Lorentz violations and Euclidean signature metrics

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We show that the families of effective actions considered by Jacobson et al. to study Lorentz invariance violations contain a class of models that represent pure general relativity with a Euclidean signature. We also point out that some members of this family of actions preserve Lorentz invariance in a generalized sense.

As we can see the fact that $u^a$ is a unit vector implies that the determinants of $g^E_{ab}$ and $g_{ab}$ are proportional to each other with a constant of proportionality that can be made either positive or negative by choosing appropriate values of $\alpha$ and $\beta$. Let us write now the Einstein-Hilbert action for $g^E_{ab}$ as a function of $g_{ab}$ and $u^a$. To this end we need the inverse metric $g^{Eab}$ and the Christoffel symbols for $g^E_{ab}$ (here $g_{ab}$ satisfies $g_{ab}g^{bc} = \delta_a^c$):

$$g^E_{ab} = \frac{2}{\sqrt{\alpha(\alpha + 2\beta)}} \left[ g_{ab} - \frac{\alpha + \beta}{\alpha + 2\beta} \frac{u^a u^b}{2} \right].$$ (4)

$$\Gamma^E_{bc} = \frac{\alpha + \beta}{\alpha + 2\beta} \left( u^c u_b - u^b u_c \right) - \frac{2(\alpha + \beta)}{\alpha(\alpha + 2\beta)} \left( u^a u^c u^d u_b - u^a u^d u_b u^c \right).$$ (5)

A tedious but straightforward computation now gives

$$S_E = \int d^4x \sqrt{|g_E|} g^E_{ab} R^E_{ab}$$

$$= \text{sgn}(\alpha) \int d^4x \sqrt{|g|} \left[ -\frac{\alpha}{2} R + \frac{(\alpha + \beta)}{\alpha + 2\beta} u^a u^b R_{ab} - \frac{(\alpha + \beta)^2}{\alpha(\alpha + 2\beta)} g_{ab} \omega_a \omega_b \right],$$ (6)

where $R^E_{ab}$ is the Ricci tensor built with $g^E_{ab}$, $R_{ab}$, and $R$ with $g_{ab}$, and $\omega_a$ is the twist of $u^a$ given by

$$\omega_a = \epsilon_{a2a3} u^2 (\nabla^a u^3).$$ (7)

It is useful to notice that

$$\omega_a \omega^a = (\nabla^a u_b) (\nabla^b u_a) - (\nabla^a u_b) (\nabla^b u_a) - u^a u^b$$

$$= \frac{1}{2} F^a_{ab} F^{ab} - u^a \dot{u}_a.$$ (8)

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As in Eq. (1), the condition that $u^a$ is a unit vector can be explicitly incorporated into the action by adding a suitable Lagrange multiplier term to Eq. (6). Another way to do that is to write $u^a = \eta^a/(g_{bc} \eta^b \eta^c)^{1/2}$, with an unconstrained, timelike, vector field $\eta^a$, in which case the action becomes invariant under the gauge transformations consisting in local rescalings of the vector field. We can readily see that Eq. (7) is a particular case of the action (1) considered in [1] with the parameter choices $a_0 = 0$, $a_1 = |a|2$, $a_2 = -\text{sgn} (\alpha) (\alpha + \beta)$, $b_1 = (\text{sgn} \alpha)(\alpha + \beta)^2/2(\alpha + 2\beta)$, $b_2 = 0$, and $b_3 = -2b_1$.

Several comments are now in order.

(i) Some of the parameter choices do not change the signature of the metric. If both $g_{ab}^E$ and $g_{ab}$ have Lorentzian signatures, the action (6) is strictly equivalent to the Einstein-Hilbert action for $g_{ab}$. It is important to realize that Eq. (6) has a gauge symmetry that is related to the fact that the variations in the vector field can always be compensated inside $g_{ab}^E$ by a suitable variation of the metric $g_{ab}$. This also means that the field equations coming from variations in the vector field are always redundant. We see then that there is a one to one correspondence between the solutions to the field equations for the Einstein-Hilbert action (Lorentzian or Euclidean) and gauge equivalence classes of solutions to the field equations derived from Eq. (6). In our opinion, it would not be justified to talk about Lorentz violating effects when $g_{ab}$ is Lorentzian.

(ii) The fact that $u^a$ is dynamical or not is irrelevant in our scheme. If $u^a$ is a fixed geometric structure, general covariance is broken but, as long as matter couples to $g_{ab}$, the physical content of the model corresponds to general relativity in the sense that there is a one to one correspondence between the solutions of the two theories and their symmetries.

(iii) Because of the presence of two different metrics $g_{ab}^E$ and $g_{ab}$, one can consider matter couplings to either of them. If matter is coupled to $g_{ab}^E$ and the parameters of the model are chosen in such a way that $g_{ab}^E$ is Lorentzian, we still have Lorentzian general relativity without breaking any Lorentz invariance in the sense discussed above. If, on the other hand, we choose the parameters to get a Euclidean signature we end up with Euclidean general relativity with matter. Finally, if matter is coupled to $g_{ab}$ we have the Lorentz violating effects described in [1–3].

(iv) The field equations obtained by varying in $u^a$ are redundant. This can be explicitly checked by varying our action (6) with respect to $u^a$ and checking that the equations thus obtained are satisfied as a consequence of the equations derived by varying with respect to $g_{ab}$. This can also be seen by noticing that a variation in $g_{ab}^E$ of the type generated by changing $u^a$ can also be obtained by a suitable variation of $g_{ab}$, as discussed above.

(v) If $u^a$ is hypersurface orthogonal the twist is not present, and we get the formulation presented in [7] in the context of real Wick rotations.

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