Bundling and Switching Costs, with Implications on Convergence

March 15, 2011

Abstract

Convergence in information technology has made bundling as a key business strategy and policy makers are very interested in knowing how bundling affects competition in the IT sector. For instances, does bundling build barrier to entry? Does head-to-head competition between bundles generate more consumer surplus than competition among individual products? Furthermore, switching cost is quite important in the IT sector. In this paper, we revisit the two questions in a framework in which consumers have heterogeneous switching costs. First, when we consider an incumbent facing $n$ specialist entrants, we find that the incumbents bundling always increases its own profit. In addition, it reduces each entrants profit if and only if consumers are heterogeneous enough in terms of switching cost. However, as $n$ goes to infinite, each entrants profit converges to zero regardless of the degree of homogeneity. Second, when we consider an incumbent facing a generalist entrant producing $n$ products, we find that competition between bundles generates higher consumer surplus than competition among individual products if and only if consumers are heterogeneous enough in terms of switching cost. As $n$ increases, the zone in which bundling increases consumers’ surplus shrinks.
1 Introduction

Does bundling build a barrier to entry? Does head-to-head competition between bundles improve consumers’ surplus compared to competition among individual products? The first is a classic question related to anticompetitive effects of bundling (or tying). The second is a question that some recent papers analyzed. In this paper, we revisit these questions in a framework in which consumers have heterogeneous switching costs. Although both bundling and switching costs are topics intensively investigated in the IO literature, to our knowledge there has been no paper that addresses the interaction between bundling and switching costs.

A main motivation for us to study bundling in relation to switching costs comes from convergence. Namely, convergence in information technology has made bundling a key business strategy and policymakers are very interested in knowing how bundling practices will affect competition and consumers’ surplus.

Examples of bundling practices originated from convergence or advancement in information technology are abundant. Many telecommunications operators in the world offer triple-play boxes that provide Internet access, TV access, and telephone service in a package. Hand-set manufacturers such as Apple, Nokia, Samsung integrate more and more diverse services and products into their handsets such as operating system, web browser, emails, camera, interactive maps, GPS, MP3, on-line games, etc. In the case of cloud computing, service providers offer a whole range of integrated services of data management such that an employee of a client company can get access to software and database necessary for her work just from the Internet.

This trend of bundling in IT sector raises two kinds of questions. First, we ask whether bundling would destabilize the level-playing field for the competition between a big company offering a whole range of services and small companies specialized in each service in favor of the former. In other words, does bundling build barriers to entry to producers of specialized services? Second, is competition between bundled products fiercer than competition that would arise when bundling is prohibited?

We address these questions in a context of switching costs. Switching costs can be very high in the above examples. For instance, a mobile phone user writes down her contacts into the phone, stocks songs, pictures, games and downloads applications. Actually, the cost of switching only an email account from one to another can be extremely high if the user intensively sends and receives files and all these files are stocked in the email account. This suggests that the scale of switching cost will be far larger in the case of cloud computing service offered to businesses. Although we described only examples in the IT sector, our framework applies to other industries as long as switching costs are significant.
To address the question whether bundling builds a barrier to entry, we study the pricing game played between an incumbent producing \( n \) products (that can be independently consumed) and \( n \) specialist entrants. There is a mass one of consumers who have a unit demand for each of \( n \) products. Every product gives each consumer a surplus equal to \( u \), which is large enough such that every consumer consumes \( n \) products. However, consuming a product offered by an entrant requires to bear a switching cost. We assume that a consumer’s switching cost is given by a non-negative constant \( a \in [0, 1) \) plus a random term which is uniformly distributed over \([0, 1]\), and the random terms are i.i.d. across products. Clearly, \( a \) can be interpreted as an index which represents consumers’ preferences for the incumbent’s products, but we show in Section 2 that \( a \) can also be interpreted as a degree of homogeneity of consumers in terms of switching costs.

We compare the case in which bundling is prohibited from the case in which it is allowed. The former is called ”independent pricing vs. independent pricing” and the latter ”mixed bundling vs. independent pricing”. Note that mixed bundling includes pure bundling and independent pricing as special cases. Since all products can be independently consumed, in the case of ”independent pricing vs. independent pricing”, we can analyze each market in isolation.

When we compare ”pure bundling vs. independent pricing” with ”independent pricing vs. independent pricing”, we find that the incumbent’s profit is always higher in the former than in the latter, implying that pure bundling is credible. When \( n = 2 \), each entrant’s profit is smaller in the former than in the latter if and only if \( a < 0.54623 \). However, as \( n \) increases, this threshold value of \( a \) increases such that in the limit as \( n \) tends to infinity, each entrant’s profit is zero for any \( a \in [0, 1) \). Therefore, for small \( n \) bundling reduces the profit of relatively strong entrants while (moderately) increasing the profit of relatively weak entrants. Since an entrant’s profit always decreases in \( a \), if there is a fixed cost of entry which does not depend on \( a \), weak entrants are unlikely to be able to enter anyway. Hence, our results suggest that bundling effectively builds barrier to entry and is credible; the barrier to entry is more important as the number of bundled products increases.

The intuition can be provided in terms of two effects: demand size effect and demand elasticity effect. The incumbent’s bundling changes competition between individual products into competition between two bundles where the price of the bundle of the entrants is the sum of each entrant’s price. Hence, under bundling, what matters is the distribution of the average switching cost, which is more skewed around the mean switching cost than the distribution of the individual switching costs. This implies that even though the incumbent maintains the same aggregate price, bundling increases its demand as long as its market share is larger than a half in the absence of bundling. This is the ”demand

\[ \text{The restriction } a < 1 \text{ is imposed otherwise each entrant realizes zero profit in the benchmark in which bundling is prohibited.} \]
size” effect which works always in favor of the incumbent. Furthermore, the incumbent has an incentive to change its price after bundling. Bundling makes the demand for the incumbent’s bundle more elastic when the marginal consumers have an average switching cost close to the mean; then bundling induces the incumbent to reduce its price. On the contrary, bundling makes the demand more rigid if the marginal consumers have an average switching cost close to the extreme values for switching costs; then bundling induces the incumbent to increase its price. As $a$ increases, the switching cost of the marginal consumer becomes smaller and closer to the lowest possible value, $a$, and this induces an increase in the price of the incumbent. Actually, we find that when $n = 2$, bundling makes the incumbent more aggressive if and only if $a < 0.12132$. When the incumbent reduces (increases) its price, the entrants respond by reducing (increasing) their prices since prices are strategic complements. Overall, from the incumbent’s point of view, the demand size effect dominates the demand elasticity effect when they work in opposite directions such that bundling always increases the incumbent’s profit. Regarding the entrant’s profit, as $a$ increases, the competition softening effect from the change in demand elasticity dominates the demand size effect such that bundling increases its profit for $a > 0.54623$ when $n = 2$. As $n$ increases, the distribution of average switching cost is more skewed toward the mean, suggesting that the demand size effect becomes stronger. In particular, for a large $n$ consumers are almost homogeneous since the distribution of the average switching cost is very concentrated near the mean. Then the incumbent can attract almost all consumers by charging a price equal to the expected switching cost, but in fact the incumbent does much better since each entrant charges a relatively high price, larger than $u - a - \frac{1}{2}$, in order to profit from the small amount of consumers which are not attracted by $I$. This allows $I$ to charge the monopoly price $u$ while attracting all consumers. It is remarkable that a large coordination failure occurs among entrants, since if each entrant reduces his price below $u - a - \frac{1}{2}$, then entrants would attract all consumers and make a significant profit.

For the case of $n = 2$ we also consider mixed bundling of the incumbent, and we find that the equilibrium outcome is the same as in the case of pure bundling. A mixed bundling strategy turns out to be a best reply for the incumbent only when $a$ is small and the entrants play very small prices, but that never occurs in equilibrium. On the other hand, when entrants’ prices are not small, mixed bundling is inferior to bundling for the incumbent because it cannibalizes the incumbent’s revenue from the sales of the bundle, and this revenue reduction is not compensated by the increase in revenue due to sales of separate products.

We now turn to the second question: is head-to-head competition between bundles fiercer than competition among individual products? To answer the question, we study the competition between a generalist incumbent and a generalist entrant; each of them
produces $n$ different products. We compare "independent pricing vs. independent pricing" with "bundle vs. bundle"; in the latter case, each firm offers only a pure bundle.

With respect to the comparison between "independent pricing vs. independent pricing" and "bundling vs independent pricing", the comparison between "independent pricing vs independent pricing" and "bundle vs bundle" brings in one new effect, "externality internalization" effect; a generalist entrant internalizes the positive externalities on other products from a reduction of the price of a given product and chooses lower prices than $n$ specialist entrants under "bundling vs independent pricing". Actually, because of this effect both firms realize a lower profit under "bundle vs bundle" than under "bundling vs independent pricing". However, the demand size effect and the demand elasticity still exist under "bundle vs bundle". Actually, when $n = 2$, for $a > 0.68794$, the competition softening effect from the change in demand elasticity dominates all other effects and every firm's profit is higher under "bundle vs bundle" than under "independent pricing vs independent pricing". Obviously, because of the positive demand size effect, the threshold value of $a$ that makes the incumbent's profit higher under "bundle vs bundle" is smaller than 0.68794, and equal to 0.20743. Consistently with this fact, consumers' surplus is lower under "bundle vs bundle" for $a > 0.37602$, which is between the two previous thresholds. For a large $n$ we find again that $I$ can attract almost all consumers by playing a price per product slightly smaller than the average switching cost. However, now the entrant's price is close to zero, since there is no coordination failure within a (single) generalist entrant, and the incumbent is unable to extract all consumers' surplus. Nevertheless, the incumbent's profit is higher under "bundle vs bundle" than under "independent pricing vs independent pricing", and the entrant's profit is close to zero for any $a \in [0, 1)$. Finally, consumers' surplus is lower under "bundle vs bundle" for $a > 0.19615$.

We conclude by considering the case in which each consumer's switching cost is perfectly correlated across different products. In this case, neither demand size effect nor demand elasticity effect exists. Actually, we first find that the outcome of "independent pricing vs independent pricing" is equivalent to that of "bundle vs bundle". In the case of "pure bundling vs independent pricing", because of non-internalization of externalities, $n$ specialist entrants charge higher prices than a generalist entrant, which in turns induces the incumbent to charge a higher price. For this reason, the incumbent's profit is always higher under "pure bundling vs independent pricing" than under "independent pricing vs independent pricing". In the case of the entrants, as $n$ increases, the loss from miscoordination of prices dominates the competition softening effect of the miscoordination such that the entrant's profit is lower under "pure bundling vs independent pricing" than under "independent pricing vs independent pricing" if and only if $n > 4$ for any $a \in [0, 1)$.

The bundling (or tying) literature can be divided into three categories.\(^2\) The first one

\(^2\)See Armstrong (2006) for a useful survey.
includes the papers that view bundling as a screening device for a monopolist (Schmalensee, 1984, McAfee et al. 1989, Salinger 1995, Armstrong 1996, Bakos and Brynjolfsson, 1999, Fang and Norman, 2006). The second category examines whether bundling builds a barrier to entry. The third one is about competitive bundling (Nalebuff, 2000, Armstrong and Vickers (2010) and Thanassoulis, forthcoming). Our paper is closely related to the second and the third category. The second one can be divided into the papers that consider one specialist entrant such as Whinston (1990) and Nalebuff (2004) and those that consider multiple specialist entrants such as Nalebuff (2000) and Choi and Stefanadis (2001). Our main contribution with respect to the literature on multiple specialist entrants is that we provide general insight based on the two effects ”demand size effect” and ”demand elasticity effect” by varying the parameter $a$ that represents the degree of consumer homogeneity in terms of switching cost. Furthermore, we show that pure bundling is profitable and credible (even if mixed bundling is allowed, the incumbent uses pure bundling).

Whinston (1990) studies leverage from tying a monopoly product to another product that faces competition. Tying induces the incumbent to be aggressive and reduces the profit of the rival firm in the market of the tied product; thus tying may induce that firm not to enter in the presence of some fixed cost. Notice however that tying reduces the profit of the incumbent if the rival enters, and in this sense it is not credible unless it succeeds in driving the rival out of the market. Our paper is not about leverage since the $n$ products are symmetric, and we find that pure bundling is profitable and credible and that pure bundling can make the incumbent more or less aggressive depending on $a$. Nalebuff (2004) shows how bundling two products can reduce the profit of an entrant in a single market, but his result applies under the assumption that the incumbent is a Stackelberg leader in setting its prices. In our paper, instead, all prices are chosen simultaneously by all players.

Nalebuff (2000) is the closest reference to our paper. The key difference is that Nalebuff considers a symmetric Hotelling model. For instance, in the case of ”independent pricing vs independent pricing”, the two firms competing in the market for a given product are symmetric in a standard Hotelling model. Then, starting from the equilibrium prices for this regime, the demand size effect is zero, and the demand elasticity makes the incumbent more aggressive, which induces the entrants to be also more aggressive under ”bundling vs independent pricing”. The result is that when $n = 2$ or $n = 3$, the incumbent’s profit is smaller than under ”independent pricing vs independent pricing”.3

Choi and Stefanadis (2001) provide a theory of dynamic leverage by studying a setting in which the incumbent sells two perfect complements and faces an entrant in each component. Each entrant decides its R&D amount, which determines the probability of

---

3In addition, Nalebuff (2000) does not analyze the case in which the incumbent practices mixed bundling.
successfully developing a product superior to that of the incumbent. Without tying, each entrant can make a profit whenever its R&D is successful, but if the incumbent ties the two components, an entrant profits only if both entrants’ R&D are successful. Thus, tying reduces the entrants’ incentives to invest in R&D and reduces the probability that the incumbent is eliminated from the market by two successful entrants. However, this uncertainty related to the success or failure of R&D is crucial in their theory: conditional on that each entrant has a superior technology, entry occurs regardless of tying. This is the main difference with respect to our paper in which bundling can deter entry even though the fixed cost is such that each entrant can enter successfully if bundling is prohibited.

Nalebuff (2000) and Thanassoulis (forthcoming) study the regime of "bundle vs bundle", and they find (as in our paper) that each firm realizes a lower profit than under "bundling vs independent pricing". They also find that competition is fiercer under "bundle vs bundle" than under "independent pricing vs independent pricing". Similar results are obtained by Matutes and Regibeau (1988, 1992) and by Armstrong and Vickers (2010). When \( n = 2 \), we find that this negative effect of bundling on profits holds only for small \( a \). Furthermore, in the case of a large \( n \) the incumbent’s profit is always larger under "bundle vs bundle" than under "independent pricing vs independent pricing", as we explained above.

## 2 Model

We consider two variations of one model to address two different questions. We first consider competition between a generalist incumbent (producing \( n \) different products) and \( n \) specialist entrants to address the question of whether bundling helps the incumbent to build a barrier to entry. And then we consider competition between a generalist incumbent and a generalist entrant to address the question of whether competition between bundles is fiercer than competition among individual products.

---

4In Thanassoulis (forthcoming), there are small customers who buy only one of the two products and large customers who buy both. Hence, pure bundling is never optimal with respect to mixed bundling. However, he obtains that when two firms producing different products merge and use mixed bundling, the rivals have no incentive to merge to use mixed bundling as competition becomes fiercer under "bundle vs bundle".

5Matutes and Regibeau (1988, 1992) assume that firms choose between offering compatible or incompatible products before competing, and find that often profits are higher under compatibility. Products are compatible by assumption in Nalebuff (2000), but compatibility is irrelevant under "bundle vs bundle".
2.1 Competition with \( n \) specialist entrants

There is one generalist incumbent \( I \) who produces \( n(\geq 2) \) different products, and there are \( n \) specialist entrants: \( E_i \) is the entrant for product \( i \), for \( i = 1, \ldots, n \). Each consumer has a unit demand for product \( i \), and the utility that he obtains from each product \( i \) is \( u > 0 \) regardless of whether it is produced by the incumbent or by an entrant, and regardless of the other products he consumes; therefore, the products have independent values.

We study static games of price competition between the incumbent and the entrants. In the period preceding the game, a mass one of consumers is assumed to have bought the \( n \) products from the incumbent. Therefore, in order to obtain a positive market share, an entrant \( i \) needs to induce some consumers to switch from the incumbent to \( E_i \). For simplicity, we suppose that the production cost is zero for all firms: later on, when we discuss the implications of our results on foreclosure, we introduce a fixed cost of entry for the entrants.

We model switching costs by assuming that the consumers are uniformly distributed over the unit hypercube \([0, 1]^n\), and for a consumer with type \((x_1, \ldots, x_n)\) [that is, located at \((x_1, \ldots, x_n)\)], the switching cost for product \( i \) is \( s_i = \alpha + \beta x_i \) with \( \alpha \geq 0 \) and \( \beta > 0 \). It turns out that equilibrium profits are homogenous of degree one in \((\alpha, \beta)\): if we denote by \( \pi^r_j(\alpha, \beta) \) the equilibrium profit of firm \( j \) under regime \( r \) (the regime could be "independent pricing vs independent pricing", for example), then \( \pi^r_j(\alpha, \beta) = \beta \pi^r_j(\frac{\alpha}{\beta}, 1) \).

Our main purpose is to investigate whether different competition regimes increase of decrease the firms' profits, and this investigation yields same results in the original model with \( s_i = \alpha + \beta x_i \) as in a model with \( s_i = a + x_i \), in which \( a = \frac{\alpha}{\beta} \). Hence, in what at follows we assume that a consumer with type \((x_1, \ldots, x_n)\) has switching cost \( s_i = a + x_i \) for product \( i \), with \( a \geq 0 \). In addition, we assume \( a < 1 \) and \( u > 1 + a \), which guarantee respectively that the equilibrium prices are strictly positive when bundling is prohibited, and that every consumer buys each product, except in the setting of Section 4.2.

A natural interpretation of an increase in \( a \) is that the asymmetry between the incumbent and the entrants increases, but in fact an alternative interpretation is possible, related to consumers heterogeneity. Precisely, given a \( \mu > 0 \), consider the original formulation with \( s_i = \alpha + \beta x_i \) in which \( \alpha = \frac{2\mu a}{1+2a} \) and \( \beta = \frac{2\mu}{1+2a} \); notice that \( a = \frac{\alpha}{\beta} \). Then an increase in \( a \) causes an increase in \( \alpha \) and a decrease in \( \beta \), but the expected switching cost, \( \alpha + \frac{\beta}{2} \), is equal to \( \mu \) for every \( a \). Therefore, as \( a \) increases consumers become more similar in terms of switching cost, but the average switching cost is unchanged. From the equality \( \pi^r_j(\alpha, \beta) = \frac{2\mu}{1+2a} \pi^r_j(a, 1) \) we infer that by studying how \( a \) affects the comparison

\[ \pi^r_j(\alpha, \beta) = \beta \pi^r_j(\frac{\alpha}{\beta}, 1). \]
between $\pi_j^r(a, 1)$ and $\pi_j^{r'}(a, 1)$ for different regimes $r$ and $r'$, we infer how a different degree of consumers’ heterogeneity affects the firms’ preferences between $r$ and $r'$.

We study two different games of simultaneous pricing played between the incumbent and the entrants, in which each entrant $i$ is assumed to choose a price $p_{Ei}$ for his product $i$ in a non cooperative way. The two games differ depending on whether or not the incumbent’s bundling is allowed. If it is prohibited, then we have

- Independent pricing versus independent pricing: the incumbent chooses $p_{II}$ and entrant $i$ chooses $p_{Ei}$ for each product $i = 1, ..., n$.

If the incumbent’s bundling is allowed, then we have

- Mixed bundling versus independent pricing (for the case of $n = 2$): The incumbent can use mixed bundling and charge $P_I$ for the bundle and $p_{I1}, p_{I2}$ for each single product; entrant $i$ chooses $p_{Ei}$ for $i = 1, 2$.

A special case of the above game is

- Pure bundling versus independent pricing: The incumbent is restricted to offer a pure bundle and charges $P_I$ for the bundle; entrant $i$ chooses $p_{Ei}$ for $i = 1, ..., n$.

We call the game of ”independent pricing versus independent pricing” game II, ”mixed bundling versus independent pricing” is game MI, and ”pure bundling versus independent pricing” is game BI. In this section, we study the two games II and MI and compare their outcomes in terms of the incumbent’s profit and each entrant’s profit. To denote equilibrium prices (and profits and consumers’ surplus) of each game, we use the superscripts II, MI, BI.

### 2.2 Competition with a generalist entrant

We also consider competition between a generalist incumbent and a generalist entrant, assuming that each of them produces the $n$ products. We compare ”independent pricing versus independent pricing” with ”bundle versus bundle” in terms of each firm’s profit and consumers’ surplus. In the case of ”bundle versus bundle”, we assume that each firm just sells a pure bundle of $n$ products. We call the game of ”bundle versus bundle” game BB and use the superscript BB to denote equilibrium prices (and profits and consumers’ surplus) of this game.

---

7 In our model social surplus is $nu$ minus the average switching cost which consumers incur, and is maximized when each entrant has zero market share. Hence, we focus on the more interesting consumers’ surplus rather than on social welfare.
Remark: Although we call one firm the incumbent and the other the entrant (denoted by $E$), the latter does not need to be an entrant. Actually, our model of "bundle vs bundle" can be interpreted as the last period of a dynamic duopoly competition in which both firms sell pure bundles. Then, in the beginning of the last period, there are two groups of consumers depending which bundle they bought in the previous period. Once we allow the firms to charge different prices depending on which bundle a consumer bought in the previous period, then we can isolate the competition in one group of consumers from the competition in the other as in our model.

3 Independent pricing vs independent pricing

Suppose that bundling is prohibited. Given that the products have independent values, each consumer can consume any combination of products regardless of who produced them. This allows us to study the competition for each product in isolation, independently of whether the incumbent faces $n$ specialist entrants or a generalist entrant. Because of the symmetry among the $n$ products, in this section we do not use the notation $i$ for $i = 1, ..., n$.

Given $(p_I, p_E)$, a consumer with switching cost $x + a$ is indifferent between buying the incumbent’s product and the entrant’s product if and only if:

$$p_I = p_E + x + a,$$

which is equivalent to

$$x = p_I - p_E - a.$$

Therefore, each firm’s profit is given by:

$$\pi_I = (1 + a - (p_I - p_E))p_I;$$
$$\pi_E = (p_I - p_E - a)p_E.$$

It is straightforward to find the unique equilibrium prices, denoted by $p_I^{II}, p_E^{II}$.

Proposition 1 (independent pricing) Consider the game of "independent pricing vs independent pricing". Then, in each product market,

(i) The incumbent and the entrant charge the following prices:

$$p_I^{II} = \frac{2}{3} + \frac{a}{3}; \quad p_E^{II} = \frac{1}{3} - \frac{a}{3}.$$  

Clearly, for some values of $p_I, p_E$ there exists no $x \in [0, 1]$ which satisfies (1), but in the proof of Proposition 1 in the appendix we prove that equilibrium prices are such that (1) has a solution in $(0, 1)$.
(ii) The incumbent’s and the entrant’s market share are equal to $p_I^H$ and $p_E^H$ respectively, and therefore the profits are

$$\pi_I^H = \frac{1}{9}(2 + a)^2; \quad \pi_E^H = \frac{1}{9}(1 - a)^2.$$ 

(iii) Consumer surplus is given by

$$CS^H = u - \frac{1}{18}(8a + 11 - a^2)$$

As it is intuitive, the price and the market share of the incumbent increase with $a$. As a consequence, $\pi_I^H$ also increases with $a$. On the other hand, the price, market share and profit of the entrant decrease with $a$. We also compute the consumers’ surplus, which will be used in the section which studies competition between two generalist firms.

4 Competition with $n$ specialist entrants

4.1 Analysis of $n = 2$

In this subsection, we consider the case of two products. We study the game of "mixed bundling versus independent pricing" and compare its outcome with that of "independent pricing versus independent pricing" in terms of the incumbent’s profit and each entrant’s profit.

4.1.1 Pure bundling

We first consider the case in which the incumbent is restricted to offer only a pure bundle. Then, each consumer needs to choose between buying the bundle of the incumbent and buying both products offered by $E1$ and $E2$. Therefore, we are essentially considering competition between two bundles when the price of the bundle of the entrants is the sum of the two prices chosen by the entrants in an uncoordinated fashion.

Given $(P_I, p_{E1}, p_{E2})$, a consumer with switching costs $(x_1 + a, x_2 + a)$ is indifferent between buying the incumbent’s bundle and the entrants’ bundle if and only if

$$P_I = p_{E1} + p_{E2} + 2a + x_1 + x_2.$$ 

When $0 \leq P_I - p_{E1} - p_{E2} - 2a \leq 1$, the demand for the bundle of the entrants is

$$\frac{1}{2}(P_I - p_{E1} - p_{E2} - 2a)^2.$$ 

$^9$The proof of Proposition 2 establishes that these conditions are satisfied in equilibrium.
Therefore, in this case, the profit functions are given by:

\[ \Pi_I = \left[ 1 - \frac{1}{2} (P_I - p_{E1} - p_{E2} - 2a)^2 \right] P_I; \]  
\[ \pi_{E1} = \frac{1}{2} (P_I - p_{E1} - p_{E2} - 2a)^2 p_{E1}, \quad \pi_{E2} = \frac{1}{2} (P_I - p_{E1} - p_{E2} - 2a)^2 p_{E2}. \]

(3) (4)

and it is immediate to find the equilibrium prices \( P_{BI}, p_{BE1}, p_{BE2} \).

**Proposition 2** \((n = 2): \) pure bundling Consider competition between the incumbent’s pure bundle and the entrants’ products when each entrant chooses the price for its product.

(i) The equilibrium prices are

\[ P_{BI} = \frac{2}{5} (3a + \sqrt{10 + 4a^2}); \quad p_{BE1} = p_{BE2} = \frac{1}{10} \left( \sqrt{10 + 4a^2} - 2a \right). \]

\( P_{BI} \) is increasing in a whereas \( p_{BE1} \) is decreasing in a.

(ii) The incumbent’s market share is increasing in a and belongs to \( [\frac{4}{5}, 1) \) for any \( a \in [0, 1) \). The profits are

\[ \Pi_{BI} = \frac{4}{125} \left( a^2 + 10 \right) \sqrt{10 + 4a^2} + \frac{8}{125} a \left( 20 - a^2 \right); \]
\[ \pi_{BE1} = \pi_{BE2} = \frac{1}{250} \left( 8a^2 + 5 \right) \sqrt{10 + 4a^2} - \frac{1}{125} a \left( 8a^2 + 15 \right). \]

4.1.2 Mixed bundling

In this section we consider the case in which the incumbent practices mixed bundling, that is he chooses a price \( P_I \) for the bundle of his products, and \( p_{I1}, p_{I2} \) for his single products. Entrants \( E1 \) and \( E2 \) choose \( p_{E1}, p_{E2} \). We find that the only equilibrium outcome is the same which is obtained in the case of “pure bundling vs independent pricing”.

First we notice that if \( P_I > p_{I1} + p_{I2} \), then no type of consumer buys the bundle of the incumbent, as buying \( I \)'s two separate products is less expensive. However, the profit of \( I \) is unchanged if instead he lowers \( P_I \) to satisfy \( P_I = p_{I1} + p_{I2} \): if a consumer chooses to buy both products of the incumbent, the latter receives the same amount of money regardless of whether the consumer buys the products separately or as a bundle. Thus, without loss of generality, we assume that \( P_I \leq p_{I1} + p_{I2} \).

As a consequence, each consumer chooses among the following four alternatives: \( II \) (which means buying the incumbent’s bundle), \( IE \) (which means buying product 1 from \( I \) and the product of \( E2 \)), \( EI \) (which means buying product 2 from \( I \) and the product of \( E1 \)), \( EE \) (which means buying the product of \( E1 \) and the product of \( E2 \)). Precisely, for a consumer with type \((x_1, x_2)\), that is with switching costs \( s_1 = x_1 + a \) and \( s_2 = x_2 + a \), the cost of alternative \( II \) is \( P_I \); the cost of \( IE \) is \( p_{I1} + p_{E2} + x_2 + a \); the cost of \( EI \) is
\( p_{E1} + x_1 + a + p_{I2}; \) the cost of \( EE \) is \( p_{E1} + x_1 + a + p_{E2} + x_2 + a. \) We denote with \( S_{II} \) (respectively, with \( S_{IE}, S_{EI}, S_{EE} \)) the set of types for which \( II \) (respectively, \( IE, EI, EE \)) is the least expensive alternative.

We find that the four sets \( S_{II}, S_{IE}, S_{EI}, S_{EE} \) are all non-empty if and only if the following conditions are satisfied

\[
\max\{p_{I1} + p_{E2} + a, p_{I2} + p_{E1} + a\} \leq P_I \leq p_{I1} + p_{I2} \\
1 + p_{E1} + a \geq p_{I1} \geq p_{E1} + a, \quad 1 + p_{E2} + a \geq p_{I2} \geq p_{E2} + a
\]

(5)

see figure 1 at the end of this document and we let \( A(S_{II}), A(S_{IE}), A(S_{EI}), A(S_{EE}) \) denote the areas of the regions \( S_{II}, S_{IE}, S_{EI}, S_{EE} \). Hence, when (5) holds the profits of \( I \) and \( E1, E2 \) are given by

\[ \Pi_I = P_I A(S_{II}) + p_{I1} A(S_{IE}) + p_{I2} A(S_{EI}) \]
\[ \pi_{E1} = p_{E1} [A(S_{EI}) + A(S_{EE})], \quad \pi_{E2} = p_{E2} [A(S_{IE}) + A(S_{EE})] \]

Obviously, mixed bundling provides greater flexibility for the incumbent with respect to bundling. Nevertheless, and somewhat surprisingly, we find that in any equilibrium the incumbent plays a pure bundling strategy and the same equilibrium outcome of Proposition 2 is obtained.\(^{10}\)

**Proposition 3** (\( n = 2 \): mixed bundling) Suppose that the incumbent can use mixed bundling while facing two entrants. Then the equilibrium outcome coincides with the equilibrium outcome of the game in which the incumbent is restricted to use pure bundling, described by Proposition 2.

This result holds because it rarely occurs that a mixed bundling strategy is a best reply for the incumbent, and in those rare cases no equilibrium exists. More in detail, assume that entrants play the same price \( p_{E1} = p_{E2} = p_E \); then in the proof of Proposition 3 we find that the profit of the incumbent depends on \( p_E \) and \( a \) only through their sum \( p_E + a \), which we denote \( k \). Given \( k \), the best pure bundling strategy of the incumbent turns out to be \( P_I(k) = \frac{4}{3}k + \frac{1}{3}\sqrt{6 + 4k^2} \) with \( p_{I1} \geq P_I(k) - k, p_{I2} \geq P_I(k) - k \). These values of \( p_{I1}, p_{I2} \) imply that \( A(S_{EI}) = A(S_{IE}) = 0 \): each consumer either buys the bundle of \( I \) (if \( x_1 + x_2 > P_I(k) - 2k \)) or both products from the entrants (if \( x_1 + x_2 \leq P_I(k) - 2k \)).

Now we study the profitability for \( I \) of lowering \( p_{I1}, p_{I2} \) such that \( p_{I1} = p_{I2} = p_I \) between \( \frac{1}{2} P_I(k) \) and \( P_I(k) - k \),\(^{11}\) keeping \( P_I = P_I(k) \). This implies that \( A(S_{EI}) >

\(^{10}\)This result contrasts with the results in the literature which allows firms to use mixed bundling, but in that literature firms are often on a symmetric footing.

\(^{11}\)The condition \( p_I \geq \frac{4}{3} P_I(k) \) is equivalent to \( P_I(k) \leq p_{I1} + p_{I2}. \)
0, \( A(S_{IE}) > 0 \), while \( A(S_{II}) \) and \( A(S_{EE}) \) are reduced because a few consumer select alternative \( EI \) or \( IE \), although they choose \( EE \) or \( II \) when \( p_I \geq P_I(k) - k \).

see figure 2 at the end of this document

Precisely, \( S_{IE} \) is the south-east rectangle in the square of figure 2. Region \( X \) represents the set of consumers which buy the bundle of \( I \) when \( p_I \geq P_I(k) - k \), but when \( p_I < P_I(k) - k \) they buy from \( I \) only product 1. On the other hand, region \( Y \) is the set of consumers which buy the products of \( E1, E2 \) when \( p_I \geq P_I(k) - k \), but when \( p_I < P_I(k) - k \) they buy product 1 from \( I \). In order to evaluate the profitability of \( p_I < P_I(k) - k \), notice that mixed bundling generates a loss for \( I \) equal to \( P_I(k) - p_I \) from each consumer in region \( X \), while \( I \) gains \( p_I \) from each consumer in region \( Y \). Since the areas of regions \( X \) and \( Y \) are \( \frac{1}{2}(2 + 3k - P_I(k) - p_I)(P_I(k) - k - p_I) \) and \( \frac{1}{2}(P_I(k) - k - p_I)^2 \) respectively, the net gain for \( I \) is \( (p_I - P_I(k))\frac{1}{2}(2 + 3k - P_I(k) - p_I)(P_I(k) - k - p_I) + p_I \frac{1}{2}(P_I(k) - k - p_I)^2 \). Standard methods shows that, for \( k > 0.0486 \), this function is negative for each \( p_I \) between \( \frac{1}{2}P_I(k) \) and \( P_I(k) - k \). Therefore the loss suffered on consumers in region \( X \) is not compensated by the gain from consumers in region \( Y \), which makes mixed bundling unprofitable for \( I \). In other terms, mixed bundling is inferior to bundling for the incumbent because it generates an internal competition among the products of \( I \) which reduces \( I \)'s revenue from the sale of the bundle more than it increases its revenue from the sale of separate products.

Finally, although mixed bundling is a best reply for \( I \) when \( k \) is small, no equilibrium exists in this case. First, it is obviously impossible that \( k \) is small unless also \( a \) is small. Second, for the case of a small \( a \) we find that no mixed bundling strategy of \( I \) induces the entrants to play \( p_{E1}, p_{E2} \) small.

4.1.3 Comparison

Before we compare the outcome of the game \( II \) with that of the game \( BI \), we here introduce the key effects that we later use to provide intuition about the comparison.

---

12 Because of symmetry, the same arguments below apply to the consumers which now select alternative \( EI \), located in a north-west region of the square which is not depicted in figure 2.

13 This analysis is only suggestive of why mixed bundling is not a best reply for \( I \) except for small values of \( k \), and not a sound proof, because it relies on fixing \( P_I \) at \( P_I(k) \). Conceivably, there could exists a mixed bundling strategy which is a best reply for \( I \) and such that \( P_I \neq P_I(k) \). Lemma 2 in the Proof of Proposition 3 provides a complete proof.

14 Figure 2 suggests an insight on why small values of \( k \) favor mixed bundling. Smaller values of \( k \) move to the right the \( -1 \) sloped line which separates the set of consumers which buy the bundle of \( I \) from the consumers which buy the entrants’ products. This implies that for a given value of \( P_I(k) - k - p_I \), which determines the area of region \( Y \), the area of region \( X \) is smaller, and thus is smaller the probability of loss for the incumbent from playing \( p_I < P_I(k) - k \). However, also the gain in region \( Y \) and the loss in region \( X \) depend on \( k \), and they are both increasing in \( k \).
For this purpose, consider the case in which bundling is prohibited and all prices are exogenously given as follows: \( p_I = p_{I2} = p_1 \) and \( p_{E1} = p_{E2} = p_E \). Then the demand for the product of each entrant is \( \sigma_E \equiv p_I - p_E - a \), and assume that \( \sigma_E \in (0, 1/2) \). Suppose now that the incumbent bundles his two products and offers only a pure bundle at a price \( P_I \).

First, assume that \( P_I = 2p_I \). Then we find that bundling increases the demand for the incumbent; we call this the "demand size effect". Precisely, bundling implies that a consumer with type \((x_1, x_2)\) buys the bundle of \( I \) if and only if \( x_1 + x_2 \leq P_I - 2p_E - 2a = 2\sigma_E \). As a consequence, bundling does not affect the choice of the consumers who bought both products from the incumbent or both products from the entrants under independent pricing. However, it affects the decision of the consumers who bought one product from the incumbent and the other from an entrant. Because of symmetry, let us consider the consumers who bought product 1 from the incumbent and product 2 from \( E2 \): the measure of this set of consumers is \( (1 - \sigma_E)\sigma_E \). Among these consumers, after bundling, only consumers of measure \( \frac{1}{2}\sigma_E^2 \) buy the bundle from the entrants and the others, with measure \( (1 - \sigma_E)\sigma_E - \frac{1}{2}\sigma_E^2 \) buy the bundle from the incumbent. The latter term is larger than the former since \( \sigma_E \in (0, 1/2) \).

Second, given \( p_{E1} = p_{E2} = p_E \), bundling affects the demand elasticity and hence the price charged by the incumbent; we call this the "demand elasticity effect". To explain it, we consider how bundling changes the best response of the incumbent in terms of unit price; notice that for any \( p_E \), this best response is always found in the interval \([p_E + a, p_E + a + 1]\). In the case of independent pricing, the incumbent’s best response is given by (if \( a + p_E \leq 1 \), as it occurs in equilibrium)

\[
BR_I^{II}(p_E) = \frac{1}{2}(1 + a + p_E).
\]

In the case in which the incumbent practices pure bundling, let \( p_I = \frac{1}{2}P_I \) denote the average price chosen by \( I \). Then, the incumbent’s optimal \( p_I \) is given by

\[
BR_I^{BI}(p_E) = \frac{2}{3}(a + p_E) + \frac{1}{6}\sqrt{4(a + p_E)^2 + 6}
\]

and we find

\[
BR_I^{BI}(p_E) \geq BR_I^{II}(p_E) \text{ iff } a + p_E \geq \sqrt{2} - 1.
\]

In particular, if \( p_E = p_E^I(a) \) then \( a + p_E^I(a) \geq \sqrt{2} - 1 \) reduces to \( a \geq 3/\sqrt{2} - 2 = 0.12132 \). Therefore, if entrants play \( p_E^I(a) \) then pure bundling makes the incumbent tougher (resp. softer) for low \( a \) (resp. for high \( a \)).

\[15\] The opposite result obtains if \( \sigma_E > \frac{1}{2} \), but whenever \( p_I \) is chosen optimally by \( I \), the inequality \( \sigma_E < \frac{1}{2} \) holds (see next paragraph).

\[16\] Actually, the incumbent’s total price is the same (i.e. \( P_I^{BI} = 2p_I^{II} \)) when \( a = 0.12132 \).
The demand elasticity effect has to do with the fact that after bundling, what matters is the average switching cost \( a + \frac{1}{2}(x_1 + x_2) \) and its distribution is more concentrated around the mean, \( a + \frac{1}{2} \), than the distribution of the individual switching costs for the single products. Therefore, compared to independent pricing, the demand for the bundle of \( I \) is more elastic for \( p_I \) close to \( p_E + a + \frac{1}{2} \) and it is less elastic for \( p_I \) close to \( p_E + a \) or close to \( p_E + a + 1 \) (i.e., for \( p_I \) far from \( p_E + a + \frac{1}{2} \)), because there are more consumers with average switching cost close to \( a + \frac{1}{2} \), and fewer with average switching cost close to \( a \) or to \( a + 1 \). Given \( p_E = p_{II}^E(a) \), as \( a \) increases we see that \( p_I = p_{II}^I(a) \) gets closer to \( p_E + a \), since for a large \( a \) the market share of each entrant is close to zero (Proposition 1), that is the marginal consumers who are indifferent between buying the incumbent’s bundle and buying the entrants’ bundle have an average switching cost which is close to \( a \), the lowest possible value. This explains why bundling makes the incumbent less aggressive for relatively high \( a \).

Third, the same kind of demand elasticity effect may arise for the two entrants as well, in the sense that the demand for their products is very elastic for \( p_{E1} = p_{E2} = p_E \) close to \( p_I - a - \frac{1}{2} \). However, since entrants choose their prices in an uncoordinated fashion, we find that they behave exactly like under ”independent pricing vs independent pricing”. Precisely, given a price per product equal to \( p_I \), in the case of independent pricing the best response of each entrant is given by:

\[ BR_{II}^I(p_I) = p_I - \frac{a}{2}. \]

Under bundling, given \( P_I = 2p_I \), we study the Nash equilibrium of the pricing game played by the entrants. Then, each entrant’s symmetric best response turns out to be

\[ BR_{II}^II(p_I) = BR_{II}^I(p_I). \]

Therefore, there is no demand elasticity effect on the side of entrants.\(^{17}\)

We now turn to the comparison. We have:

**Proposition 4** \((n=2; BI vs II)\) Suppose that the incumbent practices bundling against two specialist entrants.

(i) Bundling increases the incumbent’s profit: \( \Pi_{II}^{BI} > 2\pi_{II}^{I} \) for all \( a \in [0, 1] \).

(ii) The incumbent’s bundling reduces each entrant’s profit if and only if \( a < 0.54623 \).

The above proposition can be explained easily from the effects we introduced in the beginning of this subsection. First, the incumbent’s profit increases from the demand size

\(^{17}\)As a consequence, bundling does not affect the equilibrium prices when \( a = 0.12132 \), that is the incumbent’s total price is the same and each entrant charges the same price: \( P_I^{BI} = 2p_I^{I} \) and \( p_{E1}^{BI} = p_{E2}^{BI} = p_E^{I} \).
effect: the same effect reduces each entrant’s profit. For \( a < 0.12132 \), bundling induces the incumbent to be more aggressive, which further reduces each entrant’s profit. However, the demand size effect dominates the demand elasticity effect such that bundling increases the incumbent’s profit for \( a < 0.12132 \). On the contrary, for \( a > 0.12132 \), bundling induces the incumbent to be less aggressive with respect to “independent pricing vs independent pricing”, which in turn makes the entrants charge higher prices (since prices are strategic complements). This competition softening effect dominates the demand size effect for \( a > 0.54623 \) and makes even the entrants’ profits higher under bundling.

4.2 Analysis of the case with more than two products

In this section we extend our analysis for the game ”pure bundling vs independent pricing” to the case in which \( n \) is an arbitrary number larger than two.

Given \( n \) random variables \((x_1, ..., x_n)\) uniformly distributed over \([0, 1]^n\), let \( \bar{x}_n = \frac{1}{n}(x_1 + \ldots + x_n) \) denote the arithmetic mean of \((x_1, ..., x_n)\). We use \( F_n \) to denote the c.d.f. of \( \bar{x}_n \), and for any \( t \in [0, 1] \) we find \( F_n(t) = \frac{n}{n!} \sum_{k=0}^{n-1} (-1)^k \binom{n}{k} \max\{t - \frac{k}{n}, 0\}^n \). Given a price \( p_I \) charged by the incumbent for the bundle of his products, let \( \bar{p}_I = \frac{1}{n} \sum_{i=1}^{n} p_{Ei} \) denote the average price for each product in the bundle; likewise, let \( \bar{p}_E = \frac{1}{n} \sum_{i=1}^{n} p_{Ei} \) denote the average price charged by entrants. Then a consumer with type \((x_1, ..., x_n)\) buys the incumbent’s bundle if and only if \( p_I \leq \sum_{i=1}^{n} (p_{Ei} + x_i + a) \), which is equivalent to \( \bar{p}_I \leq \bar{p}_E + \bar{x} + a \). Thus the profit functions are given by

\[
\Pi_I = n\bar{p}_I[1 - F_n(\bar{p}_I - \bar{p}_E - a)];
\]
\[
\pi_{Ei} = p_{Ei} F_n(\bar{p}_I - \bar{p}_E - a) \quad \text{for } i = 1, ..., n
\]

The next proposition characterizes the equilibrium prices.

**Proposition 5** (finite \( n \geq 3 \): BI) Consider competition between the bundle of the incumbent and the entrants’ products given \( n \geq 3 \) number of products. Let \( \delta_n \) be the unique solution to

\[
(n + 1)F_n(\delta) + (\delta + a)F'_n(\delta) = 1
\]

We find that \( \delta_n \in (0, \frac{1}{2}) \), and if \( \frac{1-F_n(\delta_n)}{F_n(\delta_n)} \leq u \) then the equilibrium prices are such that

\[
p_{E1}^{BI} = ... = p_{En}^{BI} = p_E^{BI} \quad \text{and} \quad \bar{p}_I^{BI} - p_E^{BI} - a = \delta_n.
\]

Precisely, \( \bar{p}_I^{BI} = \frac{1-F_n(\delta_n)}{F_n(\delta_n)} \) and \( p_E^{BI} = \bar{p}_E^{BI} = \bar{p}_E = \frac{F_n(\delta_n)}{F_n(\delta_n)} \).

When \( n > 2 \), we find that the demand size effect is larger than when \( n = 2 \) (in fact, this effect is increasing in \( n \)), which increases the profit of \( I \). In addition, for a large \( n \) the average switching cost is extremely concentrated near \( a + \frac{1}{2} \), which in turn makes the demand for the bundle of \( I \) extremely concentrated around \( \bar{p}_I \) equal to \( \bar{p}_E + a + \frac{1}{2} \). This induces \( I \) to play \( \bar{p}_I \) slightly smaller than \( \bar{p}_E + a + \frac{1}{2} \), as in this way he captures almost the
complete market share, while $\bar{p}_I$ larger than $\bar{p}_E + a + \frac{1}{2}$ yields it nearly zero market share. If the entrants could coordinate their price choices, the equilibrium value of $\bar{p}_E$ would be close to zero; otherwise entrants could deviate and earn a large market share by slightly undercutting $I$. However, $n$ uncoordinated entrants are unable to do this, and in fact we find a large coordination failure such that each entrant plays a price larger than $u - a - \frac{1}{2}$; this allows $I$ to attract almost all consumers by playing the monopoly price $u$, regardless of the value of $a$.\textsuperscript{18} This result holds for any value of $a$, and thus no competition softening effect applies when $a$ increases. The consequence is that bundling is a very strong barrier to entry when $n$ is large.

**Proposition 6** (limit: II vs BI) Consider competition between the bundle of the incumbent and the entrants’ products when the number of products tends to infinity.

(i) The incumbent’s price and profit per product converge to $u$; each entrant’s price converges to a value large than $u - a - \frac{1}{2}$, thus his marked share and profit converge to zero; consumers’ surplus per product converges to 0.

(ii) Compared to “independent pricing vs independent pricing”, the incumbent’s profit is higher under bundling, each entrant’s profit and consumers’ surplus are lower under bundling.

**Remark 1** The result of Proposition 6 holds as long as $(x_1,\ldots,x_n)$ are i.i.d. (and not necessarily uniformly distributed) on $[0,1]^n$, with expected value $\mu$, and then $u - a - \frac{1}{2}$ in Proposition 6(i) is replaced by $u - a - \mu$.

5 Competition with a generalist entrant

In this section, we consider competition between a generalist incumbent and a generalist entrant.

5.1 The case of two products

Consider the case of $n = 2$. We study the pricing game in which both the incumbent and the entrant practice pure bundling.

Given $(P_I, P_E)$, a consumer of type $(x_1, x_2)$ is indifferent between buying the incumbent’s bundle and the entrant’s bundle if and only if

\[ P_I = P_E + 2a + x_1 + x_2. \]

\textsuperscript{18} Notice that the prices obtained in Proposition 5 are not the equilibrium prices when $n$ is large. Precisely, Proposition 5 implicitly assumes that $u$ is sufficiently large that all consumers buy all products, but when $n \to +\infty$ we find that $\frac{1-F_n(\delta_n)}{F_n(\delta_n)} \to +\infty$ and $n \frac{F_n(\delta_n)}{F_n(\delta_n)} \to +\infty$. Thus the implicit assumption is violated and Proposition 6 describes the result for the case of a large $n$. 

17
When $0 \leq P_I - P_E - 2a \leq 1$, the demand for the bundle of the entrants is given by
\[
\frac{1}{2}(P_I - P_E - 2a)^2.
\]
In this case, we have the following profit functions:
\[
\Pi_I = \left[ 1 - \frac{1}{2}(P_I - P_E - 2a)^2 \right] P_I;
\]
\[
\Pi_E = \frac{1}{2}(P_I - P_E - 2a)^2 P_E.
\]
It is immediate to find the unique equilibrium prices $P_{BB}^I, P_{BB}^E$.

**Proposition 7** ($n = 2$: bundle vs bundle) Consider competition between the two bundles offered by the incumbent and a generalist entrant.

(i) The incumbent and the entrant charge the following prices:
\[
P_{BB}^I = \frac{1}{4}(3\sqrt{a^2 + 2} + 5a); \quad P_{BB}^E = \frac{1}{4}(\sqrt{a^2 + 2} - a).
\]
$P_{BB}^I$ is increasing in $a$, whereas $P_{BB}^E$ is decreasing in $a$.

(ii) The incumbent’s market share is increasing in $a$ and belongs to $\left[\frac{3}{4}, 1\right)$ for any $a \in [0, 1)$.
The profits are
\[
\Pi_{BB}^I = \frac{1}{16}(9 + 2a^2)\sqrt{a^2 + 2} + \frac{1}{16}(21 - 2a^2)a;
\]
\[
\Pi_{BB}^E = \frac{1}{32}(\sqrt{a^2 + 2} - a)^3
\]

(iii) Consumers’ surplus is given by
\[
CS_{BB} = 2u - \frac{1}{24} \left( 33a + 2a^3 + (17 - 2a^2)\sqrt{a^2 + 2} \right)
\]

Before we compare ”bundle vs bundle” with ”independent pricing vs independent pricing”, we perform the easier comparison between ”bundle vs independent pricing” and ”bundle vs bundle”. Then, we have:

**Proposition 8** ($n = 2$: BI vs BB) When we compare ”bundling vs independent pricing” with ”bundle vs bundle” we find

(i) the entrants’ bundling reduces the incumbent’s profit: $\Pi_{BI}^I > \Pi_{BB}^I$ for all $a \in [0, 1]$.

(ii) the entrants’ bundling reduces each entrant’s profit: $2\pi_{BI}^E > \Pi_{BB}^E$ for all $a \in [0, 1]$.

\(^{19}\)The proof of Proposition 7 establishes that these conditions are satisfied in equilibrium.
The above proposition is easy to understand. First, each specialist entrant does not internalize the positive externality that it provides to the other entrant when it reduces its price. Therefore, given the price of the incumbent, the specialist entrants charge higher prices than a generalist entrant. This in turn induces the incumbent to charge a higher price for its bundle under "bundling vs independent pricing" than under "bundle vs bundle". Therefore, all prices are higher under "bundling vs independent pricing" than under "bundle vs bundle".

We now turn to our main comparison. Is competition fiercer with "bundle vs bundle" than with "independent pricing vs independent pricing"? We have:

**Proposition 9** ($n = 2$: II vs BB) When we compare "independent pricing vs independent pricing" with "bundle vs bundle",

(i) The incumbent’s profit is higher under "bundle vs bundle" if and only if $a > 0.20743$: $\Pi_{BB}^I > 2\pi_1^I$ for $a > 0.20743$.

(ii) Each entrant’s profit is higher under "bundle vs bundle" if and only if $a > 0.68794$.

(iii) Consumers’ surplus is higher under "bundle vs bundle" if and only if $a < 0.37602$.

Basically, with respect to the comparison between "independent pricing vs independent pricing" and "bundling vs independent pricing", there is only one new effect that is generated by the comparison between "independent pricing vs independent pricing" and "bundle vs bundle". This is the "externality internalization" effect: a generalist entrant is more aggressive than two specialist entrants because the former internalizes positive demand externalities. In other words, the combination of the "externality internalization" effect with the "demand size" and the "demand elasticity" effects from the previous comparison determine the comparison between "independent pricing vs independent pricing" and "bundle vs bundle."

First, the demand size effect means that bundling increases the incumbent’s profit while decreasing the entrants’ profits. Second, the demand elasticity effect suggests that for relatively low $a$, the incumbent becomes more aggressive under bundling. These effects together with the "externality internalization effect" makes the entrant’s profit smaller under bundling for $a$ small. Actually, the latter two effects make even the incumbent’s profit smaller under bundling for $a < 0.20743$. However, as $a$ increases, the competition softening effect dominates all other effects such that every firm’s profit is higher under bundling for $a > 0.68794$. Since the incumbent benefits for the demand size effect, the incumbent’s profit is higher under bundling for $a > 0.20743$. Finally, consumers’ surplus is lower under bundling for $a > 0.37602$. Therefore, our analysis provides quite a nuanced picture about the conventional wisdom that competition between bundles is fiercer than competition among individual products.
5.2 The case of more than two products

In this section we extend our analysis for the game “bundle vs bundle” to the case in which \( n \) is an arbitrary number larger than two.

As in Section 4.2, a consumer with type \((x_1, ..., x_n)\) buys the incumbent’s bundle if and only if \( P_I \leq P_E + \sum_{i=1}^{n} (x_i + a) \), which is equivalent to \( \bar{p}_I \leq \bar{p}_E + \bar{x} + a \), in which \( \bar{p}_I \equiv \frac{1}{n} P_I \), \( \bar{p}_E \equiv \frac{1}{n} P_E \) and \( \bar{x} \) is the arithmetic mean of \((x_1, ..., x_n)\), with c.d.f. \( F_n \). Thus

\[
\Pi_I = n \bar{p}_I [1 - F_n(\bar{p}_I - \bar{p}_E - a)], \quad \Pi_E = \bar{p}_E F_n(\bar{p}_I - \bar{p}_E - a).
\]

The next proposition characterizes the equilibrium prices.

**Proposition 10** (finite \( n \geq 3 \): BB) Consider competition between the bundle of the incumbent and bundle of the generalist entrant given \( n \geq 3 \) number of products. Let \( \hat{\delta}_n \) be the unique solution to

\[
2F_n(\delta) + (\delta + a)F'_n(\delta) = 1
\]

Then \( 0 < \hat{\delta}_n < \frac{1}{2} \) and the equilibrium prices are such that \( \bar{p}_I^{BB} - \bar{p}_E^{BB} - a = \hat{\delta}_n \). Precisely, \( \bar{p}_I^{BB} = \hat{\delta}_n + F_n(\hat{\delta}_n) + a \) and \( \bar{p}_E^{BB} = F_n(\hat{\delta}_n) \).

When \( n \) is large, a few arguments mentioned in Section 4.2 apply to this setting: the demand size effect is large and the average switching cost is very concentrated around \( a + \frac{1}{2} \), so that I can win over almost all consumers with \( \bar{p}_I \) slightly smaller than \( \bar{p}_E + a + \frac{1}{2} \). However, now we find that the equilibrium value of \( \bar{p}_E \) must be close to zero, otherwise the entrant could profitably undercut the incumbent; this leads to Proposition 11.

**Proposition 11** Consider the competition between the bundle of the incumbent and the bundle of the generalist entrant when the number products tends to infinity. Then

(i) The entrant’s market share tends to zero (and thus also his profit), the incumbent price and profit per product tends to \( \frac{1}{2} + a \), and consumers’ surplus per product tends to \( u - (\frac{1}{2} + a) \).

(ii) Compared to "independent pricing vs independent pricing"

(a) The incumbent’s profit is always higher under ”bundle vs bundle” while any entrant’s profit is always lower under ”bundle vs bundle”.

(b) Consumer surplus is lower under ”bundle vs bundle” if and only if \( a > 0.19615 \).

This proposition implies that for a large \( n \), bundling is a strong barrier to entry for a generalist entrant as it is for specialist entrants. Since this result holds for any \( a \), also in this setting there is no competition softening effect from the change in demand elasticity. Moreover, since the demand size effect is large when \( n \) is large, the incumbent’s profit is always larger under ”bundle vs bundle” than when bundling is banned. For this
reason, the consumers' surplus is likely to be smaller under "bundle vs bundle. Actually, consumers' surplus is smaller under "bundle vs bundle" for \( a > 0.19615 \) while when \( n = 2 \), consumers' surplus is smaller under "bundle vs bundle" for \( a > 0.37602 \).

**Remark 2** The result of Proposition 11 [except part (ii)] holds as long as \((x_1, \ldots, x_n)\) are i.i.d. (and not necessarily uniformly distributed) on \([0, 1]^n\), with expected value \( \mu \), and then \( \frac{1}{2} + a \) in Proposition 11(i) is replaced by \( \mu + a \).

## 6 The case of perfect correlation

In the previous sections, we assumed that the switching costs of a consumer are independently distributed across different products. We now consider the case in which the switching costs are perfectly correlated across different products: each consumer is characterized by a unique \( x \in [0, 1] \) and his switching cost for each product is \( a + x \).

Obviously, perfect correlation does not affect the outcome under "independent pricing vs. independent pricing". Furthermore, it is immediate to see that "independent pricing vs. independent pricing" is equivalent to "bundle vs bundle". To show this, let \( p_I \) and \( p_E \) denote the average prices of the incumbent of the entrant: \( p_I = \frac{1}{n} P_I \) and \( p_E = \frac{1}{n} P_E \). Then, the type \( x \) of the consumer who is indifferent between the two bundles is given by

\[
n p_I = n p_E + n(a + x)
\]

and therefore the demand of each bundle is the same as the demand of each individual product under "independent pricing vs. independent pricing". The firms’ profits are

\[
\Pi_I = n(1 - (p_I - p_E - a))p_I,
\]

\[
\Pi_E = n(p_I - p_E - a)p_E,
\]

which are just the profits under "independent pricing vs. independent pricing" times \( n \).

We now consider "pure bundling vs. independent pricing". Given \( p_I = \frac{1}{n} P_I \) and \( p_{E1}, \ldots, p_{En} \), the type \( x \) of the consumer who is indifferent between the two bundles is given by

\[
n p_I = \sum_{i=1}^{n} p_{Ei} + n(a + x).
\]

The firm’s profits are

\[
\Pi_I = n(1 + a - (p_I - 1/n \sum_{i=1}^{n} p_{Ei}))p_I,
\]

\[
\pi_{Ei} = (p_I - 1/n \sum_{i=1}^{n} p_{Ei} - a)p_{Ei}, \text{ for } i = 1, \ldots, n
\]
The equilibrium prices are given by

\[ P_{BIC}^I = n \frac{1 + n + a}{n + 2}, \quad P_{BIC}^E = \cdots = P_{BIC}^E = \frac{n}{n + 2}(1 - a), \]

where BIC refers to "pure bundling vs. independent pricing" under perfect correlation. The equilibrium profits are given by:

\[ \Pi_{BIC}^I = n \left( \frac{1 + n + a}{n + 2} \right)^2, \quad \pi_{BIC}^E = \cdots = \pi_{BIC}^E = n \left( \frac{1 - a}{n + 2} \right)^2. \]

As a consequence, we find

**Proposition 12** Suppose that each consumer’s switching costs are perfectly correlated across different products.

(i) The outcome of ”independent pricing vs. independent pricing” is equivalent to that of ”bundle vs bundle”.

(ii) If we compare the outcome of ”independent pricing vs. independent pricing” with that of ”pure bundling vs independent pricing”, we have:

(a) The incumbent’s profit is always higher under ”pure bundling vs independent pricing” than under ”independent pricing vs. independent pricing”: \( \Pi_{BIC}^I > n \pi_{II}^I \) for any \( n \geq 2 \) and \( a \in [0, 1) \).

(b) Each entrant’s profit is smaller under ”pure bundling vs independent pricing” than under ”independent pricing vs. independent pricing” if and only if \( n > 4 \) regardless of \( a \): \( \pi_{BIC}^E \preceq \pi_{II}^E \) for \( n \leq 4 \), for any \( a \in [0, 1) \).

In the case of perfect correlation, neither demand size effect nor demand elasticity effect exists. The only effect that remains is no internalization externality that arises under ”pure bundling vs independent pricing”. Therefore, the specialist entrants charge higher prices than they would if they acted cooperatively, which in turn induces the incumbent to respond by charging a higher price. For this reason, the incumbent’s profit is always higher under ”pure bundling vs independent pricing” than under ”independent pricing vs. independent pricing”. In the case of entrants, this competition softening effect from no internalization of externality is dominated by the effect of price miscoordination for \( n > 4 \) such that their profit is lower under ”pure bundling vs independent pricing” than under ”independent pricing vs. independent pricing”.

**References**


7 Appendix

Proof of Proposition 1

(i-ii) Given \( p_E \), we show that the incumbent plays \( p_I \) between \( p_E + a \) and \( p_E + a + 1 \). If \( I \) plays \( p_I \leq p_E + a \), then he captures all the consumers and earns \( p_I \); among these values of \( p_I \), the best alternative for \( I \) is \( p_I = p_E + a \). If instead \( p_I > p_E + a + 1 \), then \( I \) makes no sale and his profit is zero, but \( p_I = p_E + a \) (for instance) is a profitable deviation for him. For \( p_I \in [p_E + a, p_E + a + 1] \), the profit of \( I \) is \( \pi_I = (1 + a + p_E - p_I)p_I \) and is maximized with respect to \( p_I \) at \( p_I = \frac{1}{2}(1 + a + p_E) \) if \( \frac{1}{2}(1 + a + p_E) > p_E + a \), at \( p_I = a + p_E \) if \( \frac{1}{2}(1 + a + p_E) > p_E + a \). However, it is impossible that \( p_I = a + p_E \) and \( p_E > 0 \) in equilibrium, as then \( E \) makes no profit and has an incentive to reduce slightly his price.

A similar argument applies to the entrant: given \( p_I \), \( E \) plays \( p_E \) between \( p_I - a - 1 \) and \( p_I - a \) and thus \( \pi_E = (p_I - a - p_E)p_E \). The profit of \( E \) is maximized at \( p_E = \frac{1}{2}(p_I - a) \) and this equation, jointly with \( p_I = \frac{1}{2}(1 + a + p_E) \) yields the equilibrium prices \( p_I^E, p_E^I \) in (2). The market shares for \( I \) and \( E \) are \((1 + a + p_E^I - p_I^I) = p_I^I \) and \( p_E^I - a - p_E^I = p_E^I \), respectively, thus the equilibrium profits for \( I \) and \( E \) are \((p_I^E)^2 \) and \((p_E^I)^2 \) respectively.

(iii) The average cost borne by consumers is \( \int_{0}^{1} \pi_I^I - p_E^I - a(x + a)dx + \pi_I^I + \pi_E^I \), which is equal to \( \frac{11}{18}a + \frac{1}{18}a - \frac{1}{18}a^2 \).

Proof of Proposition 2

(i) Given \( p_{E1}, p_{E2} \), we prove that the incumbent plays \( P_I \) between \( p_{E1} + p_{E2} + 2a \) and \( p_{E1} + p_{E2} + 2a + 1 \). If \( P_I < p_{E1} + p_{E2} + 2a \), then \( I \) captures all the consumers and earns \( P_I \); among these values of \( P_I \), the best alternative for \( I \) is \( P_I = p_{E1} + p_{E2} + 2a \). If instead \( P_I \) is between \( p_{E1} + p_{E2} + 2a + 1 \) and \( p_{E1} + p_{E2} + 2a + 2 \), then \( \Pi_I = \frac{1}{2}[2 - (P_I - p_{E1} - p_{E2} - 2a)]^2P_I \) and \( \frac{d\Pi_I}{dP_I} = \frac{1}{2}(2 + 2a + p_{E1} + p_{E2} - P_I)(2a + 2 + p_{E1} + p_{E2} - 3P_I) < 0; \) thus it is profitable for \( I \) to reduce \( P_I \) to \( p_{E1} + p_{E2} + 2a + 1 \). For \( P_I \) in the interval

\[20] Playing \( P_I \) larger than \( p_{E1} + p_{E2} + 2a + 2 \) is unprofitable for \( I \) since then he makes no sale.
\[ [p_{E1} + p_{E2} + 2a, p_{E1} + p_{E2} + 2a + 1], \text{ the profit of } I \text{ is given by (3) and is maximized at } P_I = \frac{2}{3}(p_{E1} + p_{E2} + 2a) + \frac{1}{3}\sqrt{6 + (p_{E1} + p_{E2} + 2a)^2}. \]

Since we have just proved that the equilibrium prices satisfy \( p_{E1} + p_{E2} + 2a < P_I < p_{E1} + p_{E2} + 2a + 1 \), the profit functions for the entrants are given in (4) for prices close to the equilibrium prices. Therefore the first order conditions for maximization of \( \pi_{E1} (\pi_{E2}) \) with respect to \( p_{E1} \) (with respect to \( p_{E2} \)) must be satisfied: \( P_I - 3p_{E1} - p_{E2} - 2a = 0 \) and \( P_I - p_{E1} - 3p_{E2} - 2a = 0 \). Combining these conditions with \( P_I = \frac{2}{3}(p_{E1} + p_{E2} + 2a) + \frac{1}{3}\sqrt{6 + (p_{E1} + p_{E2} + 2a)^2} \) we obtain \( P_I^{BI}, p_{E1}^{BI}, p_{E2}^{BI} \).

Finally, we verify that \( p_{E1}^{BI} \) is a best reply for \( E1 \) given \( P_I^{BI}, p_{E1}^{BI}, p_{E2}^{BI} \). The demand for the bundle of the entrants is 0 if \( p_{E1} \geq P_I^{BI} - p_{E2}^{BI} - 2a = \frac{3}{10}\sqrt{10 + 4a^2} - \frac{2}{5}a \); the demand is \( \frac{3}{2}(10\sqrt{10 + 4a^2} - \frac{3}{5}a - p_{E1})^2 \) if \( p_{E1} \leq \frac{3}{10}\sqrt{10 + 4a^2} - \frac{2}{5}a \). Thus \( E1 \)'s profit is \( \frac{3}{2}(10\sqrt{10 + 4a^2} - \frac{3}{5}a) - p_{E1}^2 \) for \( p_{E1} \in [0, \frac{3}{10}\sqrt{10 + 4a^2} - \frac{2}{5}a] \) and is maximized at \( p_{E1} = p_{E1}^{BI} \).

(ii) Given \( P_I^{BI}, p_{E1}^{BI}, p_{E2}^{BI} \), the market share for \( I \) is \( 1 - \frac{1}{2}(P_I^{BI} - p_{E1}^{BI} - p_{E2}^{BI} - 2a)^2 = \frac{4}{5} + \frac{2}{5}a \sqrt{10 + 4a^2} - \frac{4}{5}a^2 \); this expression has value \( \frac{5}{4} \) at \( a = 0 \) and is simple to see that it is increasing in \( a \). Given market shares, profits are straightforward to obtain.

The average cost borne by consumers is \( \int_0^{P_I^{BI} - p_{E1}^{BI} - p_{E2}^{BI} - 2a} \int_0^{P_I^{BI} - p_{E1}^{BI} - p_{E2}^{BI} - 2a - x_1} (2a + x_1 + x_2) dx_2 dx_1 + \Pi_{E1}^{BB} + \Pi_{E2}^{BB} \), which is equal to \( \frac{16}{375}a(30 + a^2) + \frac{1}{375}(145 - 8a^2) \sqrt{10 + 4a^2} \) and is larger than \( 2(\frac{11}{18} + \frac{4}{5}a - \frac{1}{15}a^2) \), consumers’ average cost in case of independent pricing vs independent pricing.

Proof of Proposition 3

We start by describing the conditions which determine the preferred alternative of each consumer

- Type \((x_1, x_2)\) of consumer buys \( H \) if and only if \( x_1 \geq P_I - p_{I2} - p_{E1} - a, \ x_2 \geq P_I - p_{I1} - p_{E2} - a, \ x_1 + x_2 \geq p - p_{E1} - p_{E2} - 2a \). Let \( S_{HI} \) denote the set of types for which these conditions are satisfied.

- Type \((x_1, x_2)\) of consumer buys \( EI \) if and only if \( x_1 \geq p_{I1} - p_{E1} - a, \ x_2 \leq P_I - p_{I1} - p_{E2} - a, \ x_2 - x_1 \leq p_{E1} + p_{I2} - p_{E2} - p_{I1} \). Let \( S_{IE} \) denote the set of types for which these conditions are satisfied.

- Type \((x_1, x_2)\) of consumer buys \( EI \) if and only if \( x_1 \leq P_I - p_{I2} - p_{E1} - a, \ x_2 \geq p_{I2} - p_{E2} - a, \ x_1 - x_2 \leq p_{I1} + p_{E2} - p_{I2} - p_{E1} \). Let \( S_{EI} \) denote the set of types for which these conditions are satisfied.

\[ ^{21} \text{We are maximizing } P - \frac{1}{3}(P-c)^2P \text{ with respect to } P \in [c, c+1] \text{ and the derivative is } -\frac{2}{3}P^2 + 2cP + 1 - \frac{1}{3}c^2. \text{ This is positive in } [c, \frac{4}{3}c + \frac{1}{3}\sqrt{c^2 + 6}] \text{ and negative in } (\frac{4}{3}c + \frac{1}{3}\sqrt{c^2 + 6}, c + 1], \text{ thus } P = \frac{4}{3}c + \frac{1}{3}\sqrt{c^2 + 6} \text{ is the maximum point.} \]

\[ ^{22} \text{Essentially the same argument applies to } E2. \text{ Regarding } I, \text{ we have proved above that } P_I^{BI} \text{ is a best reply for } I \text{ given } p_{E1}^{BI}, p_{E2}^{BI}. \]
• Type \((x_1, x_2)\) of consumer buys \(EE\) if and only if \(x_1 \leq p_{11} - p_{E1} - a, x_2 \leq p_{12} - p_{E2} - a, x_1 + x_2 \leq P_1 - p_{E1} - p_{E2} - 2a\). Let \(S_{EE}\) denote the set of types for which these conditions are satisfied.

For ease of notation, in this proof we use \(p, p_1, p_2\) instead of \(P_1, p_{11}, p_{12}\), and \(q_1, q_2\) instead of \(p_{E1}, p_{E2}\).

We start with the study of the best reply of \(I\). First we notice that given \(q_1, q_2\) played by \(E1\) and \(E2\), without loss of generality we can restrict attention to \((p, p_1, p_2)\) such that

\[
\begin{cases}
\max\{p_1 + q_2 + a, p_2 + q_1 + a\} \leq p \leq p_1 + p_2 \\
q_1 + a \leq p_1 \leq 1 + q_1 + a, \\
q_2 + a \leq p_2 \leq 1 + q_2 + a
\end{cases}
\] (8)

We have justified in Section 8.1 the inequality \(p_1 + p_2 \geq p\), but we notice here that if \(p = p_1 + p_2\) then \(I\) is playing an independent pricing strategy since consumers get no discount from buying \(B_I\) rather than \(1_12_f\). Regarding \(p \geq \max\{p_1 + q_2 + a, p_2 + q_1 + a\}\), notice that if \(p < p_1 + q_2 + a\) then no type of consumer buys \(1_12_E\) since he prefers \(B_I\) for any switching costs \((s_1, s_2)\). But \(I\) can achieve this outcome by reducing \(p_1\) such that \(p = p_1 + q_2 + a\);\(^2\) thus without loss of generality we can assume that \(p \geq p_1 + q_2 + a\). The same argument applies to justify \(p \geq p_2 + q_1 + a\).

About \(p_1 \leq 1 + q_1 + a\), we observe that if \(p_1 > 1 + q_1 + a\) then no type of consumer buys \(1_I\) since \(1_E\) is less expensive for any \((s_1, s_2)\) and \(I\) can achieve the same outcome by reducing \(p_1\) such that \(p_1 = 1 + q_1 + a\), and thus we can assume that \(p_1 \leq 1 + q_1 + a\). Likewise, if \(p_2 > 1 + q_2 + a\) then \(I\) can equivalently set \(p_2 = 1 + q_2 + a\). Regarding \(p_1 \geq q_1 + a\), notice that if \(p_1 < q_1 + a\) then each type of consumer prefers \(1_I\) to \(1_E\), but that can also be achieved with \(p_1 = q_1 + a\). A consequence of (8) is

\[2 + q_1 + q_2 + 2a \geq p \geq q_1 + q_2 + 2a\]

Under conditions (8), each region \(S_{II}, S_{IE}, S_{EI}, S_{EE}\) is non-empty (although its measure could be zero). Precisely, we find

\[
\begin{align*}
A(S_{II}) &= (1 - p + p_2 + q_1 + a)(1 - p_2 + q_2 + a) + \frac{1}{2}(2 + 2q_1 + 2a - p + p_2 - p_1)(p_1 + p_2 - p); \\
A(S_{IE}) &= (p - p_1 - q_2 - a)(1 - p_1 + q_1 + a); \\
A(S_{EI}) &= (p - p_2 - q_1 - a)(1 - p_2 + q_2 + a); \\
A(S_{EE}) &= (p_1 - q_1 - a)(p - p_1 - q_2 - a) + \frac{1}{2}(p_1 + p - p_2 - 2q_1 - 2a)(p_1 + p_2 - p)
\end{align*}
\]

and therefore,

\[\pi_I = pA(S_{II}) + p_1A(S_{IE}) + p_2A(S_{EI})\]

\(^{23}\)In this case there are a few types of consumers which are indifferent between \(B_I\) and \(1_12_E\), but this set of consumers has zero measure.
and then

\[ \pi \]

As we have claimed above, we can think that independent pricing is never a best reply for \( p_1 \neq p_2 \). Precisely, if we set \( q_1 = q_2 = q \) in \( \pi_I \) we obtain

\[
\pi_I = (1 + q + a)^2 p - (q + a)(1 + q + a)p_1 - (1 + q + a)(q + a)p_2 - 2(1 + q + a)p^2 - p_1^2
\]

\[
- p_2^2 + 2(1 + q + a)pp_1 + 2(1 + q + a)pp_2 + \frac{1}{2} p^3 + p_1^3 + p_2^3 - 3 \frac{1}{2} pp_1^2 - 3 \frac{1}{2} pp_2^2
\]

and thus \( \pi_I \) can be written as \( f(p_1, p) + f(p_2, p) \), with

\[
f(p_1, p) = \frac{1}{2} (1 + q + a)^2 p - (1 + q + a)p^2 + \frac{4}{5} p^3 - (q + a)(1 + q + a)p_1 - p_1^2 + 2(1 + q + a)pp_1 + p_1^3 - 3 \frac{1}{2} pp_1^2, \]

which implies that \( I \) can maximize \( \pi_I \) by setting \( p_1 = p_2 \). Therefore we study the problem of maximizing

\[
\pi_I = (1 + q + a)^2 p - 2(1 + q + a)p_1 - 2(1 + q + a)p^2 - 2p_1^2 + 4(1 + q + a)pp_1 + \frac{1}{2} p^3 + 2p_1^3 - 3pp_1^2
\]

One useful property of this function is that it depends on \( q, a \) only through their sum \( q + a \), which we denote \( k \) in the following. Thus we obtain

\[
\pi_I = (1 + k)^2 p - 2k(1 + k)p_1 - 2(1 + k)p^2 - 2p_1^2 + 4(1 + k)pp_1 + \frac{1}{2} p^3 + 2p_1^3 - 3pp_1^2
\]

and in view of (8) we maximize \( \pi_I \) with respect to \( (p_1, p) \) in the set \( A = \{(p_1, p): p_1 + k \leq p \leq 2p_1 \text{ and } k \leq p_1 \leq 1 + k\} \)

see figure 3 at the end of this document

As we have claimed above, we can think that \( I \) plays independent pricing if \( p = 2p_1 \) (a point in the north-west border of \( A \)), and then \( \pi_I = 2(1 + k - p_1)p_1 \); \( I \) plays pure bundling if \( p \in (2k, 1 + 2k) \) and \( p - k = p_1 \) (the south-east border of \( A \)), and then

\[
\pi_I = [1 - \frac{1}{2}(p - 2k)^2]p, \text{ or if } p \in [1 + 2k, 2 + 2k] \text{ and } p_1 = 1 + k \text{ (the east border of } A), \text{ and then } \pi_I = \frac{1}{2} (2k + 2 - p)^2 p; \]

\( I \) plays mixed bundling if \( (p_1, p) \) is in the interior of \( A \), and then \( \pi_I \) is given in (9).

**Lemma 1** Independent pricing is never a best reply for \( I \).

**Proof** Given \( k \) we can view the pricing problem of \( I \) as the problem of a monopolist which offers two goods and faces consumers with additive valuations for the two goods which are uniformly distributed over \([k, k + 1]^2\). Then Corollary 1 in McAfee et al. (1989) implies that independent pricing is never a best reply for \( I \).

**Lemma 2** If \( k \) is such that \( I \)'s best reply is mixed bundling, then \( k \leq 0.0652 \) and the optimal \((p_1, p)\) is such that \( p_1 > \frac{2}{3}, 1 + 2k > p > p_1 + k \).
Proof In order for a mixed bundling strategy to be a best reply for $I$ it is necessary that \( \frac{\partial \pi_I}{\partial p} \) and \( \frac{\partial \pi_I}{\partial p_1} \) both vanish at a point in the interior of $A$. Precisely, we have
\[
\frac{\partial \pi_I}{\partial p} = (1 + k)^2 - 4(1 + k)p + 4(1 + k)p_1 + \frac{3}{2}p^2 - 3p_1^2
\]
\[
\frac{\partial \pi_I}{\partial p_1} = -2k(1 + k) + 4(1 + k)p - 4p_1 + 6p_1^2 - 6pp_1
\]
and we prove that \( \frac{\partial \pi_I}{\partial p} \) and \( \frac{\partial \pi_I}{\partial p_1} \) may both vanish at a point in the interior of $A$ only if $k < 0.0652$.

**Step 1** \( \frac{\partial \pi_I}{\partial p} < 0 \) for any \((p_1, p)\) in the interior of $A$ such that \( p \geq 1 + 2k \).

If \((p_1, p)\) is in the interior of $A$ and \( p \geq 1 + 2k \), then \( p_1 \in \left( \frac{1}{2} + k, 1 + k \right) \) and notice that \( \frac{\partial \pi_I}{\partial p} \) is a convex function of $p$; thus it is maximized with respect to $p \in [1 + 2k; 2p_1]$ at $p = 1 + 2k$ or at $p = 2p_1$. At $p = 1 + 2k$ we find \( \frac{\partial \pi_I}{\partial p} = -3p_1^2 + 4(1 + k)p_1 - 3 - 4k - 2k^2 \); this expression is maximized with respect to $p_1$ at $p_1 = \left\{ \begin{array}{ll} \frac{2}{3} & \text{if } k \leq \frac{1}{3} \\ \frac{1}{2} + k & \text{if } k > \frac{1}{3} \end{array} \right.$, and the resulting value of \( \frac{\partial \pi_I}{\partial p} \) is \( \left\{ \begin{array}{ll} -\frac{1}{6} - \frac{4}{3}k + \frac{1}{3}k^2 < 0 & \text{if } k \leq \frac{1}{3} \\ -\frac{1}{4} - k < 0 & \text{if } k > \frac{1}{3} \end{array} \right.$ which is negative in any case.

At $p = 2p_1$ we find \( \frac{\partial \pi_I}{\partial p} = (1 - p_1 + k)(k + 1 - 3p_1) < 0 \). Therefore \( \frac{\partial \pi_I}{\partial p} < 0 \) for any \((p_1, p)\) in the interior of $A$ such that \( p \geq 1 + 2k \).

**Step 2** If there exists \((p_1, p)\) in the interior of $A$ such that \( p < 1 + 2k \) and \( \frac{\partial \pi_I}{\partial p} = \frac{\partial \pi_I}{\partial p_1} = 0 \), then \( k \leq 0.0652 \).

If \((p_1, p)\) is in the interior of $A$ and \( p < 1 + 2k \), then \( p_1 \in (k, 1 + k) \) and \( p > p_1 + k \). From \( \frac{\partial \pi_I}{\partial p_1} = 0 \) we get \( p = \frac{k + k^2 + 2p_1 - 3p_1^2}{2 + 2k - 3p_1} \) and notice that if \( p_1 \in (k, \frac{2}{3} + \frac{k}{3}) \), then \( p_1 + k - k + k^2 + 2p_1 - 3p_1^2 = k \frac{1 + k - p_1}{2 + 2k - 3p_1} > 0 \). Thus \( p_1 + k > k + k^2 + 2p_1 - 3p_1^2 \) and \( \frac{\partial \pi_I}{\partial p_1} \) does not vanish in the interior of $A$ for \( p_1 < \frac{2}{3}(1 + k) \).

If \( p_1 = \frac{2}{3} + \frac{k}{3} \), then \( \frac{\partial \pi_I}{\partial p_1} = \frac{2}{3}k(1 + k) > 0 \).

In case that \( p_1 > \frac{2}{3}(1 + k) \), we notice that \( p = \frac{3p_1^2 - 2p_1 - k - k^2}{3p_1 - 2 - 2k} < 1 + 2k \) is equivalent to \( p_1 > \frac{2}{3} + k \). We insert \( p = \frac{3p_1^2 - 2p_1 - k - k^2}{3p_1 - 2 - 2k} \) into \( \frac{\partial \pi_I}{\partial p} = 0 \) to find that it is equivalent to \( g_k(p_1) = 0 \) with
\[
g_k(p_1) \equiv -27p_1^4 + 36(2k + 1)p_1^3 + 6(1 - 12k^2 - 13k)p_1^2 + 4(8k^3 + 13k^2 - k - 6)p_1 + 8 - 10k^3 + 3k^2 + 16k - 5k^4
\]

We show that the equation \( g_k(p_1) = 0 \) has no solution in \( (\frac{2}{3} + k, 1 + k) \) if \( k > 0.0652 \).

Notice that (i) \( g_k(\frac{2}{3} + k) = -\frac{1}{3}k^2(8k + 1) < 0 \); (ii) \( g_k(p_1) = -108p_1^4 + 108(2k + 1)p_1^2 + 12(1 - 12k^2 - 13k)p_1 + 4(-6 + 13k^2 - k + 8k^3) \); (iii) \( g_k''(p_1) = -324p_1^4 + 216(2k + 1)p_1 + 12(1 - 12k^2 - 13k) \); (iv) for \( p_1 \in [\frac{2}{3} + k, 1 + k] \), the maximum point for \( g_k'' \) is \( p_1 = \frac{2}{3} + k \)

---

24 This interval is non-empty if and only if \( k < 2 \).

25 In the case that \( k \geq 2 \), the inequality \( k < \frac{2}{3} + \frac{k}{3} \) is violated.
and \(g_k''(\frac{2}{3} k + k) = -4k^2 < 0 \) for \( k > 0.135 \). When \( k > 0.135 \), \( g_k''(p_1) < 0 \) for \( p_1 \in (\frac{2}{3} k, k) \) and hence \( g_k' \) is decreasing for \( p_1 \in (\frac{2}{3} k, k) \). Since \( g_k'(\frac{2}{3} k + k) = 0 \), this implies \( g_k'(p_1) < 0 \) for \( p_1 \in (\frac{2}{3} k, k) \) and from \( g_k(\frac{2}{3} k + k) < 0 \) we obtain \( g_k(p_1) < 0 \) for \( p_1 \in (\frac{2}{3} k, k) \). Thus \( g_k(p_1) = 0 \) has no solution for \( p_1 \) in \((\frac{2}{3} k, k, 1) \) when \( k > 0.135 \).

When \( k \in (0, 0.135) \), we find that \( g_k''(p_1) > 0 \) for \( p_1 \in \left(\frac{2}{3} k, \frac{2}{3} k + \frac{1}{3} + \frac{1}{3} \sqrt{12 - 3k} \right) \) and \( g_k''(p_1) < 0 \) for \( p_1 \in \left(\frac{2}{3} k + \frac{1}{3} + \frac{1}{3} \sqrt{12 - 3k}, 1 \right). \) Therefore \( g_k' \) is increasing for \( p_1 \in \left(\frac{2}{3} k, \frac{2}{3} k + \frac{1}{3} + \frac{1}{3} \sqrt{12 - 3k} \right) \) and decreasing for \( p_1 \in \left(\frac{2}{3} k + \frac{1}{3} + \frac{1}{3} \sqrt{12 - 3k}, 1 \right). \) We find that \( g_k'(\frac{2}{3} k + \frac{1}{3} + \frac{1}{3} \sqrt{12 - 3k}) = \frac{8}{9} (4 - k) \sqrt{12 - 3k} - 3k - 12 - 4k^2 < 0 \) for \( k > 0.0652 \), which implies that \( g_k(p_1) = 0 \) has no solution for \( p_1 \in (\frac{2}{3} k, 1) \) when \( k > 0.0652 \). When \( k \leq 0.0652 \), if a solution exists then it is such that \( p_1 > \frac{2}{3}, p > p_1 + k \).

**Lemma 3** There exists no NE in which \( I \) plays mixed bundling.

**Proof** If \( I \) plays mixed bundling, then \( E1 \) and \( E2 \) play \( q_1 = q_2 \) with \( q_1 + a \) smaller than \( p - p_1 \). Furthermore, by Lemma 2 \( q_1 + a \) needs to be smaller than 0.0652. We prove that these conditions fail to hold for any \( a > 0 \).

Since \( q_1 + a < p - p_1 \), it follows that \( \frac{1}{2} p < p_1 < p \). Given \( q_2 < p - p_1 - a \), the profit of \( E1 \) from playing \( q_1 \) is

\[
\pi_{E1} = q_1[A(S_{EI}) + A(S_{EE})] = -q_1^2 + (p + pp_1 + pq_2 + pa - p - 2p_pq_2 - a - \frac{1}{2} p^2 - 2pa)q_1
\]

This function is maximized with respect to \( q_1 \) in the interval \([0, p - p_1 - a] \) if and only if \( q_1 = \frac{1}{2} (p + pp_1 + pq_2 + pa - p - 2p_pq_2 - a) < p - p_1 - a \). The same argument applies to \( E2 \): given \( q_1 \) and \( q_2 \) smaller than \( p - p_1 - a \), the profit of \( E2 \) is

\[
\pi_{E2} = q_2[A(S_{IE}) + A(S_{EE})] = -q_2^2 + (p + pp_1 + pq_1 + pa - p - 2p_pq_1 - a - \frac{1}{2} p^2 - 2pa)q_2
\]

and \( \pi_{E2} \) is maximized with respect to \( q_2 \) in the interval \([0, p - p_1 - a] \) if and only if \( q_2 = \frac{1}{2} (p + pp_1 + pq_1 + pa - p - 2p_pq_1 - a - \frac{1}{2} p^2 - 2pa) < p - p_1 - a \). From \( q_1 = q_2 \) we find \( q_1 + a = q_2 + a = h(p, p_1) = \frac{p}{2} + \frac{a - p_1}{2p + a - p} \) and we prove that this expression is larger than 0.0652, which is inconsistent with \( I \) playing mixed bundling. First notice that \( p < 1 + 2(q_1 + a) \) in a mixed bundling equilibrium by Lemma 2, thus \( p < 2 \) since \( q_1 + a \leq 0.0652 \). Since \( \frac{\partial h}{\partial p_1} = \frac{p - 2a - 2}{(2p + 2 - p_1)^2} \), it follows that \( h \) is decreasing in \( p_1 \) and \( h(p_1, p) > h(p, p) = \frac{p^2 + 2a}{2p + 4} \) as \( p_1 < p \). The latter expression is increasing in \( p \) (given that \( p > \frac{2}{3} \)), hence \( \frac{p^2 + 2a}{2p + 4} < h(p, p) = \frac{2}{3} + \frac{\frac{2}{3} a}{2p + 4} \) for any \( a \in [0, 1] \).

**Lemma 4** The profile of strategies \( p = P_i^{RI}, p_1 \) high, \( p_2 \) high, \( q_1 = P_i^{RI}, q_2 = P_i^{RI} \) constitute a NE; each NE is outcome equivalent.

**Proof** Denote the equilibrium prices by \( \hat{p}, \hat{p}_1, \hat{p}_2, \hat{q}_1, \hat{q}_2 \). We know from Lemmas 1 and 3 that \( I \) plays pure bundling in each NE. That implies that given \( \hat{q} = \hat{q}_1 = \hat{q}_2 \), \( I \) plays

\[
\hat{p} = \frac{4}{3} (\hat{q} + a) + \frac{1}{3} \sqrt{6 + 4(\hat{q} + a)^2}
\]  

(10)
and $\hat{p}_1 \geq \hat{p} - \hat{q} - a$, $\hat{p}_2 \geq \hat{p} - \hat{q} - a$. However, the precise values of $\hat{p}_1, \hat{p}_2$ may be relevant. In case that $\hat{p}_1 > \hat{p} - \hat{q} - a$ and $\hat{p}_2 > \hat{p} - \hat{q} - a$, the profit function of $E1$ is $\frac{1}{2}(\hat{p} - q_1 - \hat{q} - 2a)^2q_1$ for $q_1 \geq \hat{p} - \hat{q} - a$ and $q_1 = \hat{q}$ strictly satisfies this inequality. Thus it is necessary that the first order condition for maximization of $\frac{1}{2}(\hat{p} - q_1 - \hat{q} - 2a)^2q_1$ is satisfied at $q_1 = \hat{q}$, that is $\hat{q} = \frac{1}{4}(\hat{p} - \hat{q} - 2a)$, or equivalently $\hat{q} = \frac{1}{4}(\hat{p} - 2a)$. Combining with (10) we obtain the equilibrium prices in Proposition 2.

Given $\hat{q} = \frac{1}{10}(\sqrt{10 + 4a^2} - 2a)$, from Lemmas 1 and 2 we know that $I$’s best reply is pure bundling with $\hat{p} = \frac{2}{5}(3a + \sqrt{10 + 4a^2})$ since $0.0652 < \hat{q} + a < 1.543$ for any $a \in [0, 1)$.

Regarding the entrants, notice that if $\hat{p}_1$ is not much larger than $\hat{p} - \hat{q} - a$, then a conceivably profitable deviation for $E1$ consists of reducing $q_1$, so that a few consumers buy $1_{E2I}$. Likewise, a similar deviation may be profitable for $E2$. Moreover, if $\hat{p}_1, \hat{p}_2$ are large then no type of consumer is interested in buying $1_{2E}$ and $1_{E2}$, and rather chooses between $1_{2I}$ and $1_{E2E}$. Therefore, $\hat{p} = \frac{2}{5}(3a + \sqrt{10 + 4a^2})$, $\hat{p}_1$ and $\hat{p}_2$ high, and $\hat{q} = \frac{1}{10}(\sqrt{10 + 4a^2} - 2a)$ is a NE.

Now we show that no different equilibrium exists such that $\hat{p}_1 = \hat{p} - \hat{q} - a$ and $\hat{p}_2 \geq \hat{p} - \hat{q} - a$ by proving that a profitable deviation exists for entrant 2.

We know that for $q_2$ slightly above $\hat{q}$, the profit of $E2$ is $\frac{1}{2}(\hat{p} - q_2 - 2a)^2q_2$ and its (right) derivative at $q_2 = \hat{q}$ is $\alpha \equiv \frac{1}{4}(\hat{p} - 2\hat{q} - 2a)(\hat{p} - 4\hat{q} - 2a)$. In equilibrium it is necessary that $\alpha \leq 0$, and $\hat{p} - 2\hat{q} - 2a > 0$ given (10). Thus $\alpha \leq 0$ if and only if $\hat{p} \leq 4\hat{q} + 2a$, or equivalently

$$\hat{q} \geq \frac{1}{6}(\sqrt{6 + 4(q + a)^2} - \frac{1}{3}(q + a)) \tag{11}$$

On the other hand, for $q_2$ slightly below $\hat{q}$ the profit of $E2$ is computed by evaluating $A(S_{1E}) = (\hat{p} - \hat{p}_1 - q_2 - a)(1 - \hat{p}_1 + \hat{q} + a)$ and $A(S_{EE}) = \frac{1}{2}(2\hat{p} - \hat{p}_1 - \hat{q} - 3a - 2q_2)(\hat{p}_1 - \hat{q} - a)$; thus $\pi_{E2} = q_2(A(S_{1E}) + A(S_{EE})) = -\frac{q_2}{2} + (\hat{p} - \hat{p}_1 - \hat{q} - a + a\hat{q} + \frac{1}{2}a^2 - \hat{p}_1 a + \frac{1}{2}\hat{q}^2)q_2$. The (left) derivative of $\pi_{E2}$ at $q_2 = \hat{q}$ is $\beta = -2\hat{q} + \hat{p} - \hat{p}_1 + \frac{1}{2}\hat{p}_1^2 - \hat{p}_1 \hat{q} - a + a\hat{q} + \frac{1}{2}a^2 - \hat{p}_1 a + \frac{1}{2}\hat{q}^2$, and taking into account (10) and $\hat{p}_1 = \hat{p} - \hat{q} - a$ we obtain $\beta = \frac{1}{3} + \frac{4}{9}(\hat{q} + a)^2 - \frac{2}{3}(\hat{q} + a)\sqrt{6 + 4(\hat{q} + a)^2} - \frac{2}{3}\sqrt{6 + 4(\hat{q} + a)^2} - \frac{1}{3}(\hat{q} + a)$, and we prove that the latter expression is negative; this implies that reducing slightly $q_2$ below $\hat{q}$ is a profitable deviation exists for $E2$. Let $k = q + a$ and notice that $\frac{1}{3} + \frac{4}{9}k^2 - \frac{2}{3}k\sqrt{6 + 4k^2} - \frac{1}{3}\sqrt{6 + 4k^2} + \frac{1}{3}k < 0$ is equivalent to $(\frac{1}{3} + \frac{4}{9}k^2 + \frac{1}{3}k)^2 < ((\frac{2}{3}k + \frac{1}{3})\sqrt{6 + 4k^2})^2$, which reduces to $\frac{2}{3}k > \frac{1}{10} > 0$.

Proof of Proposition 4

(i) The inequality $\Pi_I > 2\pi_I^H$ is equivalent to $\frac{4}{125}(a^2 + 10)\sqrt{10 + 4a^2} > \frac{2}{9}(2 + a)^2 - \frac{8}{125}a(20 - a^2)$, in which the right hand side is positive for every $a \in [0, 1)$. Squaring both
sides yields the equivalent inequality \( \frac{1308}{10125} + \frac{149}{2025} a + \frac{224}{10125} a^2 + \frac{608}{10125} a^3 + \frac{188}{2025} a^4 - \frac{32}{1125} a^5 > 0 \), which holds for any \( a \in [0, 1] \) since \( \frac{1308}{10125} > \frac{32}{1125} \).

(ii) The inequality \( \pi_{EI}^{BL} < \pi_{EI}^{I} \) is equivalent to \( \frac{1}{250} (8a^2 + 5) \sqrt{10 + 4a^2} < \frac{1}{9} (1 - a)^2 + \frac{1}{250} a (8a^2 + 15) \), and squaring both sides we obtain the equivalent inequality \( f(a) \equiv \frac{169}{20250} - \frac{46}{10125} a + \frac{14}{675} a^2 - \frac{86}{10125} a^3 - \frac{163}{10125} a^4 + \frac{16}{1125} a^5 > 0 \), with \( f(0) = 169 \frac{20250}{20250} > 0 \), \( f(1) = -\frac{1}{250} < 0 \). Furthermore, \( f'(a) = \frac{2}{10125} (1 - a) (-115 + 95a - 34a^2 - 360a^3) < 0 \) for any \( a \in [0, 1] \) and thus a unique solution to \( f(a) = 0 \) exists; numerical methods suggest that the solution is about 0.54623.

**Proof of Proposition 5**

The first order conditions are \( 1 - F_n(p_I - \tilde{p}_E - a) - \tilde{p}_I F_n'(p_I - \tilde{p}_E - a) = 0 \) and \( F_n(p_I - \tilde{p}_E - a) - \frac{1}{n} p_{EI} E_i F_n(p_I - \tilde{p}_E - a) = 0 \) for \( i = 1, ..., n \). Using \( \hat{\delta} \equiv \frac{\tilde{p}_I - \tilde{p}_E - a}{n} \) we can write them as

\[
1 - F_n(\delta) - \frac{1}{n} p_{EI} E_i F_n(\delta) = 0, \\
F_n(\delta) - \frac{1}{n} p_{EI} E_i F_n'(\delta) = 0 \quad \text{for} \quad i = 1, ..., n
\]

Add up the FOC for the \( n \) entrants and obtain \( nF(\delta) - \tilde{p}_E F'(\delta) = 0 \). Subtracting the latter equation from the FOC for the incumbent yields (6).

Let \( \hat{\delta}_n \) be such that \( F_n(\hat{\delta}_n) = \frac{1}{n+1} \). Then there is no solution to (6) for \( \delta > \hat{\delta}_n \) since \( F_n'(\delta) > 0 \). However, a unique solution \( \hat{\delta}_n \) exists in \( (0, \hat{\delta}_n) \) because \( \hat{\delta}_n < \frac{1}{2} \) and both \( F_n \) and \( F_n' \) are strictly increasing in \( (0, \hat{\delta}_n) \).

Given \( \hat{\delta}_n \), the equilibrium prices \( p_I^{BI}, p_{EI}^{BI}, ..., p_{En}^{BI} \) are straightforward to obtain.

**Proof of Proposition 6**

Assume that \( x_1, ..., x_n \) i.i.d., each with support \([0, 1]\), expectation \( \mu \) and standard deviation \( \sigma \). For large \( n \) we replace the c.d.f. of \( x_n \) with its normal approximation. We denote with \( \Phi_n \) the c.d.f. of a normal random variable with expectation \( \mu \) and standard deviation \( \sigma_n \equiv \frac{\sigma}{\sqrt{n}} \); then \( \Phi_n(x) = \frac{1}{\sigma \sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \) and \( \Phi_n(x) = \int_{-\infty}^x \frac{1}{\sqrt{2\pi}} e^{-\frac{(z-\mu)^2}{2\sigma^2}} dz \).

Thus we consider

\[
(n + 1) \Phi_n(\delta) + (\delta + a) \Phi_n'(\delta) = 1
\]

instead of (6). For each \( n \), there exists a unique solution \( \hat{\delta}_n \) to (12), and \( \hat{\delta}_n \in (0, \mu) \). For a general sequence \( \{a_n\} \), we write \( a_n \to L \) instead of \( \lim_{n \to +\infty} a_n = L \).

The proof is organized in two lemmas. In the first one we prove that (12) yields the equilibrium prices only for finitely many \( n \). In the second one we find the equilibrium prices when \( n \) is large.

**Lemma 1** For a large \( n \), \( \tilde{p}_I \) as derived in Proposition 5 is larger than \( u \); thus the prices in Proposition 5 are not the equilibrium prices.

31
We prove in two steps that \( \bar{p}_I \to +\infty \).

**Step 1** For any large \( n \), \( \delta_n < \bar{\delta}_n \equiv \mu - \sigma_n \sqrt{\frac{3}{2} \ln n} \).

We start by proving that \( (n + 1) \Phi_n(\delta_n) \to +\infty \), and in order to do this we notice that

\[
\Phi_n(\delta_n) = \Phi^s(-\sqrt{\frac{3}{2} \ln n}) = \int_{\sqrt{\frac{3}{2} \ln n} \frac{1}{2\pi} e^{-\frac{1}{2} x^2} dx,
\]

in which \( \Phi^s \) is the c.d.f. of a standard Normal random variable; the second equality holds since the density of the standard normal random variable is symmetric around 0. Thus \( (n + 1) \Phi_n(\delta_n) \to +\infty \) we can treat \( n \) as a continuous variable and apply l'Hopital's rule. Then we obtain

\[
\Phi_n(\delta_n) = \frac{\frac{1}{\sqrt{2\pi}} \int_{\sqrt{\frac{3}{2} \ln n} \frac{1}{2\pi} e^{-\frac{1}{2} x^2} dx}{n+1} = \frac{1}{(n+1)^{1/2}} e^{\frac{1}{2} \left( \frac{3}{2} \ln n \right)^2} \frac{1}{2\pi \ln n},
\]

which is equal to \( \frac{(n+1)^{1/2}}{n^{1/2} \sqrt{4\pi \ln n}} \to +\infty \). Since the left hand side of (12) is increasing with respect to \( \delta \in [0, \mu] \), we infer that the inequality \( \delta_n < \bar{\delta}_n \) holds for any large \( n \), otherwise (12) is violated.

**Step 2** \( \bar{p}_I \to +\infty \).

From the inequality \( \delta_n < \bar{\delta}_n \) we can prove that \( \Phi_n'(\delta_n) \to 0 \): it suffices to notice that

(i) \( \Phi_n'(\delta) \) is increasing with respect to \( \delta \in [0, \mu] \), thus \( \Phi_n'(\delta_n) < \Phi_n'(\bar{\delta}_n) \); (ii) \( \Phi_n'(\delta_n) = \frac{1}{\sigma \sqrt{2\pi}} e^{-\frac{1}{2} \sigma^2 \frac{1}{2} \ln n} = \frac{1}{\sigma \sqrt{2\pi \ln n}} \to 0 \).

Since \( \Phi_n'(\delta_n) \to 0 \), it follows from (12) that \( (n + 1) \Phi_n(\delta_n) \to 1 \) and thus \( \Phi_n(\delta_n) \to 0 \).

Finally, \( \bar{p}_I = \frac{1 - \Phi_n(\delta_n)}{\Phi_n'(\delta_n)} \to +\infty \) since \( \Phi_n(\delta_n) \to 0 \) and \( \Phi_n'(\delta_n) \to 0 \).

**Lemma 2** For a large \( n \), the unique equilibrium is such that \( \bar{p}_I = u \) and \( \bar{p}_E \) is about \( \frac{1}{2}(u - a - \mu) + \frac{1}{2} \sqrt{(u - a + \mu)^2 + 4\sigma^2} \). The market share of I tends to 1, therefore his profit per product tends to the monopoly price \( u \).

For a large \( n \) there exists no equilibrium in which I plays \( \bar{p}_I = u \), thus \( \bar{p}_I = u \) in any equilibrium. First we investigate the entrants’ behavior given \( \bar{p}_I = u \), and then we prove that \( \bar{p}_I = u \) is a best reply for \( I \) given the entrants’ behavior.

**Step 1** Given \( \bar{p}_I = u \), for a large \( n \) the entrants play \( \bar{p}_E \in (u - a - \frac{1}{2}, u - a) \) such that \( \bar{p}_E \) is about equal to \( \frac{1}{2}(u - a - \mu) + \frac{1}{2} \sqrt{(u - a + \mu)^2 + 4\sigma^2} \). Consumers buy the bundle of the entrants if and only if \( \bar{x}_n \leq u - \bar{p}_E - a \). Thus \( \pi_{Ei} = p_{Ei} \Phi_n(u - \bar{p}_E - a) \) and the first order conditions reduce to

\[
g_n(\alpha) = u - a - \alpha \quad (13)
\]

in which \( \alpha \equiv u - \bar{p}_E - a \) and \( g_n(\alpha) \equiv n \Phi_n(\alpha) = n \Phi^s(\frac{u - a - \alpha}{\sqrt{3/2 \ln n}}) = \frac{\sigma}{\sqrt{2\pi} \sqrt{\alpha}} e^{-\frac{1}{2} \alpha^2} dt \). It is useful to prove that \( g_n(\alpha) \to \frac{\alpha^2}{\mu - \alpha} \) for any \( \alpha \in (0, \mu) \); to this end, we can treat \( n \) as a continuous variable and apply l'Hopital’s rule. Then we obtain

\[
\frac{\mu - \alpha}{\sqrt{\alpha^2} \sqrt{3/2 \ln n}}, \text{ which converges to } \frac{\alpha^2}{\mu - \alpha}.
\]

32
Now we consider (13) and notice that $g_n(0) \to \frac{\sigma^2}{\mu} \leq 1$,\footnote{The inequality $\sigma^2 \leq \mu$ holds since $\mu = E(x) \geq E(x^2) - \mu^2$, and $E(x) \geq E(x^2)$ given that the support is $[0, 1]$.} $u-a > 1$, and $g_n(\mu) = \sigma \sqrt{n}$; thus we infer that there exists a solution to (13) in $(0, \mu)$ for a large $n$. Moreover, since any normal distribution is log concave, it follows that the left hand side is increasing in $\alpha$ while the right hand side is decreasing; thus there exists a unique solution $\hat{\alpha}_n$ to (13). Precisely, let $\hat{\alpha} \equiv \frac{1}{2}(u-a+\mu) - \frac{1}{2}\sqrt{(u-a-\mu)^2 + 4\sigma^2}$ be the unique solution to $\frac{\sigma^2}{\mu-\alpha} = u-a-\alpha$ which is smaller than $\mu$; then $\hat{\alpha}_n \to \hat{\alpha}$ and $\bar{p}_E$ is close to $u-a-\hat{\alpha} = \frac{1}{2}(u-a-\mu) + \frac{1}{2}\sqrt{(u-a+\mu)^2 + 4\sigma^2}$ for a large $n$, which implies that $u-a-\mu < \bar{p}_E < u-a$.

**Step 2** Given $\bar{p}_E$ between $u-a-\mu$ and $u-a$, the demand for the bundle of $I$ is $1 - \Phi_n(\bar{p}_I - \bar{p}_E - a)$, with $\frac{d\Pi}{dp_I} = 1 - \Phi_n(\bar{p}_I - \bar{p}_E - a) - \bar{p}_I \Phi_n'(\bar{p}_I - \bar{p}_E - a)$ which is decreasing in $\bar{p}_I$. Since $u-a-\mu < \bar{p}_E < u-a$, we find that $0 < u-\bar{p}_E < u-a$ and thus $\Phi_n(u-\bar{p}_E-a) \to 0$, $\Phi_n'(u-\bar{p}_E-a) \to 0$. Therefore $\frac{d\Pi}{dp_I} > 0$ at $p_I = u$ and this implies that $p_I = u$ is a best reply for $u$. The market share of $I$, $1 - \Phi_n(u - \bar{p}_E - a)$, tends to 1.

**Proof of Proposition 7**

(i) Given $P_E$, we can argue as in the proof of Proposition 2 to prove that $I$ plays $P_I$ between $P_E + 2a$ and $P_E + 2a + 1$, and that $P_I$ is maximized with respect to $P_I$ at $P_I = \frac{2}{3}(P_E + 2a) + \frac{1}{3}\sqrt{6 + (P_E + 2a)^2}$. Since we have just proved that the equilibrium prices satisfy $P_E + 2a < P_I < P_E + 2a + 1$, the profit function of the entrant is $P_E = \frac{1}{2}(P_I - 2a - P_E)^2 P_E$ for prices close to equilibrium prices. Therefore the first order condition for maximization of $P_E$ with respect to $P_E$ needs to be satisfied: $P_I - 3P_E - 2a = 0$, and combining with $P_I = \frac{2}{3}(P_E + 2a) + \frac{1}{3}\sqrt{6 + (P_E + 2a)^2}$ we obtain $P_I^{BB}, P_E^{BB}$.

In order to verify that $P_I^{BB}, P_E^{BB}$ is an equilibrium, we show that $P_E^{BB}$ is a best reply for $E$ given $P_I^{BB}$. The demand for the bundle of the entrant is 0 if $P_E \geq P_I^{BB} - 2a = \frac{3}{4}\sqrt{(a^2 + 2 - a)}$, it is $\frac{1}{2}(P_I^{BB} - P_E - 2a)^2$ if $P_I^{BB} - 2a - 1 \leq P_E < P_I^{BB} - 2a$, and it is $1 - \frac{1}{2}(2 - P_I^{BB} + P_E + 2a)^2$ if $P_I^{BB} - 2a - 2 \leq P_E < P_I^{BB} - 2a - 1$. However, if $a \geq \frac{1}{12}$ we find that $P_I^{BB} - 2a - 1 \leq 0$, and thus $E$ needs to choose $P_E$ between 0 and $P_I^{BB} - 2a$; in this case the optimal $P_E$ is $P_E^{BB}$. If $a < \frac{1}{12}$, then $\Pi_E = P_E[1 - \frac{1}{2}(2 - P_I^{BB} + P_E + 2a)^2]$ for $P_E < P_I^{BB} - 2a - 1$, with $\frac{d\Pi_E}{dp_E} = -\frac{3}{2} P_E^2 + (\frac{3}{2}\sqrt{a^2 + 2} - \frac{5}{2}a - 4)P_E - \frac{25}{16} - \frac{5}{2}a - \frac{9}{16}a^2 + \frac{9}{16}a\sqrt{a^2 + 2} + \frac{3}{2}\sqrt{a^2 + 2}$ which is positive for any $P_E \in [0, P_I^{BB} - 2a - 1]$. Thus also in this case the optimal $P_E$ is $P_E^{BB}$.

(ii) The incumbent’s market share is $1 - \frac{1}{2}(P_I^{BB} - P_E^{BB} - 2a)^2$, which is $\frac{1}{4}(3 - a^2 + a\sqrt{a^2 + 2})$. It is simple to see that $\frac{1}{4}(3 - a^2 + a\sqrt{a^2 + 2})$ is increasing and $\frac{1}{4}(3 - a^2 + a\sqrt{a^2 + 2})$ is decreasing for large $n$. Therefore also in this case the optimal $P_E$ is $P_E^{BB}$.\footnote{Regarding $I$, we have proved above that $P_I^{BB}$ is a best reply for $I$ given $P_E^{BB}$.}
\[ a\sqrt{a^2 + 2} \geq \frac{3}{4} \] for any \( a \in [0, 1) \). The equilibrium profits are \( \Pi_B = \frac{1}{4}(3 - a^2 + a\sqrt{a^2 + 2})P_B \) and \( \Pi_E = (1 - \frac{1}{4}(3 - a^2 + a\sqrt{a^2 + 2}))P_E \).

(iii) The average cost borne by consumers is \( \int_0^{p_n} \Phi_z(\hat{1} - p_E - a) - \int_0^{\hat{p}_1} \Phi_z(\hat{1} - \hat{p}_{E} - a) = 0 \) and \( \Phi_z(\hat{1} - \hat{p}_{E} - a) - \Phi_z(\hat{1} - \hat{p}_{E} - a) = 0 \). Using \( \hat{p}_I = p_{E}\hat{I} - a \) we can write them as

\[
1 - F_n(\delta) - \hat{p}_{I} F_n'(\delta) = 0, \\
F_n(\delta) - \hat{p}_{E} F_n'(\delta) = 0
\]

Subtracting the FOC for the entrant from the FOC of the incumbent we obtain (7). Notice that there is no solution to (7) for \( \delta > \frac{1}{2} \) since \( F_n'(\delta) > 0 \) and \( 2F_n(\delta) > 1 \). However, a unique solution \( \hat{\delta}_n \) exists in \((0, \frac{1}{2})\) because both \( F_n \) and \( F_n' \) are strictly increasing in \((0, \frac{1}{2})\) and the left hand side of (7) has value \( 1 + \left(\frac{1}{2} + a\right) F_n'(\frac{1}{2}) > 1 \) at \( \delta = \frac{1}{2} \).

Proof of Proposition 11

(i) Assume that \( x_1, ..., x_n \) i.i.d., each with support \([0, 1]\), expectation \( \mu \) and standard deviation \( \sigma \). For large \( n \) we replace the c.d.f. of \( \bar{x}_n \) with its normal approximation. We denote with \( \Phi_n \) the c.d.f. of a normal random variable with expectation \( \mu \) and standard deviation \( \sigma_n \equiv \frac{\sigma}{\sqrt{n}} \); then \( \Phi_n'(x) = \frac{1}{\sigma} \sqrt{\frac{2}{\pi}} e^{-\frac{1}{2}(\frac{x-\mu}{\sigma})^2} \) and \( \Phi_n(x) = \int_{-\infty}^{x} \frac{1}{\sigma} \sqrt{\frac{2}{\pi}} e^{-\frac{1}{2}(\frac{z-\mu}{\sigma})^2} dz \).

Thus we consider

\[ 2\Phi_n(\delta) + (\delta + a) \Phi_n'(\delta) = 1 \]  (14)

instead of (7). For each \( n \), there exists a unique solution \( \hat{\delta}_n \) to (14), and \( \hat{\delta}_n \in (0, \mu) \). For a general sequence \( \{a_n\} \), we write \( a_n \to L \) instead of \( \lim_{n \to +\infty} a_n = L \).

We start by proving that \( \hat{\delta}_n \to \mu \). Given an arbitrary \( \varepsilon > 0 \), consider \( \delta = \mu - \varepsilon \) and notice that (i) \( (\mu - \varepsilon + a) \Phi_n'(\mu - \varepsilon) \to 0 \); (ii) \( \Phi_n(\mu - \varepsilon) \to 0 \). Thus for any \( \varepsilon > 0 \) the inequality \( \hat{\delta}_n > \mu - \varepsilon \) holds for a large \( n \); this reveals that \( \hat{\delta}_n \to \mu \).

Given an arbitrary \( k > 0 \), the equation \( \Phi_n'(\delta) = k \) has a unique solution in \((0, \mu)\) which is \( \hat{\delta}_{n,k} \equiv \mu - \sigma_n \sqrt{\ln \frac{1}{2nk^2 \sigma_n^2}} \). Furthermore, if we consider \( k > \frac{1}{\mu + a} \) then we infer that \( \hat{\delta}_n < \hat{\delta}_{n,k} \) for any large \( n \), otherwise (14) is violated. Therefore \( \Phi_n(\hat{\delta}_n) < \Phi_n(\hat{\delta}_{n,k}) = \Phi^*(-\sqrt{\frac{1}{2nk^2 \sigma_n^2}}) \), in which \( \Phi^* \) is the c.d.f. of a standard Normal random variable. Since \( \Phi^*(-\sqrt{\frac{1}{2nk^2 \sigma_n^2}}) \to 0 \), it follows that \( \Phi_n(\hat{\delta}_n) \to 0 \). In view of (14) we infer that \( \Phi_n'(\hat{\delta}_n) \to \frac{1}{\mu + a} \) and thus \( \bar{p}_{E} = \frac{\Phi_n(\hat{\delta}_n)}{\Phi_n'(\hat{\delta}_n)} \to 0 \), \( \bar{p}_{I} = \frac{1}{\Phi_n'(\hat{\delta}_n)} \to \mu + a \).
(ii) It is straightforward to see that $\frac{1}{2} + a > \frac{1}{3}(2 + a)^2$ for any $a \in [0, 1)$. Furthermore, the inequality $u - (\frac{4}{9}a + \frac{11}{18} - \frac{5}{18}a^2) > u - (\frac{1}{2} + a)$ is equivalent to $\frac{5}{9}a + \frac{11}{18}a^2 - \frac{5}{9} > 0$, which holds if and only if $a > 0.19615$.

Proof of Proposition 12

Given $\bar{p}_E$, we can argue like in various other proofs to show that $I$ plays $\bar{p}_I$ between $a + \bar{p}_E$ and $a + \bar{p}_E + 1$, thus $\Pi_I = n(1 + a - \bar{p}_I + \bar{p}_E)\bar{p}_I$. The optimal $\bar{p}_I$ as a function of $\bar{p}_E$ is $\bar{p}_I = \frac{1}{2}(1 + a + \bar{p}_E)$ if this expression is larger than $a + \bar{p}_E$, otherwise the optimal $\bar{p}_I$ is $a + \bar{p}_E$. However, it is impossible that $\bar{p}_I = a + \bar{p}_E$ and $\bar{p}_E > 0$ in equilibrium, as then each entrant makes no profit and has an incentive to reduce slightly his price. Thus $a + \bar{p}_E < \bar{p}_I < a + \bar{p}_E + 1$ in equilibrium, and thus $\pi_{Ei} = (p_I - \bar{p}_E - a)p_{Ei}$ prices close to equilibrium prices. Therefore the first order condition for maximization of $\pi_{Ei}$ needs to hold: $(p_I - \bar{p}_E - a) - \frac{1}{n}\bar{p}_{Ei} = 0$ for $i = 1, \ldots, n$. Combining with $\bar{p}_I = \frac{1}{2}(1 + a + \bar{p}_E)$ we obtain $p_E^{BIC} = n\frac{1-a}{2+n}$, $p_I = \frac{a+n+1}{2+n}$ and thus $P_I^{BIC} = \frac{a+n+1}{2+n}$.