2} \right) .
\]

Equation (2.17) can be now rewritten

\[
S = \int dt \left[ V_{D-1}Na^{D-1}p \left( \frac{1+\dot{a}^2/N^2}{a^2} \right) + \dot{a} \right] .
\]

Assuming an implicit relation \( \dot{a}(p) \), the Hamiltonian of the system becomes

\[
H = \int dt (\dot{a}p - L) = -\int dt V_{D-1}Na^{D-1}p \left( \frac{1+\dot{a}^2/N^2}{a^2} \right) .
\]

According to (1.4), varying (2.10) with respect to the lapse function we obtain the Hamiltonian constraint

\[
a^{D-1}V_{D-1}p \left( \frac{1+\dot{a}^2/N^2}{a^2} \right) + \frac{\partial L_{\text{mat}}}{\partial \dot{N}} = 0 ,
\]

where we have introduced a matter Lagrangian \( L_{\text{mat}} \) defined as \( S_{\text{mat}} = \int dt L_{\text{mat}} \). (2.22) is what one should expect from (2.21). A classical solution should satisfy Eq. (2.22).

It can be readily seen that, for homogeneous and isotropic models,

\[
\frac{1}{a^{D-1}V_{D-1}} \frac{\partial L_{\text{mat}}}{\partial \dot{N}} = \frac{T_{11}}{N^2} ,
\]

in which, according to our notation, \( T_{11} \) is the energy component of the energy-momentum tensor for the matter content of the system.

The functional relation between \( p \) and \( \dot{a}/N \) is given in (2.19) by a polynomial of degree \( 2M - 1 \) in the time derivative. Consequently, for \( M > 1 \) the momentum inversion is multivalued, and defines, for a fixed \( a \), an algebraic function \( (\dot{a}/N)(p) \) with \( 2M - 1 \) branches, taking on \( 2M - 1 \) possible values for each point \( p \). Hence, the Hamiltonian (2.21) is also multivalued and so is not well defined classically. Moreover, for \( M > 1 \), since \( P \) is a polynomial of degree \( M \), there are several kinds of classical solutions to (2.22), and the system cannot evolve uniquely.

We consider now the classical solutions of our system for the pure gravity case. For no matter fields and zero cosmological constant, the Hamiltonian constraint (2.22) is equivalent to

\[
P \left( \frac{1+\dot{a}^2/N^2}{a^2} \right) = 0 .
\]

Denoting

\[
\frac{1+\dot{a}^2/N^2}{a^2} = x ,
\]

the classical solutions can be obtained by factorizing the polynomial \( P(x) \) in monomials:

\[
P(x) = L_M \sum_{i=1}^{M} (x - A_i) ,
\]

where the \( A_i \) \( (i = 1, \ldots, M) \) are the \( M \) roots of the polynomial \( P(x) \). The vacuum classical solutions are then given by

\[
1 + \frac{\dot{a}^2}{N^2} - A_i a^2 = 0, \quad (i = 1, \ldots, M) .
\]

In this paper we will concentrate on polynomials \( P(x) \) which are strictly increasing functions for all \( x \geq 0 \). Thus, \( P(x) \) has a simple zero at \( x = 0 \), and no positive zeros, so that we can choose \( A_M = 0 \) and then all other \( A_i \)'s must be either complex or strictly negative. Therefore, in this case we will have a Euclidean flat solution, \( (\dot{a}^2/N^2)_{\text{Eucl}} = 1 \), and \( M - 1 \) Euclidean anti-de Sitter \( (A_i < 0) \) or complex \( (A_i \) complex) solutions, \( (\dot{a}^2/N^2)_{\text{Eucl}} = 1 - A_i a^2 \).

**III. THE CHOICE OF A CLASSICAL SOLUTION**

If all \( L_n \) except \( L_1 \) are set equal to zero, (2.22) reduces to

\[
\left. \frac{1+\dot{a}^2/N^2}{a^2} \right|_{\text{L}} = \frac{T_{11}}{L_1 N^2} ,
\]

corresponding to the Hilbert-Einstein gravity, for which there exists only one classical solution. However, if the Lovelock terms are not zero but interpreted as corrections to the Hilbert-Einstein action, then (a) the classical solution to (3.1) is modified, except when this solution corresponds to a flat one in Euclidean time, which remains unmodified, and (b) since we are dealing with singular perturbations, there appear now unphysical solutions that did not exist initially.

The Hamiltonian constraint

\[
P \left( \frac{1+\dot{a}^2/N^2}{a^2} \right) = \frac{T_{11}}{L_1 N^2} ,
\]

defines an algebraic function \( \dot{a}^2/N^2 \) of \( M \) branches (Appendix B). We shall show that, under sufficiently general conditions [i.e., when \( P(x) \) is a strictly increasing function in \( x \geq 0 \)], we can always choose one inversion branch for (3.2) which is analytic at least throughout the region \( \Gamma: a^2 > 0, T_{11} \geq 0 \), in such a way that, as far as the Lovelock terms can be interpreted as corrections, the classical solution in \( \Gamma \) corresponding to the chosen inversion is also interpretable as a perturbed Einstein solution. The other inversion branches give rise then to the new additional solutions in the above region.

On the other hand, from (2.19),
\[ p^2 = \mathcal{R} \left( a^2 \frac{\dot{a}^2}{N^2} \right) \]
\[ = 4V^2_{D-1}a^{2D-6} \frac{\dot{a}^2}{N^2} \int_0^1 dy \mathcal{Q} \left( 1 + \frac{\dot{a}^2 y^2}{a^2} \right)^2, \]
(3.3)

which defines an algebraic function of $2M - 1$ branches $\dot{a}^2/N^2$ for fixed $a^2$.

Let $(a^2/N^2)_i$ be the value of $\dot{a}^2/N^2$ given by the chosen inversion of (3.2) for a given point $\xi$ in $\Gamma$ and let $p^2_i = \mathcal{R}(a^2,(\dot{a}^2/N^2)_i)$, so that $p^2_i$ can be considered as an image of $\xi$. We shall show that, keeping the given value of $a^2 > 0$, we can always choose an inversion branch of (3.3) which is analytic at least for all $p^2_i$ image of $\Gamma$ by the above procedure and such that the inversion at each of these points produces the corresponding values $(\dot{a}^2/N^2)_i$. So, once the given inversion of (3.3) is assumed, at least in the whole region $\Gamma$ the gravitational classical solution is equivalently determined by either $(a^2,(\dot{a}^2/N^2)_i)$ or $(a^2,p^2_i)$.

The inversion branch of (3.3) can be analytically continued to the entire complex plane $p^2_i$, except for existing singularities and cuts. Thus, we can obtain a monovalued function, depending on $a^2$, that we shall denote $\mathcal{R}^{-1}_i(a^2,p^2)$. The range of $\mathcal{R}^{-1}_i(a^2,p^2)$, for a fixed $a^2$, is constrained to a particular region of the complex plane and so is, then, the derivative $\dot{a}^2/N^2$.

From (2.21), we can define a monovalued gravitational Hamiltonian
\[ H = -\int dt V_{D-1}N a^{D-1} \mathcal{P} \left( 1 + \mathcal{R}^{-1}_i(a^2,p^2) a^2 \right). \]
(3.4)

Associated to (3.4) is the Hamiltonian constraint (3.2) evaluated at $\dot{a}^2/N^2 = \mathcal{R}^{-1}_i(a^2,p^2)$. For given $a^2 > 0$, the constrained range of $\mathcal{R}^{-1}_i(a^2,p^2)$ determines then the unique solution $(\dot{a}^2/N^2)_i = \mathcal{R}^{-1}_i(a^2,p^2_i)$. Therefore, the gravitational part of the system becomes well defined.

The formalism outlined in this section will be implemented in what follows by explicitly constructing the wanted inversions for the momentum and the Hamiltonian constraint.

**IV. INVERSION OF THE HAMILTONIAN CONSTRAINT AND MOMENTUM**

We want to invert first the Hamiltonian constraint (3.2) so that we get a unique classical solution. Let us consider first the case where the energy component of the energy-momentum tensor for the matter content, $T_{11}$, is independent of $a^2$. We denote
\[ \hat{L}_m = \frac{L_m}{L_1}, \]
(4.1)
\[ \hat{P}(x) = \sum_{m=1}^M \hat{L}_m x^m, \]
(4.2)
\[ \hat{Q}(x) = \sum_{m=1}^M \hat{L}_m x^{m-1}, \]
(4.3)
\[ w = 1 + \frac{a^2}{N^2}, \]
(4.4)
\[ z = \frac{T_{11}}{L_1 N^2}. \]
(4.5)

Then, (3.2) takes the form
\[ \hat{P}(w,z) \equiv \hat{P}(w) - z = 0. \]
(4.6)

Equation (4.6) defines an algebraic function with $M$ branches $w = \hat{P}^{-1}(z)$ (Appendix B).

For $T_{11} = 0$ ($z = 0$) we recover the Hamiltonian constraint for the no matter case, which always admits the Euclidean flat solution, i.e., the solution to vacuum Einstein case. Thus, the desired inversion branch, denoted as $\hat{P}^{-1}_i(z)$, should satisfy the condition
\[ \hat{P}^{-1}_i(0) = 0, \]
(4.7)
so that the associated classical solution corresponds then to the Euclidean flat solution.

$\hat{P}^{-1}$ is uniquely determined by condition (4.7), because $\hat{P}(w,z)$ has a single zero at $w=0, z=0$ (see Appendix B):
\[ \partial_w \hat{P}(w,z) |_{w=0} = \hat{Q}(0) = 1. \]

In order to extend monovaluedness in the inversion to the cases of physical interest $T_{11} > 0$, we should also require of our branch to be analytically continuable from the initial point $z = 0$ to all the points $z$ in the positive real axis.

In Appendix B we show that the above condition is equivalent to having a strictly increasing polynomial $\hat{P}(w)$ in the whole real semiaxis $w \geq 0$. In this case, by defining
\[ w_1 = \max \{ w \in \mathbb{R} / \hat{Q}(w) = 0 \} \]
(4.8)
[obviously $w_1 < 0$ and $\hat{Q}(w) > 0 \forall w > w_1$], similar arguments as those in Appendix B can be used to show that the inversion branch $\hat{P}^{-1}$ is analytically continuable from $z = 0$ along $\mathbb{R}^-$ up to the point $z_1 < 0$, where
\[ z_1 = \hat{P}(w_1). \]
(4.9)
$z_1$ is the first point from $z = 0$ at which $\partial_w \hat{P}(\hat{P}^{-1}_i(z), z)$ becomes zero. If $z_1 \neq -\infty$, $z_1$ is an ordinary algebraic singularity of $\hat{P}^{-1}_i(z)$.

For $T_{11}$ independent of $a^2$, we immediately obtain, from the inversion of (4.6),
\[ \frac{\dot{a}^2}{N^2} + 1 = a^2 \hat{P}^{-1}_i \left( \frac{T_{11}}{L_1 N^2} \right). \]
(4.10)

In Appendix B we generalize the inversion of the Hamiltonian constraint to all those cases where $T_{11}$ is a polynomial of $1/a^2$: $T_{11}(1/a^2)$. This involves models for the most usual matter fields (cosmological constant, conformal field, axionic field, etc.). If
for all $a^2 > 0$, the inversion (4.10) is straightforwardly generalized to
\[
\frac{a^2}{N^2} = \partial_a (a^2) = a^2 \hat{P}_1^{-1} \left[ \frac{T_{11}(1/a^2)}{L_1 N^2} \right], \quad (4.11)
\]
where $\partial_a (a^2)$ is an analytic function at all points $a^2 > 0$.

The following comments are worth noting.

1. $\hat{P}_1^{-1}(z)$ restricted to the interval $(z_1, \infty)$ is a real function with range $(w_1, \infty)$. Then, for fixed $a^2 > 0$, the derivative $a^2 / N^2$ given by (4.10) or (4.11) verifies that $a^2 / N^2 \in (a^2 w_1 - 1, \infty)$.

2. Equations (4.10) and (4.11) are classical evolution equations as that of the Einstein general relativity, with $\hat{P}_1^{-1}(z)$ playing the role of the energy density
\[
\frac{T_{11}(1/a^2)}{L_1 N^2},
\]
and having the same sign as $z$ in the relevant interval $z \in (z_1, \infty)$. If the Lovelock terms can be interpreted as corrections, $\hat{P}_1^{-1}(z)$ corresponds to the energy density $z$, when corrected by the gravitational reaction due to the presence of higher-curvature terms. Then, (4.10) and (4.11) give the evolution for a classical solution which is close to that of the corresponding Einstein case.

3. For a generic dimension $D$, $z_1 = -\infty$, for if
\[
D = \begin{cases} 4q + 2 & \text{(even)} \\ 4q + 1 & \text{(odd)} \end{cases}
\]
and $q \in \mathbb{N}$ being a positive integer, then $\hat{P}(w)$ is a polynomial of degree even and, at least at a point on the real axis, its derivative vanishes. If $z_1 < \infty$, the analytic continuation to $z < z_1$ will depend on the path followed, because $C - \{z_1\}$ is not simply connected. Moreover, $\hat{P}_1^{-1}(z)$ for $z < z_1$ defined by analytic continuation along a given path is no longer real. Generally, for sufficiently negative $T_{11}$ (i.e., for $T_{11} < z_1 L_1 N^2$), our formalism loses its physical meaning and is no longer treatable. For $T_{11}$ independent of $a^2$, these unphysical cases would correspond to Einstein gravity with a very large negative cosmological term.

This slightly restricts the potential applicability of our procedure. I think, however, that this limitation is not very much of a shortcoming; after all, even in the more conventional four-dimensional framework, anti-de Sitter spaces pose different problems both in the classical and quantum formalisms.

Let us invert now the functional relation (3.3) between $p^2$ and $a^2 / N^2$ for fixed $a^2 > 0$. By using the results of Appendix C, we invert this relation with a monovalued branch $\hat{R}_1^{-1}(a^2, p^2)$ which is analytic at all points $p^2 \in I_{p_1} a^2 \in [R(a^2, a^2 w_1 - 1), \infty]$ and whose range in $I_{p_1} a^2$ is precisely $(a^2 w_1 - 1, \infty)$.

Restricting to the interval $a^2 > 0$ and assuming that the energy density $z \in (z_1, \infty)$ for $a^2 > 0$, the derivative $a^2 / N^2$ in the classical solution given by (4.11) varies in $(a^2 w_1 - 1, \infty)$ with associated classical momentum
\[
p^2 = \hat{R}_1^{-1}(a^2, p^2).
\]

We can thus define the inversion of this momentum by
\[
\frac{a^2}{N^2} = \hat{R}_1^{-1}(a^2, p^2), \quad (4.12)
\]
Our formalism is consistent as, for the variation range of $\hat{R}_1^{-1}$, we recover from (4.12) the same value $a^2 / N^2$ as given by (4.11) in the studied region.

It can be also shown that the so-defined inversion branch $\hat{R}_1^{-1}(a^2, p^2)$ is an analytic function in its two arguments $a^2, p^2$ at all the points in the open interval
\[
I_{a^2 \times p^2} = \{a^2 > 0, p^2 \in I_{p_1} a^2\} \subset \mathbb{R}^2
\]
(Appendix C).

V. WHEELER-DEWITT EQUATION

We have built up a procedure which provides us with the value of the classical momentum associated to the gravitational variable $a$:
\[
p^2 = \hat{R}_1^{-1}(a^2, p^2). \quad (5.1)
\]
This relation is equivalent to the Hamiltonian constraint once the chosen inversions have been accepted.

If $\Psi$ is the wave function of the system, we can write now a Wheeler-DeWitt equation of the form (up to an operator ordering)
\[
- \frac{\partial^2 \Psi}{\partial a^2} = 4L_1^2 V_{P_1} a^{2D - 6} \left[ a^2 \hat{P}_1^{-1} \left( T_{11}(1/a^2) \right) \right] \left[ 1 + a^2 \hat{P}_1^{-1} \left( T_{11}(1/a^2) \right) \right] \int_0^y dy \hat{Q} \left[ \left[ a^2 \hat{P}_1^{-1} \left( T_{11}(1/a^2) \right) \right] \right] \Psi. \quad (5.2)
\]
If $T_{11}(1/a^2)$ is given in terms of the momenta associated to the matter fields, this can be formally quantized by using the conventional procedure. The quantization of matter fields is implicitly assumed in (5.2), and may be highly non-trivial.

Assuming the applicability of the semiclassical approximation to (5.2), the semiclassical solution will be given by $\Psi \approx e^{-i}$, with
\[ I = \mp \int d^3a 2L_1 V_{D-1} a^{D-3} \left[ 1 - a^2 \hat{P}^{-1} \hat{L}_1 \left( \frac{T_{11}(1/a^2)}{L_1 N^2} \right) \right]^{1/2} \int_0^1 dy \hat{Q} \left[ 1 + a^2 \hat{P}^{-1} \hat{L}_1 \left( \frac{T_{11}(1/a^2)}{L_1 N^2} \right) - 1 \right] y^2 + \text{const} \ . \]  

(5.3)

However, since the semiclassical approximation gives an asymptotic expansion at the limit \( \hbar \to 0 \), if the \( \hat{L}_m \) depend on \( \hbar \) it becomes possible that some of the terms in the semiclassical action (5.3) will not be dominant in the asymptotic expansion and its inclusion loses its sense. In general, we should be very careful in performing the semiclassical approximation in this case. We will consider, nevertheless, the general case in which the coefficients \( \hat{L}_n \) do not depend on \( \hbar \).

There exists a correspondence between the semiclassical action (5.3) and the action corresponding to the Euclidean version of the classical solution (4.11):

\[ \text{L_{\text{mat}} = } N \frac{\partial L_{\text{mat}}}{\partial N} \ . \]  

(5.5)

This correspondence can be illustrated in the case in which \( L_{\text{mat}} \) is homogeneous of degree one in the lapse function \( N \), so that

\[ L_{\text{mat}} = N \frac{\partial L_{\text{mat}}}{\partial N} \ . \]  

(5.6)

VI. APPLICATIONS TO SIMPLE MODELS

In this section we apply the procedure outlined in the preceding sections to the cases of Lovelock gravity plus cosmological constant or a massive scalar field.

The action corresponding to a cosmological constant can be written

\[ S_{\Lambda} = - \int dt L_1 V_{D-1} a^{D-1} \Lambda \ . \]  

(6.1)

It follows that \( T_{11}/L_1 N^2 = \Lambda \), which is independent of \( a^2 \). We restrict then to \( \Lambda > 0 \), covering all positive values and an interval of negative values of the cosmological constant. In this case, the inversion of the Hamiltonian constraint can be obtained by using the chosen branch. We obtain

\[ \frac{\dot{a}^2}{N^2} = a^2 \hat{P}^{-1}(\Lambda) - 1 \ . \]  

(6.2)

The sign of \( \Lambda \), which does coincide with that of \( \hat{P}^{-1}(\Lambda) \), will then dictate the cosmological behavior of the classical solution.

The corresponding Wheeler-DeWitt equation is then

\[ -\frac{\partial^2 \Psi}{\partial a^2} = 4L_1^2 V_{D-1} a^{2D-6} [a^2 \hat{P}^{-1}(\Lambda) - 1] \left[ \int_0^1 dy \hat{Q} \left[ 1 + a^2 \hat{P}^{-1}(\Lambda) - 1 \right] y^2 \right]^2 \Psi \ . \]  

(6.3)

The semiclassical action is now

\[ I = \mp \int d^3a 2L_1 V_{D-1} a^{D-3} \left[ 1 - a^2 \hat{P}^{-1}(\Lambda) \right]^{1/2} \int_0^1 dy \hat{Q} \left[ 1 + a^2 \hat{P}^{-1}(\Lambda) - 1 \right] y^2 + \text{const} \ . \]  

(6.4)
Since in this model $L_{\Lambda}$ satisfies the condition (5.5), action (6.4) is the Euclidean action for the classical solution given by (6.2) subjected to Wick rotation
\[ \frac{\alpha^2}{N^2} = 1 - a^2 \tilde{P}_1^{-1}(\Lambda). \] (6.5)

For $a^2 > 1/\tilde{P}_1^{-1}(\Lambda)$, the first term in (6.4) becomes imaginary, and the semiclassical approximation leads to an oscillatory solution of the Wheeler-DeWitt equation
\[ \Psi \approx e^{-i + e^{+i}}. \] (6.6)

Let us consider now a model with a massive scalar field $\phi$. In this case
\[ S_{\text{mat}} = -\int dt L_D - i\sigma a^{D-1} \left[ m^2 \phi^2 - \frac{\dot{\phi}^2}{N^2} \right], \] (6.7)
where $\sigma$ is a positive coupling constant and $m$ is the mass of the scalar field. We have then a momentum for the scalar field
\[ p_\phi = 2L_D - i\sigma a^{D-1} \phi N. \] (6.8)

Hence, the Hamiltonian constraint (2.22) becomes
\[ \tilde{P} \left[ \frac{1 + \dot{a}^2/N^2}{a^2} \right] = \sigma m^2 \phi^2 - \frac{p_\phi^2}{(2L_D - i\sigma a^{D-1} \phi N^2). \] (6.9)

We generalize now the Hawking massive model. The condition $\sigma m^2 \phi^2 >> 1$ would translate then into
\[ \tilde{P}_1^{-1}(\sigma m^2 \phi^2) >> 1. \] (6.10)

Now, we ought to accept moreover the additional condition
\[ \sigma m^2 \phi^2 >> \frac{p_\phi^2}{4L_D - i\sigma a^{D-1} \phi N^2}, \] (6.11)
at least in the region $a^2 \leq \tilde{P}_1^{-1}(\sigma m^2 \phi^2)^{-1}$. We will see that these conditions are consistent; i.e., once the wave function has been conveniently approximated, (6.11) can be obtained from (6.10).

We expect the approximations
\[ 1 + \frac{a^2}{N^2} = a^2 \tilde{P}_1^{-1} \left[ \sigma m^2 \phi^2 + \frac{p_\phi^2}{4L_D - i\sigma a^{D-1} \phi N^2} \right] \approx a^2 \tilde{P}_1^{-1}(\sigma m^2 \phi^2), \] (6.12)
\[ p^2 = \tilde{P} \left[ \frac{a^2}{N^2} \right] \approx \tilde{P}(a^2, a^2 \tilde{P}_1^{-1}(\sigma m^2 \phi^2)^{-1} - 1) \] (6.13)
to hold at least in the region $a^2 \leq \tilde{P}_1^{-1}(\sigma m^2 \phi^2)^{-1}$. (6.12) follows from (6.11) except for polynomials $\tilde{P}(x)$ showing a wide interval, $I_x$, in $x >> 1$ where its derivative nearly vanishes. These polynomials, which are close to the applicability limit of our procedure, map the interval $I_x$ to a considerably small interval $I_p$. Hence, if $\sigma m^2 \phi^2 \in I_p$, a small relative variation of $\sigma m^2 \phi^2$ may induce, under $\tilde{P}_1^{-1}$, a large relative variation of $\tilde{P}_1^{-1}(\sigma m^2 \phi^2) \in I_x$. In any case, for sufficiently large $\tilde{P}_1^{-1}(\sigma m^2 \phi^2)$ and thus sufficiently large $\sigma m^2 \phi^2$, approximation (6.12) is valid; for the highest order term in $x$ of $\tilde{P}(x)$ dominates $\tilde{P}_1^{-1}$:
\[ \tilde{P}_1^{-1}(\sigma m^2 \phi^2) \approx \left[ \frac{\sigma m^2 \phi^2}{L_M} \right]^{1/M}. \] (6.14)

Clearly, denoting
\[ \epsilon = \frac{p_\phi^2}{4L_D - i\sigma a^{D-1} \phi N^2} << 1, \]
\[ \left[ \frac{\sigma m^2 \phi^2}{L_M} \right]^{1/M} \approx \left[ \frac{\sigma m^2 \phi^2}{L_M} \right]^{1/M}. \] (6.15)

It appears then that the region where the above can be applied is by no means empty. The particular form of the polynomial $\tilde{P}(x)$ will dictate thus the minimal value of $\tilde{P}_1^{-1}(\sigma m^2 \phi^2)$ for which (6.12) holds.

If (6.12) is applicable, then by explicitly calculating the discarded terms in approximation (6.13) we check that these terms are, in general, much less than unity in the considered region $a^2 \leq \tilde{P}_1^{-1}(\sigma m^2 \phi^2)^{-1}$ for $\tilde{P}_1^{-1}(\sigma m^2 \phi^2) >> 1$.

Consequently, given (6.12) and (6.13), and parallel to the case of a massive scalar field in four dimensions, our model is treatable as a model with a positive cosmological constant $\Lambda = \sigma m^2 \phi^2$.

The Wheeler-DeWitt equation is thereby
\[ -\frac{\partial^2 \Psi}{\partial a^2} = 4L_D^2 - a^2 \phi \left[ a^2 \tilde{P}_1^{-1}(\sigma m^2 \phi^2) - 1 \right] \int_0^1 \frac{dy \tilde{Q}}{1 + a^2 \tilde{P}_1^{-1}(\sigma m^2 \phi^2) - 1} \left[ \frac{1 + a^2 \tilde{P}_1^{-1}(\sigma m^2 \phi^2) - 1}{a^2} \right]^2 \Psi. \] (6.16)

The effective action in the semiclassical approximation satisfies
\[ \frac{\partial L}{\partial a} = \frac{2L_D - a^2 \phi}{\sqrt{J}} \int_0^1 \frac{dy}{-a^2 \tilde{Q} \left[ 1 - J y^2 \right] a^2} + 2 J y^2 \tilde{Q} \left[ 1 - J y^2 \right] a^2. \] (6.17)

Hence,
\[ \frac{\partial J}{\partial a} = \frac{2L_D - a^2 \phi}{\sqrt{J}} \int_0^1 \frac{dy}{-a^2 \tilde{Q} \left[ 1 - J y^2 \right] a^2} + 2 J y^2 \tilde{Q} \left[ 1 - J y^2 \right] a^2 \].
where \( U = a m^2 \phi \hat{P}_1^{-1}(\sigma m^2 \phi^2) \) and \( J = 1 - a^2 \hat{P}_1^{-1}(\sigma m^2 \phi^2) \). Integrating by parts the last term and applying 
\[ \hat{P}_1^{-1}(\sigma m^2 \phi^2) \hat{Q} [ \hat{P}_1^{-1}(\sigma m^2 \phi^2) ] = 1, \]
we get finally

\[
\frac{\partial I}{\partial \phi} = \mp \int da \frac{2L_1 V_{D-1} a^{D-1} \sigma m^2 \phi}{[1 - a^2 \hat{P}_1^{-1}(\sigma m^2 \phi^2)]^{1/2}}. 
\] 

(6.18)

In order for (6.11) to be a good approximation, we must have

\[
\left| \frac{\partial \phi}{\partial \phi} \left( \frac{1}{2L_1 V_{D-1} a^{D-1} \sigma m \phi} \right) \right| = \left| \frac{\partial I}{\partial \phi} \frac{1}{2L_1 V_{D-1} a^{D-1} \sigma m \phi} \right| \leq T \ll 1, 
\] 

(6.19)

provided that \( \Psi = e^{-I} \).

If we evaluate the wave function at \( a_f^2 \leq [\hat{P}_1^{-1}(\sigma m^2 \phi^2)]^{-1} \equiv a_0^2 \) and assume “no boundary” condition, integration of (6.17) should produce a real action, so that we expect \( \Psi = e^{-I} \) to be valid in the semiclassical approximation. Then

\[
T = \left| \frac{m}{a_f} \int_0^{a_f} da \frac{a^{D-1}}{[1 - a^2 \hat{P}_1^{-1}(\sigma m^2 \phi^2)]^{1/2}} \right|. 
\] 

(6.20)

As \( (a/a_f)^{D-2} \leq 1 \) if \( a \in [0, a_f] \), it follows that

\[
T \leq \frac{m}{a_f} \int_0^{a_f} da \frac{a}{[1 - a^2 \hat{P}_1^{-1}(\sigma m^2 \phi^2)]^{1/2}} = \frac{m}{a_f} \left( \frac{1}{[\hat{P}_1^{-1}(\sigma m^2 \phi^2)]^{1/2}} \right), 
\] 

(6.21)

or, since \( \sqrt{1-x} \geq 1-x \) for \( x \in [0,1] \),

\[
T \leq ma_f \leq \frac{m}{[\hat{P}_1^{-1}(\sigma m^2 \phi^2)]^{1/2}}, 
\]

which is much smaller than unity if (6.10) holds.

If \( a_f \) is slightly greater than \( [\hat{P}_1^{-1}(\sigma m^2 \phi^2)]^{-1/2} \), then the action is no longer real. In this case, a real function can be obtained by combining the two complex conjugate amplitudes in the semiclassical approximation:

\[
\Psi = e^{-I} \cos(S_1 + \theta),
\] 

(6.22)

where

\[
I_0 = \int_{0}^{a_0} da \frac{\partial I}{\partial a}, \quad S_1 = \int_{a_0}^{a_f} da \frac{\partial I}{\partial a}
\]

with \( a_0 \) defined as before and \( \theta \) being a constant phase. The continuity of the wave function implies that

\[
\rho_\phi = \frac{1}{\Psi(a_f, \phi)} \frac{\partial \Psi(a_f, \phi)}{\partial \phi} \approx \frac{1}{\Psi(a_0, \phi)} \frac{\partial \Psi(a_0, \phi)}{\partial \phi} = \frac{\partial I_0}{\partial \phi},
\]

i.e., the momentum as evaluated at \( a_0 \). Since \( a_f \approx a_0 \), it follows from (6.19) that

\[
T \leq \left| \frac{\partial I_0}{\partial \phi} \right| \leq \left| \frac{\partial I_0}{\partial \phi} \right| \leq \frac{2L_1 V_{D-1} a_0^{D-1} \sigma m \phi}{2L_1 V_{D-1} a_0^{D-1} \sigma m \phi}.
\]

the right-hand side of this expression is just the value of \( T \) at \( a_0 \) which, as we have seen, is much smaller than unity.

It is worth noticing that all the above arguments are valid as long as the semiclassical approximation is applicable. We have considered the case when the coefficients \( L_n \) are independent of \( \hbar \). If \( L_n \) would actually depend on \( \hbar \), we said that some terms included in the used semiclassical approximation could be disregarded; we would still expect their subdominance to be preserved in the asymptotic expansion of the action derivatives as \( \hbar \rightarrow 0 \), so that our conclusions would not be changed.

VII. CONCLUSIONS AND FURTHER COMMENTS

We have considered a \( D \)-dimensional (\( D \geq 3 \)) homogeneous and isotropic minisuperspace model with Lovelock gravitational action. The entire gravitational dynamics has been described by introducing an associated polynomial to the Lovelock theory; this polynomial plays a fundamental role in all our considerations.

Using a monovalue branch of an algebraic function we have inverted the Hamiltonian constraint, obtaining a unique classical solution. The inversion branch has been chosen by the physical requirement that when no matter is present one recovers Euclidean flat space as the classical solution. We have also shown that this condition fixes uniquely the inversion branch, and then the classical solution, at least whenever one is dealing with positive Lorentzian energy densities, provided that the associated Lovelock polynomial \( P(x) \) is a strictly increasing function for all \( x \geq 0 \). The classical solution can be interpreted in these cases as a perturbed Einstein solution. Furthermore, the functional relation between the momentum and the time derivative of the scale factor has been consistently inverted by using also a monovalue branch of an algebraic function.

Assuming the validity of this procedure, we have gone one step beyond, introducing the corresponding Wheeler-DeWitt equation. We have applied the Wheeler-DeWitt equation to a model containing a cosmological constant and to a massive scale field model. We have thus succeeded in generalizing the analogous four-dimensional models worked out so far.19,20 Moreover, it is worth mentioning that all the results obtained by this procedure are very similar to the results of conventional four-dimensional classical and quantum cosmology, the main difference being the substitution of
the energy density by an effective density, taking account of the Lovelock corrections.

However, for sufficiently negative energy densities our procedure loses its physical meaning and is no longer treatable. This is perhaps the kind of problem that one could expect when dealing with a model with negative energy density. After all, in conventional four-dimensional anti-de Sitter spaces, we also have to confront serious conceptual and technical problems which have not yet been solved. Of course, one should be especially careful with some aspects of the quantum version of the multidimensional model, particularly with our formal quantization procedure itself and also with the factor ordering problem and the semiclassical approximation if the Lovelock coefficients depend on \( \hbar \). Furthermore, as our classical formalism is not properly defined for sufficiently negative energy densities, which give rise to large Euclidean time derivatives of the scale factor, one should expect some kind of boundedness in the quantum operators associated to these variables. All these problems should be studied in more detail in order to check the consistency of our procedure.

Before closing, we want to briefly comment on the possibility of defining quantum amplitudes in Lovelock gravity by means of a convergent path integral. In four dimensions, this is obtained by integrating the gravitational variables over suitable complex contours, so that the gravitational Euclidean action turns out to be positive definite. For higher-dimensional Lovelock actions, there is no similar prescription. Actually, the consideration of particular cases suggests that a direct generalization from the four-dimensional prescription does not exist. In fact, in trying to rotate any gravitational variable in the complex plane to reach the desired contour, the monovalued inversion of the relation between momenta and velocities generally breaks down, leaving the system ill defined. From the perturbation point of view, a Lagrangian with corrective terms would only keep its meaning in a certain complex region of the given gravitational variables.

Our procedure is restricted to a particular minisuperspace. It appears to be of interest to generalize it to other types of minisuperspaces. This would allow us to use other topologies in which one could eventually uncover physical phenomena which are not present in Einstein four-dimensional gravity, so as to test the validity of the proposed procedure.

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APPENDIX A: SURFACE TERMS

The Lovelock Lagrangian is formed up by the dimensionally extended Euler indices. For manifolds with boundaries, the simple extension of the Euler numbers includes boundary corrections given by the surface integral of the Chern-Simons form.16,22 The Euler density of order \( m \) has the expression

\[
L_m = P_m(R_1, \ldots, R_m),
\]

where \( P_m \) is an invariant polynomial122 and \( R \) is the curvature two-form for the connection \( \omega \) of manifold \( \mathcal{M} \).14 If the manifold \( \mathcal{M} \) has a boundary \( \partial \mathcal{M} \), one can choose local Gaussian coordinates \( \{y, x^\mu\} \), so that \( y = 0 \) is the local equation of \( \partial \mathcal{M} \), and

\[
ds^2 = -dy^2 + g_{\mu\nu}(x^\delta y)dx^\nu dx^\mu.
\]

One can choose a product metric near the boundary which coincides with (A2) at \( \partial \mathcal{M} \):

\[
ds^2 = -dy^2 + g_{\mu\nu}(x^b, 0)dx^\nu dx^\mu.
\]

The connection \( \omega_0 \) for this metric has only tangential components.16

Let \( \Theta = \omega - \omega_0 \) be the second fundamental form, related to the extrinsic curvature tensor \( K^a_b \) by

\[
\Theta^{ab} = (N^i N_j)K^a_c K^b_c e^c,
\]

where \( N \) is the unit normal to \( \partial \mathcal{M} \).

We can then introduce the connection

\[
\omega_s = \omega - s \Theta,
\]

and the corresponding curvature

\[
R_s = d\omega_s + \omega_s \omega_s.
\]

The boundary correction to the Euler number is then

\[
- \int_{\partial \mathcal{M}} Q_m, \text{ where the so-called Chern-Simons form } Q_m \text{ is}
\]

\[
Q_m = \int_0^1 ds m P_m(\Theta_s(R_s)_1, \ldots, (R_s)_m-1).
\]

For manifolds with dimension larger than \( 2m \), \( Q_m \) can be dimensionally extended.

The surface terms for our Lovelock Lagrangian are given by

\[
surface \ terms = -\int_a \sum_{m=1}^M \frac{L_m m}{(D-1)! (D-2m)!} \int_0^1 ds \epsilon_{a_1} \cdots a_D \Theta^a_{a_1} \cdots a_D R_{a_2}^a a_3 \cdots R_{a_4}^a a_5 \cdots R_{a_{2m+1}}^a a_{2m+2} \cdots e^{a_{2m+1}} \cdots e^{a_D}.
\]

Denoting the spatial indices by \( i \) and the temporal index by 1, we have, for our minisuperspace model,

\[
N^a = \delta^a_i, \quad K_i = \frac{\dot{a}}{a N} \delta_i^a, \quad \Theta^i = \frac{\dot{a}}{a N} e^i, \quad \Theta^i = 0 = \Theta^{11}.
\]
Hence

\[ \text{surface terms} = - \int_{\partial \mathcal{R}} d\Omega_{D-1} \sum_{m=1}^{M} \frac{L_m 2m}{D-1!!(D-2m)} a^{D-1} d\sigma_{D-1} \left( \frac{a}{N} \right)^{m-1} \left[ 1 + \frac{a^2}{N^2} (s-1)^2 \right]^{m-1} \]  

(A10)

Using (A9) and \( R_s = R - s \Theta - s \Theta \omega + s^2 \Theta \Theta \),

\[ R_i^{(n-i)} = R^{(n-i)} - s \Theta^{(n-i)} - s \Theta^{(n-i)} + s^2 \Theta^{(n-i)} \Theta^{(i)} \]  

(A11)

We have, moreover,

\[ R_i^{(n-i)} = \frac{1}{a^2} \left( 1 + \frac{a^2}{N^2} \right) e^{s-i} e^{s} \]

and, since \( \omega_0 \) has no normal components, \( \omega = \Theta^{(i)} \). Whence

\[ R_i^{(n-i)} = \frac{1}{a^2} \left( 1 + \frac{a^2}{N^2} (s-1)^2 \right) \]  

(A12)

Inserting (A12) into (A10), we obtain

\[ \text{surface terms} = - \int_{\partial \mathcal{R}} d\Omega_{D-1} \sum_{m=1}^{M} \frac{L_m 2m}{D-1!!(D-2m)} a^{D-1} d\sigma_{D-1} \left( \frac{a}{N} \right)^{m-1} \left[ 1 + \frac{a^2}{N^2} (s-1)^2 \right]^{m-1} \]  

(A13)

expanding the binomial by Newton’s formula and integrating in \( s \), (A13) becomes finally

\[ \text{surface terms} = - \int_{\partial \mathcal{R}} d\Omega_{D-1} \sum_{m=1}^{M} \frac{L_m 2m}{D-2m} a^{D-2m} \sum_{n=0}^{m-1} \left( \frac{m-1}{n} \right) \left( \frac{a}{N} \right)^{2n+1} \frac{1}{2n+1} \]  

(A14)

**APPENDIX B: ALGEBRAIC FUNCTIONS: THE INVERSION OF THE HAMILTONIAN CONSTRAINT**

In this appendix we first briefly review some relevant notions of complex analysis and algebraic functions, and develop those of them which are to be subsequently used in the inversion of the Hamiltonian constraint. This is done first for an energy density independent of the scale factor, generalizing then to some cases where it does depend on \( a^2 \).

A pair \((f, \Omega)\), where \( f \) is a function which is analytic in the complex region \( \Omega \), constitutes a functional element; a global analytic function is a collection of functional elements which are analytic continuations of each other.

Given a functional element \((f, \Omega)\) and a point \( \xi \in \Omega \) we can form a pair \((f, \xi)\). Two pairs \((f_1, \xi_1)\) and \((f_2, \xi_2)\) are considered equivalent whenever \( \xi_1 = \xi_2 \) and \( f_1 = f_2 \) in a neighborhood of \( \xi_1 \). This defines an equivalence relation whose equivalence classes are called germs of the analytic functions.

If \((f, \xi)\) is a germ that can be analytically continued along any arc in a certain region \( \Omega \), its continuation along two homotopic arcs in \( \Omega \) provide the same germ at the final point of these arcs. Thus, given an initial germ of a global analytic function which can be continued along any arc in a simply connected region \( \Omega_0 \), we get by analytic continuation a monovalued analytic function in this region. If \( f \) is a global analytic function in such a region \( \Omega_0 \) which admits \( N \) different germs at a given point \( \xi \in \Omega_0 \), the analytic continuation of these germs provides us then with \( N \) different monovalued analytic functions in \( \Omega_0 \) each being called a branch of \( f \) in \( \Omega_0 \).

Consider now a polynomial

\[ P(w, z) = a_0(z)w^n + a_1(z)w^{n-1} + \cdots + a_n(z), \quad n \geq 1, \]  

(B1)

which is irreducible in both \( w \) and \( z \) [i.e., \( P(w, z) \) cannot be factorized as a product of two nonconstant polynomials]. The relation \( P(w, z) = 0 \) defines a global analytic function \( w(z) \). Functions defined in this way are called algebraic functions.

If \( a_0(z) \neq 0 \) and \( P(w, z) \) has no multiple roots in \( w, z \) is a regular point of the algebraic function \( w(z) \); then, there exist an open disk \( \Delta \) around \( z_0 \) and \( n \) functional elements \( (w_1, \Delta), \ldots, (w_n, \Delta) \) having the following properties: (i) \( P(w_i(z), z) = 0 \) in \( \Delta \), (ii) \( w_i(z_0) = w_0^i \), with \( w_i^0 \) \((i = 1, \ldots, n)\) being the \( n \) roots of \( P(w, z_0) \), (iii) if \( P(w, z) = 0 \) for \( z \in \Delta \), then \( w = w_i(z) \) for some \( i \).

When \( z_0 \) is a zero of \( a_0(z) \) with multiplicity \( m \), by dividing \( P(w, z) \) by \((z - z_0)^m \) and taking the limit as \( z \to z_0 \), it can be shown that there exists the limit

\[ \exists \lim_{z \to z_0} w_i(z) \sim (z - z_0)^m, \]  

(B2)

so that \( w(z) \) has at most an algebraic pole at \( z_0 \).

A similar argument can be used to show that, if (1) \( a_i(z) \) has a degree \( r_i \) in \( z, i = 0, \ldots, n \), and (2) we define \( q = \max \{1/k(r_i - r_0), 1 \leq k \leq n\} \),

\[ \exists \lim_{z \to \infty} w_i(z) \sim q \]  

(B3)

as \( z \) tends to infinity. Then, \( w(z) \) has at most an algebraic pole
pole at infinity.
Likewise, if \( z_0 \) is not a zero of \( a_0(z) \), but \( P(w,z_0) \) has multiple roots in \( w \), then
\[
\exists \lim_{z \to z_0} w_j(z) = w_j(z_0),
\]
which can be checked if one divides \( P(w,z) \) by \( w_j(z) \) and takes the limit \( z \to z_0 \). Therefore, \( z_0 \) is at most an ordinary algebraic singularity of \( w(z) \).

On the other hand, if \( w_j \) is such that \( P(w_j,z_0) = 0 \) and \( \frac{\partial w_j}{\partial w} P(w_j,z_0) \neq 0 \), then \( w_j \) is a simple root of \( P(w,z_0) \) and there exists a unique branch \( w(z) \) such that \( \lim_{z \to z_0} w(z) = w_j(z_0) \). Since \( \frac{\partial w_j}{\partial w} P(w_j,z_0) \neq 0 \), it follows, from the implicit function theorem, that the branch \( w(z) \) is analytically continued to \( z_0 \).

Let us now apply the above concepts to the inversion of the Hamiltonian constraint (3.2). In the notation given by (4.1)–(4.5), (3.2) can be written as (4.6), i.e.,
\[
\hat{P}(w,z) = \hat{P}(w) - z = 0.
\]

We first discuss the case \( z \neq z_0(a^2) \); \( w \) and \( z \) can therefore be treated as independent variables. (B5) defines then an algebraic function of \( M \) branches \( \hat{P}^{-1}(z) \). As for that particular case, \( a_0(z) = L_M^0 \neq 0 \), all the possible finite singular points correspond to ordinary algebraic singularities. At these points, which we denote \( z_j \), the polynomial \( \hat{P}(w,z_j) \) has at least one multiple root \( w_j \), so that \( \frac{\partial w}{\partial w}(\hat{P}(w,z_j))w_j = \hat{Q}(w_j) = 0 \). Then, all the algebraic singularities are determined by \( z_j = \hat{P}(w_j) \) with
\[
w_j \in \{ w \in \mathbb{C} \mid \hat{Q}(w) = 0 \}.
\]

Among all the \( M \) branches of \( \hat{P}^{-1}(z) \), we can choose a unique branch \( \hat{P}^{-1}(z) \) by condition (4.7): \( \hat{P}^{-1}(z_0) = 0 \). This is made possible by the fact that \( \hat{P}(w,0) \) has a simple zero at \( w = 0 \). We will show that the corresponding germ \( \hat{P}^{-1}(z_0,0) \) can be analytically continued along the entire positive real axis \( \mathbb{R}^+ \) if and only if the polynomial \( \hat{P}(w) \) is a strictly increasing function for all \( w \geq 0 \).

Let \( I = \mathbb{R}^+ \) be the maximum open interval where the germ \( \hat{P}^{-1}(z_0,0) \) can be analytically continued along \( \mathbb{R}^+ \). Since \( \hat{P}(w) \) has real coefficients, it follows from the functional relation (B5) and condition (4.7) that \( \hat{P}^{-1} \) defines a real function in \( I \). Furthermore, as \( \hat{P}^{-1}(z_0) = \{ \hat{Q}[\hat{P}^{-1}(z_0)] \}^{-1} \) because of the inverse function theorem, \( \hat{P}^{-1} \) cannot change sign in \( I \); otherwise \( \hat{Q} \) should be singular at a finite point. The sign of \( \hat{Q} \) is also preserved in \( \hat{P}^{-1}(I) \) as the derivative of \( \hat{P}^{-1} \) is finite in \( I \). Then, as \( \hat{Q}[\hat{P}^{-1}(0)] = 0 \), \( \hat{P}^{-1}(z) \) and \( \hat{P}(w) \) have to be strictly increasing functions in \( I \) and \( \hat{P}^{-1}(I) \), respectively.

Now, if \( \hat{P}^{-1}(z) \) is analytically continuables along the entire positive axis, we conclude from (B5) that \( \hat{P}^{-1}(\mathbb{R}^+) \) runs on the entire positive axis and that \( \hat{P}(w) \) has to be a strictly increasing function in \( \mathbb{R}^+ \).

On the other hand, if \( \hat{P}(w) \) is a strictly increasing function in \( \mathbb{R}^+ \), let us assume that \( \hat{P}^{-1}(z_0) \) can be analytically continued along \( \mathbb{R}^+ \) only up to some \( z_0 < \infty \). As the only possible singularities are algebraic, there always exists the limiting value \( \hat{P}^{-1}(z_0) \), which is positive because \( \hat{P}^{-1}(z_0) \) is strictly increasing. Therefore, \( \frac{\partial w_j}{\partial w}(\hat{P}^{-1}(z_0),0) = \hat{Q}(\hat{P}^{-1}(z_0)) > 0 \) and, by the implicit function theorem, \( \hat{P}^{-1}(z) \) can be analytically continued to \( z_0 \), contrary to our assumption. This closes up our demonstration.

For \( \hat{P}(w) \) a strictly increasing function for all \( w \geq 0 \), let us define
\[
w_1 = \max \{ w \in \mathbb{R} \mid \hat{Q}(w) = 0 \}.
\]
Then, similar arguments can be used to show that \( \hat{P}^{-1}(z_0,0) \) is analytically continuables along \( \mathbb{R}^+ \) up to the point \( z_1 = \hat{P}(w_1) \).

We now generalize our results to the case where \( T_{11} \) depends on \( a^2 \) as a polynomial in \( 1/a^2 \):
\[
\frac{T_{11}(1/a^2)}{L_1 a^2} = a_M(1/a^2),
\]
with \( a_M \) being some polynomial. If we denote \( \omega = 1 + \hat{a}^2/N^2 \), \( x = 1/a^2 \), (B5) can be rewritten in the form
\[
\hat{P}(\omega, x) = \hat{P}(\omega x) - a_M(x) = 0.
\]

This functional relation defines an algebraic function of \( M \) branches, \( \omega(z) \), with singular points: \( x = 0 \), the only point at which \( a_M(x) = L_M^0 x^M \) vanishes, \( x = \infty \), and some algebraic points \( x_i \).

These \( x_i \) are the points at which \( a_M(x_i) = z_j \), where \( z_j \) are the ordinary algebraic singularities of the above algebraic function \( \hat{P}^{-1}(z) \). To see that, we note that, for \( x_i \) to be an ordinary algebraic singularity, \( \hat{P}(\omega, x_i) \) must have at least one multiple root \( \omega_j \), so that \( \frac{\partial \hat{P}(\omega, x_i)}{\partial \omega} = x_i \hat{Q}(x_i, 0) = 0 \). As \( x_i \neq 0 \), \( \hat{Q}(x_i, \omega) \) has to vanish and \( \omega_j \) must be equal to some \( w_j \in \{ w \in \mathbb{C} \mid \hat{Q}(w) = 0 \} \). We obtain then \( a_M(x_i) = \hat{P}(x_i, w_j) = z_j \).

The \( M \) branches of \( \omega(z) \) are given in terms of the \( M \) branches of \( \hat{P}^{-1}(z) \):
\[
\omega_i(x) = \frac{1}{x} \hat{P}^{-1}[a_M(x)], \quad i = 1, \ldots, M.
\]

These \( M \) branches are analytic at the regular points, because \( a_M(x) \) is analytic in \( x \), \( \hat{P}^{-1}[a_M(x)] \) provides \( M \) different analytic solutions whenever \( a_M(x) \neq z_j \) for all \( j \), and \( x^{-1} \) is analytic except for \( x = 0 \).

In almost all interesting cases, \( a_M(x) \geq 0 \) for positive \( x \); particularizing our discussion to polynomials \( \hat{P}(w) \) which are strictly increasing in \( \mathbb{R}^+ \cup \{0\} \), \( \omega(x) \) is then analytic, clearly, in the entire positive real axis.

Using the inversion branch \( \omega_1 \) as a function of \( 1/a^2 \),
\[
1 + \frac{a^2}{N^2} = \omega_1(a^2) = \omega_1(1/a^2)
\]
\[
= a^2 \hat{P}^{-1} \left( \frac{T_{11}(1/a^2)}{L_1 N^2} \right).
\]

Thus, when \( a_M(x) \geq 0 \) for \( x > 0 \) and \( \hat{Q}(w) > 0 \) for \( w \geq 0 \), \( \omega_1(a^2) \) is an analytic function of \( a^2 \) in the entire positive real axis.
Furthermore, it follows from (B10) that there exists the limit
\[
\lim_{x \to 0} x \omega_l(x) = \hat{P}^{-1}_l[a_m(0)].
\] (B12)
If \(a_M(0) = 0\), then \(a_M(x) = x \hat{a}_M(x)\), with \(\hat{a}_M(x)\) a certain polynomial. The irreducible functional relation associated to (B9) is now
\[
\sum_{n=1}^{M} \hat{a}_n x^{n-1} \omega^n - \hat{a}_M(x) = 0,
\] (B13)
which is equivalent to (B9) except for \(x = 0\). (B13) defines the algebraic function of \(M\) branches \(\omega_l(x) = x^{-1} \hat{P}^{-1}_l(x \hat{a}_M(x))\). Taking the limit \(x \to 0\) in (B13) it is easy to check that there exists a branch verifying \(\lim_{x \to 0} x \omega_l(x) = \hat{a}_M(x)\). This branch is precisely \(\omega_l(x)\), because \(\lim_{x \to 0} x \omega_l(x)\) vanishes only for \(\omega_l(x)\), as \(\hat{P}^{-1}_l(0) = 0\) fixes this branch uniquely. Therefore, in this case,
\[
\lim_{x \to 0} \omega_l(x) = \hat{a}_m(0).
\]
Finally, we discuss the behavior of \(\omega_l(x)\) as \(x \to \infty\). Using our previous notation for the degrees of polynomials \(a_l(x)\) appearing in (B9), we have \(r_k = M - k\) for \(0 \leq k < M\) and \(r_M\) for the degree of \(a_M(x)\), from which we conclude
\[
\exists \lim_{x \to \infty} \omega_l(x) x^{M - r_M}.
\] (B14)
if \(r_M > M\), \(x = \infty\) is an algebraic pole of \(\omega_l(x)\); if \(r_M \leq M\), there exists the limit \(\lim_{x \to \infty} \omega_l(x)\), and \(x = \infty\) is at most an ordinary algebraic point.

**APPENDIX C: INVERSION OF THE MOMENTUM-VELOCITY RELATION**

In this appendix we apply the concepts introduced in Appendix B to invert the functional relation (3.3) between the momentum associated to the scale factor and the time derivative of the latter:
\[
R\left[\frac{a}{N^2}, \frac{\dot{a}}{N^2}, p^2\right] \equiv \dot{R}\left[\frac{a}{N^2}, \frac{\dot{a}}{N^2}\right] - p^2 = 0.
\] (C1)
We consider the particular case in which the polynomial \(\hat{P}(w)\) is strictly increasing for all \(w \geq 0\). We will show that then there exists an inversion branch of (C1) such that, for fixed positive \(a^2\), its range always contains the interval \((a^2 w_1 - 1, \infty)\), with \(w_1\) given by (4.8).

We take for the moment a fixed \(a^2 > 0\). (C1) defines then an algebraic function of \(2M - 1\) branches \(\tilde{R}^{-1}(a^2, p^2)\).

We choose the branch \(\tilde{R}^{-1}(a^2, p^2)\) by imposing the condition \(\tilde{R}^{-1}(a^2, 0) = 0\). Using
\[
\partial_{\frac{a}{N^2}} R\left[\frac{a}{N^2}, \frac{\dot{a}}{N^2}, p^2\right] = 4L^2 V^2_{\hat{D}} - a^{2D - 6} \hat{Q}\left[1 + \frac{\dot{a}}{a^2} \frac{\dot{a}}{N^2}\right]
\times \int_0^1 \frac{dy}{1 + \frac{\dot{a}}{a^2} \frac{\dot{a}}{N^2}} \hat{Q}\left[1 + \frac{\dot{a}}{a^2} \frac{\dot{a}}{N^2}\right],
\] (C2)
we easily check that \(\tilde{R}^{-1}(a^2, p^2)\) is then uniquely determined; for, as
\[
\partial_{\frac{a}{N^2}} R\left[\frac{a}{N^2}, \frac{\dot{a}}{N^2}, p^2\right] \bigg|_{\frac{\dot{a}}{N^2} = 0} = 4L^2 V^2_{\hat{D}} - a^{2D - 6} \hat{Q}\left[1 + \frac{\dot{a}}{a^2}\right]^2 > 0,
\] (C3)
\(\frac{\dot{a}}{N^2} = 0\) is a simple root of \(R(a^2, \frac{\dot{a}}{N^2} w_1, 0)\) for \(a^2 > 0\).

Moreover, since \(\hat{Q}(w) > 0\) when \(w > w_1\), for \(a^2 > 0\) and \(\frac{\dot{a}}{N^2} \in (a^2 w_1 - 1, \infty)\), \(\partial_{\dot{a}} R, \frac{\dot{a}}{N^2} \in (a^2 w_1 - 1, \infty)\).

Then, \(R(a^2, \frac{\dot{a}}{N^2} w_1, 0)\), when given as a real function of \(\frac{\dot{a}}{N^2} w_1, 0)\), is a strictly increasing function in \((a^2 w_1 - 1, \infty)\). Furthermore, the chosen branch, \(\tilde{R}^{-1}(a^2, p^2)\), takes on real values when it is analytically continued along the real axis of \(p^2\). Using a similar argument to that for the inversion of the Hamiltonian constraint in Appendix B, we conclude that the germ \(\tilde{R}^{-1}(a^2, p^2)\) can be analytically continued along the real axis of \(p^2\) to the entire interval:
\[
I_{p^2 > 0} \equiv (R(a^2, a^2 w_1, 1) = 0, x) .
\] (C4)

Note that the range of \(\tilde{R}^{-1}(a^2, p^2)\) in \(I_{p^2 > 0}\) is precisely \((a^2 w_1 - 1, \infty)\).

Let us allow \(a^2\) to vary now in \(R^+\). As \(R(a^2, \frac{\dot{a}}{N^2} w_1, 0)\) is also a polynomial in \(a^2\), the functional relation (C1) defines an algebraic function of \(2M - 1\) branches \(\tilde{R}^{-1}(a^2, p^2)\), which depends on two complex variables. Choosing \(a_0^2 > 0\), let us then form the germ \((\tilde{R}^{-1}(a^2, p^2), (a_0^2, p^2) = 0)\) which, for the same reason as before, is uniquely defined by the condition \(\tilde{R}^{-1}(a_0^2, 0) = 0\). This germ can be analytically continued to all the points in the interval:
\[
I_{a^2 > 0} \equiv \left\{ a^2 \in R^+, p^2 \in I_{p^2 > 0} \left(\tilde{R}^{-1}(\tilde{a}^2, 0)\right) \right\} \subset R^2.
\] (C5)
\(I_{a^2 > 0}\) is simply connected in \(C^2\) and the analytic continuation does not depend on the path followed in \(I_{a^2 > 0}\). Thus, going from \((a_0^2, 0)\) to a point \((a_f^2, p_f^2)\) \(\in I_{a^2 > 0}\), let us choose the path
\[
(a_0^2, 0) \longrightarrow (a_f^2, 0) \longrightarrow (a_f^2, p_f^2).
\]
Along \(C_1\), analytic continuation gives \(\tilde{R}^{-1}(a^2, 0) = 0\) for all \(a^2 \in (a_0^2, a_f^2)\); this continuation can always be done because
\[
\partial_{\dot{a}} R\left[\frac{a}{N^2}, \frac{\dot{a}}{N^2}, p^2\right] \bigg|_{\dot{a} = 0, p^2 = 0} = 4L^2 V^2_{\hat{D}} - a^{2D - 6} \hat{Q}\left[1 + \frac{\dot{a}}{a^2}\right]^2 > 0
\] for \(a^2 > 0\). The analytical continuation along \(C_f\) is a particular case of the continuation for fixed \(a_f^2 > 0\) and, therefore, it is always possible for \(p_f^2 > R(a_f^2, \frac{\dot{a}}{N^2} w_1 - 1)\).

The so-built inversion branch is an analytic function of its two variables in \(I_{a^2 > 0}\). Keeping \(a_f^2\) fixed, \(a^2 = a_f^2 \in R^+\), we recover the result that the range of \(\tilde{R}^{-1}(a^2, p^2)\) in \(I_{p^2 > 0}\) equals the interval \((a_f^2 w_1 - 1, \infty)\).
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