Variational approach for walking solitons in birefringent fibers

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ABSTRACT

We use the variational method to obtain approximate analytical expressions for the stationary pulselike solutions in birefringent fibers when differences in both phase velocities and group velocities between the two components and rapidly oscillating terms are taken into account. After checking the validity of the approximation we study how the soliton pulse shape depends on its velocity and nonlinear propagation constant. By numerically solving the propagation equation we have found that most of these stationary solutions are stable.

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1 Introduction

Pulse propagation in nonlinear birefringent optical fibers is presently an active area of research [1-12]. Even “single-mode” fibers support two degenerate modes polarized in two orthogonal directions. The degeneracy can be broken introducing deliberately a large amount of birefringence through design modifications giving place to the so called polarization-preserving fibers. Weak light pulses propagating in these media maintain its state of polarization when they are initially polarized along either of the principal axes. However at high intensities nonlinear birefringence can change drastically the propagation dynamics. Birefringence can also be a residual effect due to imperfections in the manufacturing process.

The equations that govern pulse propagation in nonlinear birefringent pulses have been derived by Menyuk [13]. Two main cases of birefringence: i) high and ii) low, have been usually considered separately. In the approximation of high birefringence [13, 14], phase and group velocities of both field components are considered to be different. At the same time rapidly oscillating nonlinear terms are neglected. On the other hand, the approximation of low birefringence takes into account the difference in phase velocities between the two linearly polarized components, but neglects their difference in group velocities. The full polarization dynamics of solitons in this last approximation has been studied in [11].

The general case has been analyzed in just a few papers [1, 8, 15]. New vectorial stationary solitons for this case were first discovered in [6], and later on the whole family of stationary pulselike solutions was numerically determined by Torner et al. [1]. These
families of solitons correspond to mutually trapped pulses that propagate with a common group velocity and which therefore can be called walking vector solitons [16]. Stationary solutions as singular points of this infinite dimensional dynamical system play a pivotal role to understand the propagation dynamics of any arbitrary input pulse. Here we generalize the method described in [14] to find accurate analytical approximations of the solitary wave solutions for the case of high birefringence. We generalize it for the more general case, when no terms are neglected. Our approximate results are compared with existing exact ones [1, 6] to prove the validity of our approach. We then examine the behavior of the two pulse components when the propagation constant \((q)\) or the velocity \((v)\) change. We find that the width of the pulses is determined by \(q\) whereas the amplitudes ratio \(a_1/a_2\) is determined by \(v\) when the energy is higher than a certain value. By means of numerical propagation of the stationary solutions found in this way, we observe that most of them are stable.

The rest of the paper is organized as follows. The problem to be addressed by the variational method is described in section 2. Our variational approach is developed in section 3. In section 4 we present the numerical results and finally in section 5 we briefly summarize the main conclusions.

2 Statement of the problem

Pulse propagation in a birefringent optical fiber can be described in terms of two non-linearly coupled nonlinear Schrödinger equations. In a reference frame traveling along the \(\xi\) axis with the average group velocity, and in normalized units, this set takes the
form [13, 15]

\[-iU_\xi = i\delta U_\tau + \frac{1}{2} U_{\tau\tau} + \left(|U|^2 + A|V|^2\right) U + \beta U + BV^2 U^*,\]

\[-iV_\xi = -i\delta V_\tau + \frac{1}{2} V_{\tau\tau} + \left(|V|^2 + A|U|^2\right) V - \beta V + BU^2 V^*, \tag{1}\]

where $U$ and $V$ are the slowly varying envelopes of the two linearly components of the field along the $x$ and $y$ axis respectively, $\xi$ is the normalized propagation coordinate, $\delta$ is the inverse group velocity difference, $\beta$ is half the difference between the propagation constants, $A$ is the normalized ratio of the nonlinear susceptibilities, $B = 1 - A$, $\tau$ is the normalized retarded time, and the asterisk denotes complex conjugation. In linearly birefringent fibers $A = 2/3$, as we set here. The set of Eqs. [1] has at least three integrals of motion [6], i) the action (total energy), ii) the momentum and iii) the Hamiltonian, which are a consequence of i) the translational invariance of Eqs. (1) relative to phase shifts, ii) invariance in $\tau$ and iii) translational invariance in $\xi$.

Eqs. [1] have two simple linearly polarized pulselike solutions, viz., linearly polarized soliton waves along the slow axis

\[U = \frac{\sqrt{2(q - \beta)}}{\cosh \left[\sqrt{2(q - \beta)}(\tau - \delta \xi)\right]} \exp (i q \xi), \quad V = 0 \tag{2}\]

and linearly polarized soliton waves along the fast axis

\[U = 0, \quad V = \frac{\sqrt{2(q + \beta)}}{\cosh \left[\sqrt{2(q + \beta)}(\tau + \delta \xi)\right]} \exp (i q \xi). \tag{3}\]

Their corresponding values for the total energy of the pulse are:

\[Q = \int_{-\infty}^{\infty}(|U|^2 + |V|^2) d\tau = 2\sqrt{2(q \pm \beta)} \tag{4}\]
The stability of these solutions has been determined in Ref. [11] for $\delta = 0$, and for arbitrary nonzero values of $\delta$ in Ref. [8].

Usually Eqs. (1) are written without the last two terms [13], as terms with coefficient $\beta$ can be eliminated by $z$ dependent phase transformations of $U$ and $V$, and this in turn gives the last nonlinear terms rapid phase variations with $z$ which in principle allow their neglect. We must however bear in mind that the last terms in Eq. (1) are the only ones responsible for energy transfer between both polarizations. And although averaging out the fast oscillatory terms has proven to be a good approximation to describe most of the observed phenomena [8, 19, 17] in the picosecond regime, we will go further in retaining these terms in the analysis.

3 Variational approach

The variational approach when applied to the single Nonlinear Schrödinger Equation (NLS) was introduced by Anderson [18]. Since then it has been widely used for coupled NLS equations. For the specific case of birefringent fibers in the high birefringence approximation it has been used for studying dynamical behaviors [4, 8, 19, 20, 21], as well as the stationary case [14].

We look for stationary solutions moving at a common velocity $v$ in our frame of reference. They can be written [4] as

$$U(\tau', \xi) = P_1(\tau') \exp (i q \xi)$$

$$V(\tau', \xi) = P_2(\tau') \exp (i q \xi)$$

(5)
where \( \tau' = \tau - v\xi \) is the common retarded time. Inserting Eqs. 5 into Eqs.1, we get a set of ordinary differential equations (ODE’s) for \( P_1, P_2 \), which reads

\[
\begin{align*}
i(\delta - v)\dot{P}_1 + (\beta - q)P_1 + \frac{1}{2}\ddot{P}_1 + (|P_1|^2 + A|P_2|^2)P_1 + BP_2^2P_1^* &= 0 \\
-i(\delta + v)\dot{P}_2 - (\beta + q)P_2 + \frac{1}{2}\ddot{P}_2 + (|P_2|^2 + A|P_1|^2)P_2 + BP_1^2P_2^* &= 0
\end{align*}
\]

(6)

where the overdots indicate derivative respect to \( \tau' \). Eqs. (6) can be derived (via the Euler-Lagrange equations) from the following Lagrangian

\[
\mathcal{L} = \int_{-\infty}^{\infty} \mathcal{L} \, d\tau',
\]

where the lagrangian density \( \mathcal{L} \) is given by

\[
\mathcal{L} = |\dot{P}_1|^2 + |\dot{P}_2|^2 - i(\delta - v)\left(P_1^*\dot{P}_1 - P_1\dot{P}_1^*ight)
\]

\[
- i(\delta + v)\left(-P_2^*\dot{P}_2 + P_2\dot{P}_2^*ight) + 2(q - \beta)|P_1|^2
\]

\[
+ 2(q + \beta)|P_2|^2 - |P_1|^4 - |P_2|^4 - 2A|P_1|^2|P_2|^2
\]

\[
- B\left(P_1^2P_2^* + P_1^2P_2^2\right).
\]

(7)

We now assume the following ansatz for \( P_{1,2} \)

\[
P_i = a_i \text{sech}(b_i \tau') \exp(i c_i \tau'), \quad i = 1, 2.
\]

(8)

We know that the exact solution has a complex phase chirp induced by the four wave mixing term (last term in Eqs.1), which we neglect for simplicity but also because from previous numerical calculations we observed that the phase dependence on \( \tau' \) was mainly lineal.

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Where the variational parameters \( a_{1,2}, b_{1,2}, \) and \( c_{1,2} \) are assumed to have real values. The above ansatz is inspired from solutions 2, and 3, as well as from the work of Paré [14]. We know that the exact solution has a complex phase chirp induced by the four wave mixing term (last term in Eqs.1), which we neglect for simplicity but also because from previous numerical calculations we observed that the phase dependence on \( \tau' \) was mainly lineal [3].

By introducing Eqs. 8 into Eqs.7, and applying the Euler-Lagrange equations to the averaged Lagrangian one obtains, after some straightforward algebra, a set of six nonlinear
coupled equations, viz,

\[ c_1 = -\delta + v + B b_1 a_2^2 I(c_-) \] (9)

\[ c_2 = \delta + v - B b_2 a_1^2 I(c_-) \] (10)

\[ \frac{b_1}{3} + \frac{2D_1}{b_1} - \frac{4a_1^2}{3b_1} - A \frac{a_2^2}{b_2} m_0(\eta) - B a_2^2 M(b_1, b_2, c_-) = 0 \] (11)

\[ \frac{b_1}{3} - \frac{2D_1}{b_1} + \frac{2a_1^2}{3b_1} - A \frac{a_2^2}{b_2} m_0(\eta) - B a_2^2 b_1 \frac{\partial}{\partial b_1} M(b_1, b_2, c_-) = 0 \] (12)

\[ \frac{b_2}{3} + \frac{2D_2}{b_2} - \frac{4a_2^2}{3b_2} - A \frac{a_1^2}{b_1} m_0(\eta^{-1}) - B a_1^2 M(b_1, b_2, c_-) = 0 \] (13)

\[ \frac{b_2}{3} - \frac{2D_2}{b_2} + \frac{2a_2^2}{3b_2} - A \frac{a_1^2}{b_1} m_0(\eta^{-1}) - B a_1^2 b_2 \frac{\partial}{\partial b_2} M(b_1, b_2, c_-) = 0 \] (14)

where

\[ I(x) = \int_{-\infty}^{\infty} \tau \sin(2x\tau) \text{sech}^2(b_1\tau) \text{sech}^2(b_2\tau) d\tau \] (15)

\[ c_- = c_2 - c_1 = 2\delta - \frac{b_1 b_2}{2} QBI(c_-) \] (16)

\[ m(x) = \int_{-\infty}^{\infty} \text{sech}^2 t \text{sech}^2(x t) dt \] (17)

\[ M(b_1, b_2, c_-) = \int_{-\infty}^{\infty} \cos(2c_- t) \text{sech}^2(b_1 t) \text{sech}^2(b_2 t) dt. \] (18)

\[ D_1 = q - \beta + c_1^2/2 + (\delta - v)c_1, \quad D_2 = q + \beta + c_2^2/2 - (\delta + v)c_2, \quad \eta = b_1/b_2, \quad \text{and} \quad Q = 2(a_1^2/b_1 + a_2^2/b_2) \]

In order to simplify the set of equations 9 - 14, we need to do some assumptions and approximations. We first take into account [1, 6] that the main contribution to \( c_- \) becomes from \( 2\delta \), while the integral is a minor correction. Therefore, using the first order Böhrn approximation [24], \( c_- \) becomes

\[ c_- \approx 2\delta + \frac{b_1 b_2}{2} QBI(2\delta) \] (19)

Introducing Eq.19 into Eqs.11 - 14 we get a reduced set of four nonlinear equations. The next step must consist in finding accurate analytical expressions for the integrals \( I, m, \)
and $M$ which can not be exactly integrated. Following ref. [14] we expand $m$ in a Taylor series around $\eta = 1$ up to second order, and obtain:

$$m(x) = \frac{4}{3} + \frac{2}{3}(1 - x) + \frac{2}{3}(1 - \frac{\pi^2}{15})(1 - x)^2$$

$$\dot{m}(x) = \frac{2}{3} - \frac{4}{3}(1 - \frac{\pi^2}{15})(1 - x) + (\frac{\pi^2 - 10}{5})(1 - x)^2$$  \hspace{1cm} (20)

The same Taylor expansion for $M$ or $I$ gives much more complicated expressions in terms of Polygamma functions [13], which should be evaluated numerically. Instead of this approach we choose to approximate $M$ and $I$ by replacing $b_i$ by $(b_1 + b_2)/2$. In this way the integrals can be analytically performed to give:

$$M \approx \frac{16\pi c_-}{3b^2} \left[1 + \left(\frac{2c_-}{b}\right)^2\right] \text{csch}\left[\frac{2\pi c_-}{b}\right]$$  \hspace{1cm} (21)

$$\frac{\partial M}{\partial b_i} = \frac{-108\pi c_+^3}{3b^5} \text{csch}\left[\frac{2\pi c_-}{b}\right] - \frac{32\pi c_-b^2 + 4c_+^2}{3b^5} \text{csch}\left[\frac{2\pi c_-}{b}\right]$$

$$+ \frac{32\pi^2 c_-^2[b^2 + 4c_+^2]}{3b^5} \text{coth}\left[\frac{2\pi c_-}{b}\right] \text{csch}\left[\frac{2\pi c_-}{b}\right]$$  \hspace{1cm} (22)

$$I(x) \approx \frac{16\pi x[b^2 + x^2]}{3b^5} \left[\pi \text{coth}\left(\frac{2\pi x}{b}\right) - \frac{b}{2x} - \frac{4bx}{b^2 + 4x^2}\right] \text{csch}\left[\frac{2\pi x}{b}\right]$$  \hspace{1cm} (23)

where $b = b_1 + b_2$.

From here the procedure goes as follow. Given the equation coefficients $\delta$, $\beta$, $A (= 2/3)$, and $B (= 1/3)$, together with the parameters $v$ and $q$, we solve the Eqs. [11] - [14] to obtain $b_{1,2}$, and $a_{1,2}$. Then $c_{1,2}$ are obtained from Eqs. [3] - [10].
4 Numerical results

We have numerically solved the four coupled nonlinear algebraic equations by a Powell hybrid method \cite{24}. In all cases a unique solution was instantaneousy obtained. In order to check the validity of the approximations used to solve the integrals we sometimes evaluated numerically $M$, $m$, and $I$ at each step getting almost identical results but with a much higher CPU time consumption.

Fig.1 shows the dependence of the soliton energy $Q$ on the nonlinear propagation constant $q$. This diagram was obtained by means of the above described variational method. The solid lines represent the solutions given by Eqs. 4. Different sets of almost parallel curves correspond to different values of the parameter $\delta$ whilst the value of $v$ hardly influences the shape of these curves, fixing solely the minimum allowed value of $q$. These curves coincide totally with those shown in fig.1(a) of Ref.\cite{3}, which were obtained exactly by numerically solving the propagation equation for certain values of $q$ whatever $v$ was.

In addition to the solutions approximated by Eq.8, where $a_{1,2}$ were assumed to be real, and therefore represent solutions where $U$ and $V$ are in phase, there exist solutions where $U$ and $V$ are $\pi/2$ out of phase \cite{1}. They can be obtained directly from our variational approach if we do not imposed $a_{1,2}$ to be real. Alternatively our approach can account for them just by changing the ansatz (Eq.8 for $P_i$). If we choose the following one:

\begin{equation}
\begin{align*}
\tilde{P}_1 &= a_1 \text{sech}(b_1 \tau') \exp(i c_1 \tau') \\
\tilde{P}_2 &= a_2 \text{sech}(b_2 \tau') \exp(i(c_2 \tau' + \pi/2)),
\end{align*}
\end{equation}

\hfill (24)
and repeat the process, it is easily realized that the only change produced in the nonlinear
algebraic equations is to replace $B$ by $-B$. This indicates that the existence of these
independent solutions is intimately related to the inclusion of the four wave mixing term
in the propagation equations.

Figure 2 shows the fraction of energy in the slow mode ($Q_1/Q$) versus the total en-
ergy ($Q$) of the two-parameter family of solutions for $\delta = \beta = 1$ as obtained from our
variational method. In (a) and (b) $P_i$ is given by Eq.8 and Eq. 24 respectively. The results
can be compared with the exact ones in Fig. 1 of Ref.[1], which were found numerically.
The symmetry of these curves is a result of the symmetry that possesses the Lagrangian
under:

$$q \rightarrow \tilde{q} = q + \frac{\tilde{v}^2 - v^2}{2}, \quad v \rightarrow \tilde{v} = \frac{2\beta}{\delta} - v,$$

$$c_{2,1} \rightarrow \tilde{c}_{2,1} = \frac{2\beta}{\delta} - c_{1,2}, \quad a_2 \leftrightarrow a_1, \quad b_2 \leftrightarrow b_1$$ (25)

Figs. 1, and 2 illustrate the accuracy of the variational approach as developed in the
section 3. We must remark that solving the set of four nonlinear algebraic equations is an
easy task that can be done really fast. In our computer (Alpha Dec 2100/500) we obtain
thousands of these variational solutions in just a few CPU seconds.

Fig. 3 shows the variation of the width ratio ($\eta = b_1/b_2$) vs. the propagation constant
$q$ for the stationary solutions when $U$ and $V$ are (a) in phase, and (b) in quadrature.
We take $\beta = \delta = 1$, and three values of $v$ are considered, which are written close to the
corresponding curves. In all cases $\eta$ tends to 1 as $q$ increases, i.e. for high values of $Q$
(see Fig. [1]). The departure of $\eta$ from unity is higher when $U$ and $V$ are in quadrature
than when they are in phase. In any case it is small except around the minimum allowed
value of $q$, where the curves emerge and one of the components ($V$ in the cases of figure 3) is almost zero. This figure serves also to verify a posteriori that our approximations of integrals $I$, $m$ and $M$ had good basis.

Similarly Fig. 4 shows (a) $c_2 - c_1$ and (b) $c_2$ vs. $q$ for the same values of the parameters. As expected, at the minimum value allowed for $q$, where the solution is a fast soliton (Eq.3), $c_2$ is exactly $\delta + v$. As it occurred for $Q$ vs. $q$, the value of $c_-$ hardly depends on $v$, being mainly determined by the value of $q$. It is also remarkable that while the solutions for $U$ and $V$ in phase decrease its frequency difference (i.e. $c_-$) as we increase $q$ (and therefore $Q$), the opposite happens for the solutions with $U$ and $V$ in quadrature. In these last cases $c_-$ becomes more different to $2\delta$ than when $U$ and $V$ are in phase. The same happened with $\eta$ respect to 1 (see Fig. 3). In fact we observe that the variational solutions were much more accurate for the solutions with $U$ and $V$ in phase than for those in quadrature.

Fig. 5 shows the variation of the widths and peak amplitudes vs. $q$ for the solutions with $U$ and $V$ in phase. It can be seen that for a given value of $\delta$ and $\beta$, the widths depend almost exclusively on $q$, being almost completely independent on $v$. On the other hand the asymptotic behavior of $a_1/a_2$ does not depend on $q$ but only on $v$ (see Fig. 5c).

We have numerically solved Eqs. taking as initial conditions the variational solutions corresponding to different values of the parameters $v$ and $q$ and fixed values of $\delta$ and $\beta$, viz. $\beta = 1 = \delta$. In all cases that we propagate in-phase solutions, we obtained stable propagation, whilst $\pi/2$-dephased solutions were sometimes unstable. Fig. 6 shows the stable propagation of the in-phase variational solution for $q = 3.2$ and $v = 0.9,$
whereas Fig. 7 shows the same for the $\pi/2$-dephased solution corresponding to $q = 6$ and $v = 0.9$. Contrary to the previous case, a small difference can be appreciated between the variational solution and the stationary one. The exact stable stationary solution (which of course is $\pi/2$-dephased) is reached after a very short propagation distance. Finally Fig. 8 illustrates the unstable behavior of a $\pi/2$-dephased solution ($q = 7.2$, $v = 0.8$). The solution oscillates around the stationary solution with increasing amplitude emitting a small quantity of radiation. Eventually, after the emission of a large quantity of radiation, a stable solution is reached (not shown in the figure). In Figs. 6-8 $\tau'' = \tau + \delta$.

5 Summary

We have developed an accurate variational approach to derive analytical approximations of the coupled pulselike stationary solutions in birefringent fibers in the most general case when differences in both phase velocities and group velocities between the two components and fastly oscillatory terms are taken into account (they are important in the subpicosecond regime). As particular cases it includes those where any term in Eqs. 1 can be neglected. We have shown that the difference between the central frequencies of the components, their width and their energies are almost independent on $v$, being mainly determined by $q$. We have also shown that the ratio $a_1/a_2$ is mainly determined by $v$. The stability of these solutions has been briefly considered, by numerically propagating them. In all the cases we propagated solution in-phase, they happened to be stable whilst those $\pi/2$ out of phase present some intervals of stability. A global stability analysis of these solutions remains to be done. This variational method could be useful to design
soliton-dragging logic gates. \[25\]

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References


Figure 1: Dependence of the soliton energy $Q$ on the nonlinear propagation constant $q$ for the fast and slow linearly polarized solitons (continuous lines) and coupled soliton states for $\beta = 1, \delta = 2$ and $4$, and several values of the parameter $\nu$, which is written close to the point where the curves emerge. This point is marked with a filled circle.
Figure 2: Fraction of energy carried by the walking soliton in the slow polarization vs. the total energy of the solution. (a) $U$ and $V$ are in phase. (b) $U$ and $V$ have a relative phase difference of $\pi/2$. They correspond to $\delta = \beta = 1$, and different values of $v$, which is written in the figure.
Figure 3: Width ratio ($\eta$) vs $q$ for $\delta = \beta = 1$, and three values of the parameter $v$. a) $U$ and $V$ are in phase. b) $U$ and $V$ are in quadrature.
Figure 4: a) Central frequency difference between the soliton components $V$ and $U$ vs. $q$. b) Frequency shift of the fast component as a function of the propagation constant $q$. The dashed lines represent the solutions where $U$ and $V$ are in phase and the dotted ones those in quadrature. The values of the parameters are the same than in Fig. 3.
Figure 5: a) Inverse widths ($b_{1,2}$), b) peak amplitudes ($a_{1,2}$) and c) amplitude ratio ($a_1/a_2$) of the variational solution for $U$ and $V$ in phase vs. $q$. In (a) and (b) the dotted lines are for $U$ and the dotted lines for $V$. $\delta = \beta = 1$ and the values of $v$ are written in the figures.
Figure 6: Stable propagation of a walking soliton with its components $U$ and $V$ in phase.

$\beta = \delta = 1$, $q = 3.2$, and $v = 0.9$
Figure 7: Stable propagation of a walking soliton with its components $U$ and $V$ in quadrature. $\beta = \delta = 1$, $q = 6$, and $v = 0.9$
Figure 8: Unstable propagation of a stationary solution with its components in quadrature. \( \beta = \delta = 1, q = 7.2, \) and \( v = 0.8 \)