EQUILIBRIUM WAGE/TENURE CONTRACTS[1]

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Abstract

The objective of this study is to analyze and characterize equilibrium in a labor market where firms post wage contract offers and workers - both employed and unemployed - search for better paid job opportunities. Given the environment faced, it is shown that in equilibrium the contract offered by a firm to any employee implies the worker's wage increases with tenure at that firm. Further, although different firms offer different contracts, in equilibrium all contracts can be related to a single wage/tenure contract with different starting points. The results lead to several predictions about the labor market histories of workers and the nature of labor market equilibria that have not been exploited to date.

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During the last ten years or so a literature has developed based on the equilibrium analysis of labor markets where employed workers continue search for better job opportunities. This has led to significant theoretical and empirical insights which have deepened our understanding of how real world markets work (see, Van den Berg (1999) for a survey). Much of this work has been based on the framework developed by Burdett and Mortensen (1989,1998) (hereafter termed B/M).[1]

A critical feature of the B/M framework is that each firm posts a single price - a wage which it pays all of its employees at every point in time. In the context of a relatively standard matching framework with identical firms and workers, equilibrium with on-the-job search implies a non-degenerate distribution of wage offers. The logic is relatively simple. Firms offering a high wage make less profit per employee than those firms offering a lower wage. On-the-job search, however, implies those firms offering higher wages attract more workers and so enjoy a larger steady-state labor force. In equilibrium, all firms obtain the same steady-state profit flow, even though they offer different wages.

The objective is to extend the B/M framework to equilibria where firms do not post a single wage, but post wage contracts. These contracts specify the wage paid to any employee as a function of that worker's tenure at the firm. It will be shown that in equilibrium each firm offers a wage/tenure contract that imply an employee's wage increases smoothly with tenure. As with the B/M study, the market equilibrium is characterized by wage dispersion in that there is a non-degenerate distribution of

[1] An alternative approach, developed by Burdett and Judd (1983), has recently been investigated by Acemoglu and Shimmer (1997).

wage/tenure contracts offered by firms. There is wage dispersion within a firm due to different tenures as well as across the market when controlling for tenure. This leads to new and testable predictions about the nature of markets.

Three restrictions play a critical role in obtaining these results. First, as is common in labor market modelling, we assume there is an imperfect capital market where workers cannot borrow against future earnings. Second, workers are assumed to be risk averse. Third, each firm is assumed not to respond to outside offers received by any of its employees. The first two restrictions are relatively standard, the third is not. Clearly, this latter restriction is not satisfied in some labor markets such as the academic labor market in the U.S.. Nevertheless, there are several reasons to suspect our restriction holds in other labor markets, especially those market where workers are homogenous. First, outside offers may not be observable (or verifiable) by firms. Indeed, why should a firm verify to another firm that it has made a particular offer to a worker? Of course, given offers from other firms are not verifiable, they will be ignored. Second, as we assume throughout that all workers are equally productive, some employees may well become disgruntled should a no more talented and more junior employee receive a higher wage on the basis of some random outside offer. Third, as we know in bargaining theory, it is best to have all the bargaining power. By precommiting to a fixed company wage policy, the firm refuses to bargain with employees over their search rents.

In contrast, suppose a firm does respond to an offer received by one of its workers. How would this process then proceed? Clearly, several alternative modelling restrictions can be made. In an insightful study Robin and Postel-Visnay (2001) propose that the two firms - the worker's current employer and the firm that made the new offer - enter into a Bertrand pricing game. In such a case with identical firms, both firms offer a wage equal to the worker's marginal product. The worker then stays at his or her current employer but yields no further profit to the firm. In addition, the firm making the 'verifiable' outside offer makes no profit.

The basic structure considered here is that a firm precommits to a fixed company wage policy where, with identical workers, the wage paid depends only on the worker's current tenure (or seniority). The firm realises that its employees will receive alternative (non-verifiable) job offers from time to time and will quit if a preferred offer is received. A primary issue is what form does the firm's optimal wage contract take?

It turns out there are two basic forces at work. First, there is an incentive effect where a firm prefers to backload wages in any wage/tenure profile. By offering a worker a smaller wage today but a greater wage at some future date, a firm reduces its current wage bill and also increases an employee's expected return to staying with the firm. This, in turn, reduces the probability the worker quits to a preferred job contact. Second, there is an insurance effect. As workers cannot borrow against future earnings then, ceteris paribus, each risk averse worker prefers a wage/tenure contract which implies a constant wage per period. An optimal contract trades off these two competing effects and implies wages rise smoothly with tenure, limiting to a wage which is strictly less than marginal product.

The above result is of course a partial - about how a given firm behaves in a particular environment. Later in this study we show that such behavior is consistent with a steady-state market equilibrium. At such an equilibrium each firm offers an optimal wage/tenure contract given its correct beliefs about the contracts offered by other firms and the behavior of workers. Further, workers utilize their best search strategy.

An important element of our results is that the equilibrium identified can be characterized by a baseline salary scale. A baseline salary scale is the equilibrium wage/tenure profile of a firm offering the lowest initial wage to those it employs. It is shown that any other firm's wage tenure profile can be described by the baseline salary scale with a different starting point. For example, suppose a firm offers a starting wage that is the same as the baseline salary at (say) 7 months tenure. An optimal contract implies employees at this firm are paid a wage after 3 months equal to the wage paid at 10 months according to the baseline salary scale. And so on. This, of course, leads to a strong testable restriction on possible wage/tenure profiles offered by firms.

In equilibrium, wage dispersion arises for three reasons. First, different unemployed workers obtain different acceptable job offers - some are lucky and obtain a job with a high initial wage, others are not. Second, a worker's wage at a particular firm increases with tenure according to the baseline salary scale. Finally, on-the-job search implies a worker will change job if a more valuable contract offer is received.

A particularly convenient result for applied work is that equilibrium also implies that an employee's current wage is a sufficient statistic describing that worker's contemporaneous quit rate. Of course when receiving an outside offer, a worker considers the relative value of accepting the alternative job offer, which depends on expected future wages at the respective firms. But the baseline salary scale property implies a worker will quit if and only if the initial wage offered at the new firm is strictly greater than the worker's current wage.

A closely related paper is Stevens (2000) who assumes risk neutral workers. When workers are indifferent to risk, there is no reason to smooth payments with tenure. The optimal wage contract is then a step contract where a zero wage is paid until some finite tenure date, after which the worker is paid marginal product. A surprising feature is that a market equilibrium then implies all firms offer the same step contract, and this degenerate outcome then implies no quit turnover.

In the next Section we specify the basic elements of the model. As the derivation of the results is not trivial, things are kept reasonably simple. After briefly describing the optimal quit behavior of workers, we first derive the optimal contract a firm offers within a particular matching environment. We then construct a market equilibrium using a two-step procedure. Given the assumed search behavior of workers, we first describe a non-cooperative wage contract posting game played by firms. Given the equilibrium to that game, we then identify a market equilibrium where the assumed search behavior of workers is indeed optimal (given firm behaviour). Finally the implications of our results are discussed.

1 Basic Framework

Although the major results of this study are presented within the context of a continuous time model, for the present assume that time can be divided into small discrete intervals $dt > 0$.

Suppose a unit mass of workers and firms participate in a labor market. Workers and firms are assumed to be homogeneous such that any firm generates revenue pdt for each worker it employs in interval dt. Both unemployed and employed workers obtain new job offers from time to time. Let λdt denote the probability a new job offer is received by a worker at the end of interval dt. Any job offer is fully described by the wage contract offered by the firm. Such a contract specifies the wage the worker receives as a function of his or her tenure at that firm. As employees are identical, anti-discrimination legislation requires that a firm offers all new hires the same contract.

An unemployed worker becomes employed on receiving a job offer that yields an expected return at least as great as unemployment, whereas an employed worker changes employer on receiving an offer that yields a strictly greater expected return than remaining with his or her current employer.

Employed workers continue to work and receive new offers until they die. Let δdt denote the probability any worker dies (leaves the market for good) in interval dt. Such workers are instantly replaced by new unemployed workers. Unemployed workers obtain bdt unemployment insurance payment in interval dt.

2 The Workers

All we need to know about the firms at present is that each firm is assumed to post a contract $\widehat{w} = \{w(\tau)\}_{\tau=0}^{\infty}$, where $w(\tau)dt$ specifies the amount it pays any employee with tenure τ for the following period dt.

As stated previously, we assume throughout that the worker cannot borrow against future earnings. As wages will be monotonically increasing with age, it will not be optimal for workers to save for future consumption. Hence we assume workers are always liquidity constrained. The worker maximizes expected lifetime utility, where given the worker consumes income wdt in period t, the worker obtains one period utility $u(w)dt$. Assume u is strictly increasing, strictly concave, twice differentiable and

$$\lim_{w \to 0^+} u(w) = -\infty. \tag{1}$$

Note these restrictions are consistent with utility functions which have constant relative risk aversion $u(x) = x^{1-\sigma}/[1-\sigma]$ with parameter $\sigma \geq 1$.[2] Aside from their death rate $\delta > 0$, assume workers do not discount the future.

[2]this assumption ensures that the corner constraint $w \geq 0$ is never binding.

Any job offer received is assumed to be the realization of a random draw from F, where $F(V)$ is the probability any contract offered yields an expected lifetime utility no greater than V if it is accepted. Let $[\underline{V}, \overline{V}]$ denote its support.

Given a worker with tenure t is employed by a firm offering contract $\widehat{w}$, let $V(t;\widehat{w})$ denote the worker's expected lifetime payoff when using an optimal quit strategy. Similarly, let V_u denote an unemployed worker's expected lifetime payoff. For now, take V_u as given.

Given any contract $\widehat{w}$ and any tenure t where $V(t+dt;\widehat{w}) \geq V_u$, the Bellman equation implies

$$\begin{aligned} V(t;\widehat{w}) &= u(w(t))dt + (1-\delta dt)[(1-\lambda dt)V(t+dt;\widehat{w}) \\ &\quad +\lambda dt F(V(t+dt;\widehat{w})V(t+dt;\widehat{w}) + \lambda dt \int_{V(t+dt;\widehat{w})}^{\overline{V}} x dF(x)] \end{aligned} \tag{2}$$

where the interpretation is standard - the worker quits to another firm if an outside offer is received whose value x exceeds $V(t+dt;\widehat{w})$.

Of course given some arbitrary contract $\widehat{w}$, it may happen that $V(t+dt;\widehat{w}) < V_u$ for some t. In that case the worker quits to unemployment at the end of the period and so

$$V(t;\widehat{w}) = u(w(t))dt + (1-\delta dt)[(1-\lambda dt)V_u + \lambda dt F(V_u)V_u + \lambda dt \int_{V_u}^{\overline{V}} x dF(x)].$$

Define $T(\widehat{w}) = \min\{t \geq 0 : V(t;\widehat{w}) < V_u\}$ which is the tenure date at which the worker quits to unemployment. If no such tenure date exists, then set $T(\widehat{w}) = \infty$. From (2) it can be seen that $V(.;\widehat{w})$ satisfies the difference equation

$$\begin{aligned} &\frac{\delta V(t;\widehat{w})}{1-\delta dt} - \frac{V(t+dt;\widehat{w}) - V(t;\widehat{w})}{dt} \\ &= \frac{u(w(t))}{1-\delta dt} + \lambda \int_{V(t+dt;\widehat{w})}^{\overline{V}} [x - V(t+dt;\widehat{w})] dF(x), \end{aligned} \tag{3}$$

while $V(t+dt;\widehat{w}) \geq V_u$, and the boundary condition $V(T;\widehat{w}) = V_u$ if $T < \infty$.

Consider an employee at a firm offering contract $\widehat{w}$. At any tenure t where $t+dt < T(\widehat{w})$, then $(\delta + \lambda(1 - F(V(t+dt;\widehat{w}))dt + o(dt)$ is the probability he or she leaves at the end of the period. Hence for dt arbitrarily small and for any t such that $t < T(\widehat{w})$,

$$\psi(t;\widehat{w}) = e^{-\int_0^t [\delta + \lambda(1-F(V(s;\widehat{w}))]ds}$$

is the probability an employee does not leave before tenure t. The next section considers a firm's optimal wage contract $\widehat{w}$, given this quit strategy.

3 The Firms

The objective in this Section is to derive the contract that maximizes a firm's expected profit given it yields an expected lifetime utility of at least V_p to any new worker who accepts it. Such a contract is termed an optimal contract. It is shown that there are two forces which determine the nature of this contract. First, as capital markets are imperfect, there is an insurance problem where, ceteris paribus, risk averse workers prefer a constant wage stream. The insurance effect implies workers value a smoother wage stream more highly. Second, there is a moral hazard problem where an employee quits if a better outside offer is received.

When designing an optimal contract, each firm takes as given (a) F, the distribution of contracts offered by other firms in the market, (b) V_u, the expected lifetime utility of an unemployed worker, and (c) the quit strategy of an employed worker given the contract offered .

We start by making two preliminary points. First, as the arrival rate of further job offers is independent of a worker's state, an unemployed worker accepts a contract which offers $w(t) = b$ for all t. As $b < p$ by assumption, a firm can always obtain strictly positive profit by offering this contract. Hence, the following only considers situations where firms make strictly positive profit.

Second, note that a firm which offers a contract that yields an expected lifetime utility V_p to a worker, where $V_p < V_u$, hires no workers. As such a contract makes zero profit, strictly positive profit therefore requires that each firm offers a $V_p \geq V_u$. As F describes the distribution of contract offers in the market, we only consider situations where $\underline{V} \geq V_u$; i.e. the lowest value wage contract offered in the market has value at least as great as V_u.

The distribution of outside offers faced by workers plays a central role in what follows. We construct an equilibrium where F has the following properties.

A1: (a) $\overline{V} < u(p)/\delta$, *and (b) for all* $V \in (\underline{V}, \overline{V})$, F *is continuously differentiable and satisfies* $F'(V) > 0$.

Note, A1(a) must hold, otherwise $\overline{V} \geq u(p)/\delta$ would require that some firms offer contracts which make negative profits. The second assumption, A1(b), is more difficult to justify at this stage. This restriction and potential alternatives are discussed later.

Fix an F satisfying A1, and a V_u satisfying $V_u \leq \underline{V}$. In equilibrium both F and V_u are endogenously determined, but each firm takes these objects as given. Given such a V_u and F, suppose the firm offers a starting wage contract that yields an expected utility V_p to any worker it hires.

Clearly $V_p < V_u$ implies this contract does not attract any workers, and so consider V_p satisfying $V_p \geq V_u$. Suppose the firm utilizes contract $\widehat{w}$. Assuming firms do not discount the future, then for dt arbitrarily small the firm's expected return to hiring a worker is

$$\int_0^{T(\widehat{w})} \psi(t;\widehat{w})[p - w(t)]dt,$$

where $\psi(t;\widehat{w})$ and $T(\widehat{w})$ are defined in the previous section. Of course if $T(\widehat{w}) < \infty$, the worker quits at tenure $T(\widehat{w})$ and the firm makes no further profit from this hire.

Hence given the worker's optimal quit strategy, the firm's formal optimal contracting problem (for dt arbitrarily small) can be written as

$$\max_{\widehat{w}} \int_0^{T(\widehat{w})} \psi(t;\widehat{w})[p - w(t)]dt \tag{4}$$

subject to

$$V(0;\widehat{w}) \geq V_p \tag{5}$$

The first result is immediate.

Claim 1. Fix an F satisfying A1 and a $V_u \leq \underline{V}$. If $V_p \geq \overline{V}$ and $dt > 0$, the optimal wage contract implies $w(t) = w_p$ for all t where $u(w_p) = \delta V_p$, and an employee never quits.

Proof : Fix any $V_p \geq \overline{V}$. Now consider the wage contract $\widehat{w} \equiv \{w(t) = w_p\}_{t=0}^{\infty}$ where w_p is defined in the Claim. The Bellman equation (3) implies $V(t;\widehat{w}) = V_p$ for all t, where $V_p \geq \overline{V}$ implies the worker never quits. As this contract is jointly efficient and also extracts maximal employee rents (given V_p) it therefore maximizes the firm's profit.

As it plays a most important role in what follows, define $\overline{w}$ where

$$u(\overline{w}) = \delta\overline{V}.$$

Claim 1 establishes that a firm which offers the contract that yields the greatest lifetime utility to new hires (i.e. sets $V_p = \overline{V}$), provides perfect income insurance - it offers a constant wage $w(t) = \overline{w}$ for all t. As an employee never quits at such a firm, the firm's expected profit $\overline{\Pi}$ per hire is

$$\overline{\Pi} = [p - \overline{w}]/\delta.$$

Note that $A1(a)$ implies $\overline{w} < p$ and $\overline{\Pi} > 0$.

We now turn to a firm that offers a contract which yields V_p, where $\underline{V} \leq V_p < \overline{V}$. Such a firm faces a positive risk that its employee quits to competing firms. What follows establishes that in the optimal contract, a firm will gradually increase wage payments with tenure.

The necessary conditions describing the optimal wage contract are derived in Appendix A. Here we simply describe the solution for the limiting case where $dt \to 0$.

Given V_p, let $w^* = \{w^*(\tau|V_p)\}_{\tau=0}^{\infty}$ denote the firm's optimal contract. Let $V^*(\tau|V_p)$ describe the value of being employed with tenure τ at a firm which offers contract w^* with expected (starting) payoff V_p. [3] Similarly, let $\Pi^*(\tau|V_p)$ describe the firm's expected profit given a current employee with tenure τ.

Theorem 1.

Fix an F satisfying A1 and a $V_u \leq \underline{V}$. In the limit as $dt \to 0$ and for any $V_p \in [\underline{V}, \overline{V})$, the optimal contract w^* and corresponding worker and firm payoffs $\{V^*, \Pi^*\}$ are solutions to the differential equation system $\{w, V, \Pi\}$:

$$\frac{-u''(w)}{u'(w)^2}\frac{dw}{dt} = \lambda F'(V)\Pi, \tag{6}$$

$$\delta V - \frac{dV}{dt} = u(w) + \lambda \int_V^{\overline{V}} [x - V]F'(x)dx, \tag{7}$$

$$[\delta + \lambda(1 - F(V))]\Pi - \frac{d\Pi}{dt} = [p - w], \tag{8}$$

subject to the boundary conditions:

(a) $\lim_{t\to\infty}\{w(t), V(t), \Pi(t)\} = (\overline{w}, \overline{V}, \overline{\Pi})$, and

(b) the initial condition $V(0) = V_p$

Proof is in Appendix A.

A formal existence proof of an optimal contract is provided later (see the proof of Theorem 2 below). For now, we provide the relevant insights.

[3] Note, that $V^*(\tau|V_p)$ is equivalent to $V(\tau; \widehat{w})$ when $\widehat{w} = w^*$.

Equations (7) and (8) are standard flow equations describing the continuation payoffs V^* and Π^*. For example, (7) follows directly from the Bellman equation (3), while integration of (8) and boundary condition (a) (which implies boundedness) gives

$$\Pi^*(t|.) = \int_t^\infty \frac{\psi(\tau; w^*)}{\psi(t; w^*)}[p - w^*(\tau|.)]d\tau \tag{9}$$

which is simply the firm's expected future profit given an employee with current tenure t (where Appendix A establishes that $T(w^*) = \infty$; a worker never quits into unemployment).

The central economic insight is provided by (6) which describes how wages change with tenure. As $\Pi > 0$ and $F' > 0$ by assumption, wages are strictly increasing with tenure and converge asymptotically to $\overline{w}$. Given an employee with current expected payoff $V = V^*(\tau|.)$, the density function $F'(V)$ measures the number of firms whose outside offer will marginally attract this worker. If there are no such firms, i.e., $F'(V) = 0$, marginally raising the worker's wage $w^*(\tau|.)$ at tenure τ has no marginal effect on the worker's quit rate at τ. Optimal insurance then implies the firm pays a (locally) constant wage. However as $F'(V) > 0$ by assumption, there is a trade-off between increasing pay at tenure τ, which reduces marginally the worker's quit rate at τ, and worsening ex-ante income insurance. Indeed, integrating (6) over $[0, \tau]$ implies

$$\frac{u'(w^*(0|.))}{u'(w^*(\tau|.))} = 1 + u'(w^*(0|.)) \int_0^\tau \lambda F'(V^*(t|.))\Pi^*(t|.)dt. \tag{10}$$

By marginally increasing the wage paid at tenure τ, the firm reduces marginally the worker's quit rate over $[0,\tau]$. The integral term in (10) measures the firm's overall return to that decreased quit rate, which distorts the optimal wage contract away from full insurance.

As is standard with moral hazard, the optimal contract rewards those who do not quit - in this case the principal increases payments for those with higher tenure. Being liquidity constrained new employees are potentially made worse off (as they cannot borrow against future earnings), but the promise of higher earnings in the future lowers their quit rate and increases joint surplus (where a quit is jointly inefficient).

Of course, as wages increase with tenure, so does the value of employment $V^*(t|.)$. Further, as wages keep rising while $V^* < \overline{V}$, there comes a point where V^* equals (or is at least very close to) $\overline{V}$. But Claim 1 establishes the optimal contract for $V = \overline{V}$, and so boundary condition (a) ensures that the limiting contract is optimal. Backward induction using (6)-(8) then backs out the optimal transitional wage dynamics, where the iteration stops at the point where $V = V_p$.

More formally, the optimal wage contract corresponds to the saddle path implied by the differential equations (6)-(8) and the stationary point $(\overline{w}, \overline{V}, \overline{\Pi})$.[4] In fact, a backward induction argument using A1 and Theorem 1 establishes that $w^*(\tau|V_p)$ and $V^*(\tau|V_p)$ are continuous and strictly increasing with τ, converging to $\overline{w}$ and $\overline{V}$ respectively.

The above has characterized the wage contract a profit maximizing firm offers its employees given the distribution of outside offers, and that it chooses to offer new employees a starting payoff V_p. Not surprisingly, it can be shown that the firm's expected payoff per newly hired employee is strictly decreasing in V_p. Clearly, if this were the only consideration, firms would choose the lowest V_p that is acceptable. However, with on-the-job search, the greater the starting payoff V_p offered by a firm, the more workers it will attract.

The next section now determines F, the distribution of starting payoffs V_p, as part of a non-cooperative wage contract posting game.

4 A Firm Replication

Throughout we focus on steady state and so define the following steady-state variables. First consider steady state unemployment U. Strictly positive profit implies $\underline{V} \geq V_u$, and so each unemployed worker matches with the first job offer received. Also an optimal contract implies a worker never quits into unemployment. Hence steady state unemployment satisfies $\delta(1-U) = \lambda U$, and so

$$U = \frac{\delta}{\delta + \lambda},$$

while steady state employment is $\lambda/(\delta + \lambda)$.

Now define $G(V)$ as the steady-state probability a randomly selected *employed* worker has current expected lifetime utility no greater than V. This distribution function depends not only on F, but also on wage-tenure effects and quit turnover. As we do not rule out mass points in G, use the notation $G(V^-) \equiv \lim_{\varepsilon>0,\varepsilon\to 0} G(V-\varepsilon)$.

[4] note, equations (7),(8) have unstable forward looking roots.

Now note that if the firm offers starting payoff $V_p \geq V_u$, its steady state hiring inflow, denoted $h(V_p)$, is

$$h(V_p) = \lambda[\frac{\delta}{\delta+\lambda} + \frac{\lambda}{\delta+\lambda} G(V_p^-)]$$

where $V_p \geq V_u$ implies $\lambda\delta/(\delta+\lambda)$ is the firm's inflow from the unemployment pool, while $\lambda^2 G(V_p^-)/(\delta+\lambda)$ is the firm's inflow from the pool of employed workers whose current lifetime utility is strictly less than V_p.

Of course given an arbitrary contract $\widehat{w}$ which offers starting payoff V_p, the firm's steady state profit flow is

$$\theta(V_p) = \int_0^{T(\widehat{w})} [h\psi(t;\widehat{w})dt][p - w(t)],$$

where $h\psi(t;\widehat{w})dt$ is the measure of employees with tenure t, and $[p-w(t)]$ is the firm's profit flow given the wage paid to those workers. But given V_p and the corresponding hiring inflow h, note that an optimal contract maximizes this flow payoff; i.e. the firm's maximal steady state profit flow is

$$\begin{aligned} \theta^*(V_p) &= h(V_p) \int_0^\infty \psi(t;w^*)[p - w^*(t \mid V_p)]dt \\ &= h(V_p)\Pi^*(0|V_p). \end{aligned}$$

Definition: Given V_u, a **replication** is a distribution of optimal wage contracts $\{w^*, V^*, \Pi^*\}$, with corresponding steady state distributions F, G, and a flow profit $\overline{\theta} > 0$ where

(i) $\theta^*(V_p) = \overline{\theta}$ for all V_p in the support of F;

(ii) $\theta^*(V_p) \leq \overline{\theta}$ otherwise.

A replication requires that all firms make the same steady state profit flow $\overline{\theta} > 0$ and that any other wage contract results in lower profit.

The following establishes necessary conditions for a replication. The first step in accomplishing this task is to consider the wage contract of firms offering the lowest value contract in the market; i.e., those firms that offer $V_p = \underline{V}$.

Definition : Given an F satisfying A1, the **baseline salary scale**, denoted $\{w^s(t), V^s(t), \Pi^s(t)\}$, is the solution to the differential equations and boundary conditions defined in Theorem 1 with $V_p = \underline{V}$.

Assumption $A1$ and the equations of Theorem 1 imply that $w^s(.)$ and $V^s(.)$ are both continuous and strictly increasing functions which converge to $\overline{w}$ and $\overline{V}$ respectively. Hence, given any other starting payoff $V_p \in [\underline{V}, \overline{V})$, a salary point $t_p \geq 0$ exists where $V^s(t_p) = V_p$. Further, reflecting the saddle path property of the differential equations described in Theorem 1, optimality of w^s implies that the optimal wage contract given starting payoff $V_p = V^s(t_p)$ corresponds to the wage tenure payments $w^s(.)$ described for tenures $t \geq t_p$.

This yields the convenient result that $w^*(t|V_p) \equiv w^s(t + t_p)$; i.e. given starting salary point t_p an optimal wage contract pays a worker with tenure t a wage commensurate with point (t_p+t) on the baseline salary scale. Such a worker then only quits if offered a starting point t'_p strictly greater than his/her current salary point. Note this also implies the firm's continuation payoff can be written as $\Pi^*(t|V_p) \equiv \Pi^s(t + t_p)$.

Due to the argument made above, rather than consider firms as competing on starting payoffs V_p, we consider instead firms competing on starting points t_p on an endogenously determined baseline salary scale. The least generous firms set $t_p = 0$ and so offer wage contract $\{w^s(t)\}_{t=0}^{\infty}$ which implies starting payoff $V_p = V^s(0) \equiv \underline{V}$. A firm offering a higher starting point $t_p > 0$ offers a wage contract $\{w^s(t_p+t)\}_{t=0}^{\infty}$ which is the optimal wage contract given alternative starting payoff $V_p = V^s(t_p) \in (\underline{V}, \overline{V}]$.

Define $F^s(t_p)$ as the distribution of starting points t_p on the baseline salary scale in a replication, and note that V^s strictly increasing implies

$$F(V^s(t_p)) = F^s(t_p).$$

Also note that A1 and the definition of the baseline salary scale imply $dF^s/dt = F'(V^s)dV^s/dt > 0$ for all $t > 0$ and therefore F^s has a connected support $[0,\infty)$ and is differentiable for $t > 0$. Also by definition $V^s(0) \equiv \underline{V}$ and $\lim_{t\to\infty} V^s(t) = \overline{V}$.

Similarly, the distribution of worker payoffs in a steady state, $G(V)$, can instead be considered as a distribution of workers along the baseline salary scale. However, it is mathematically convenient to define $1 - G^s(t_p)$ as the measure of employed workers on points $t \geq t_p$ on the baseline salary scale, so that $G^s(t_p)$ is the total measure of those unemployed and those employed at a point strictly below t_p.

In the rest of this study we consider a replication as a quintuple $\{w^s, V^s, \Pi^s, F^s, G^s\}$ where:

(a) $\{w^s, V^s, \Pi^s\}$ jointly describe the baseline salary scale,

(b) F^s describes the distribution of starting points t_p offered by firms, and so $w^s(t_p+t)$ and F^s describes the distribution of wage contracts $w^*(t|.)$, and

(c) G^s describes the distribution of workers across that salary scale.

The next three claims characterize each component in turn.

Claim 2. A replication satisfying A1 implies the baseline salary scale $\{w^s, V^s, \Pi^s\}$ satisfies the differential equations

$$\frac{-u''(w^s)}{u'(w^s)^2}\frac{dw^s}{dt} = \lambda\Pi^s \frac{dF^s/dt}{dV^s/dt}, \tag{11}$$

$$\delta V^s - \frac{dV^s}{dt} = u(w^s) + \lambda \int_t^\infty [V^s(\tau) - V^s(t)]dF^s(\tau), \tag{12}$$

$$\{\delta + \lambda[1 - F^s]\}\Pi^s - \frac{d\Pi^s}{dt} = [p - w^s], \tag{13}$$

for all $t \geq 0$, and the boundary conditions

(a) $\lim_{t\to\infty}(w^s(t), V^s(t), \Pi^s(t)) = (\overline{w}, \overline{V}, \overline{\Pi})$ where $\overline{V} = u(\overline{w})/\delta, \overline{\Pi} = [p - \overline{w}]/\delta$;

(b) $V^s(0) = \underline{V}$.

Proof : Follows from the definition of the baseline salary scale and Theorem 1, using $F(V^s) = F^s$, $F'(V^s) = [dF^s/dt]/[dV^s/dt]$ (which exists by A1) and using the transform $x = V^s(\tau)$ to establish

$$\int_{V^s(t)}^{\overline{V}} [x - V^s(t)]F'(x)dx = \int_t^\infty [V^s(\tau) - V^s(t)]dF^s(\tau).$$

This completes the proof.

Next we compute steady state G^s.

Claim 3. A replication satisfying $A1$ implies

$$G^s(0) = \delta/(\delta + \lambda)$$

$$[\delta + \lambda[1 - F^s]]G^s + \frac{dG^s}{dt} = \delta \text{ for } t > 0. \tag{14}$$

Proof: Note that the definition of G^s implies $G^s(0)$ is equal to the number unemployed, which in a (steady state) replication is $\delta/(\delta+\lambda)$. The differential equation for G^s follows from standard steady state flow arguments which are provided in Appendix B.

Finally we compute the steady state profit condition. As F^s has connected support $[0, \infty)$, the constant profit condition requires

$$\lambda G^s(t)\Pi^s(t) = \overline{\theta} \text{ for all } t \geq 0$$

where steady state implies $\lambda G^s(t)$ is the hiring rate of a firm that offers starting point $t \geq 0$, i.e., it attracts all those workers who are not employed on a higher salary point, and $\Pi^s(t)$ is the firm's expected profit per worker by posting the corresponding optimal wage contract.

Claim 4. A replication satisfying A1 implies:

(i) $\lim_{t\to\infty} F^s(t) = 1$ (i.e. there is no mass point in F at $\overline{V}$),

(ii) $\lim_{t\to\infty} G^s(t) = 1$,

(iii) $\overline{\theta} = \lambda[p - \overline{w}]/\delta$ and so positive profit requires $\overline{w} < p$ and

$$G^s(t)\Pi^s(t) = [p - \overline{w}]/\delta \text{ for all } t \geq 0. \tag{15}$$

Proof in Appendix B.

Claims 2-4 establish the following Proposition.

Proposition 1 (Necessary Conditions for a Replication)

Necessary conditions for a replication satisfying $A1$ is a quintuple $\{w^s, V^s, \Pi^s, F^s, G^s\}$ satisfying the four differential equations (11)-(14), the constant profit condition (15), the boundary conditions;

(a) $\lim_{t\to\infty}(w^s(t), V^s(t), \Pi^s(t), G^s(t), F^s(t)) = (\overline{w}, \overline{V}, \overline{\Pi}, 1, 1)$ where $\overline{w} < p, \overline{V} = u(\overline{w})/\delta, \overline{\Pi} = [p - \overline{w}]/\delta$,

(b) $G^s(0) = \delta/(\lambda + \delta)$,

where F^s, G^s have the properties of distribution functions; i.e., are positive and (strictly) increasing for all $t \geq 0$, and F has the assumed properties $A1$ where F satisfies $F(V^s) = F^s$ for all $t \geq 0$.

Identifying a solution to the conditions stated in Proposition 1 is relatively straightforward. First pick some arbitrary value for $\overline{w}$ satisfying $\overline{w} < p$. Using the limit point defined in boundary condition (a), the proof of Lemma A below shows we can iterate the differential equation system (11)-(15) backwards through time along the saddle path. When $G^s = \delta/(\lambda + \delta)$ (should it occur), the iteration is stopped and time is then renormalized so that $t = 0$ at this point. This ensures boundary condition (b) is satisfied. As long as this path also implies F^s, G^s are positive and increasing and that the implied F has properties $A1$, we then have a candidate replication - a system $\{w^s, V^s, \Pi^s, F^s, G^s\}$ which satisfies the conditions stated in Proposition 1. Formally, we define any such solution as a *Candidate Replication* and denote it as a quintuple

$\{\widetilde{w}^s, \widetilde{V}^s, \widetilde{\Pi}^s, \widetilde{F}^s, \widetilde{G}^s\}$ where, for example, $\widetilde{w}^s(t|\overline{w})$ denotes the wage paid at point t on the baseline salary scale in Candidate Replication $\overline{w}$.

Lemma A. A Candidate Replication exists and is unique for any $\overline{w} \in (w_1, p)$ where $w_1 = \lambda(\lambda + 2\delta)p/[\lambda + \delta]^2 < p$. A Candidate Replication does not exist for $\overline{w} < w_1$.

Proof in Appendix B.

This restriction on $\overline{w}$ (the highest wage in the market) is discussed in detail later. w_1 is a critical point where a Candidate Replication with initial value $\overline{w} < w_1$ does not exist (it would require $w^s(0) < 0$). From now on, we restrict attention to initial values $\overline{w} \in (w_1, p)$.

Claim 5. A replication satisfying A1 and $\overline{w} \in (w_1, p)$ implies $\underline{V} = V_u$.

Proof in Appendix B.

Claim 5 establishes that the firm offering the lowest value contract in the market extracts full rents from the unemployed. This last Claim now allows us to fully characterize a replication.

Proposition 2. (Characterization of a Replication)

Given V_u, necessary and sufficient conditions for a replication satisfying A1 with $\overline{w} \in (w_1, p)$ are those conditions described in Proposition 1 and the boundary condition

$$V^s(0) = V_u.$$

Proof. Proposition 1 and Claims 2(b) and 5 establish that these conditions are necessary. To see that they are sufficient note that the constant profit condition (15) (and $\overline{w} < p$) ensures that all optimal contracts with payoffs $V_p \in [\underline{V}, \overline{V}]$ generate the same payoff $\overline{\theta} = \lambda[p - \overline{w}]/\delta > 0$. Obviously a contract which offers $V_p > \overline{V}$ generates less profit (it pays a higher wage $w_p > \overline{w}$ and $\lim_{t\to\infty} G_s(t) = 1$ implies it attracts no more workers than a contract offering $\overline{w}$). Also any contract offering $V_p < \underline{V} = V_u$ attracts no workers and so makes zero profit. Hence any solution to the conditions stated must describe a replication.

Theorem 2 now establishes existence of a replication. However to simplify that proof assume[5]

A2 : $\int_0^a u(x)dx = -\infty$ for any $a > 0$ (finite).

[5] otherwise we have to worry much more about the limiting properties of the equations described in Proposition 1 as $\overline{w} \to w_1$ which implies $w^s(0) \to 0$.

which with constant relative risk aversion would now require $\sigma \geq 2$.

Theorem 2. (Existence of a Replication)
Given A2, a replication satisfying A1 exists for any $V_u < u(p)/\delta$.

The proof is in Appendix B. It establishes that for any $V_u < u(p)/\delta$, there exists a $\overline{w} \in (w_1, p)$ where the corresponding Candidate Replication implies $\widetilde{V}^s(0 \mid \overline{w}) = V_u$. This Candidate Replication then satisfies the conditions of Proposition 2 and so describes a replication.

5 A Market Equilibrium

The previous section has described equilibrium firm behavior assuming the unemployed have some value V_u to being unemployed. But if F describes the distribution of job offers, then in the limit as $dt \to 0$, optimal job search implies

$$\delta V_u = u(b) + \lambda \int_{V_u}^{\overline{V}} [x - V_u] dF(x). \tag{16}$$

Of course, equilibrium requires that V_u must not only satisfy this job search condition, but F must be consistent with a replication. Further we know that $V^s(0) = \underline{V} = V_u$ in any such replication. Hence by transforming variable $x = V^s(t)$, we can define a market equilibrium as follows.

Definition : A Market Equilibrium is a replication where V_u also satisfies

$$\delta V_u = u(b) + \lambda \int_0^{\infty} [V^s(t) - V_u] dF^s(t). \tag{17}$$

(17) of course ties down V_u.

Theorem 3. (Existence of a Market Equilibrium)

Given A2, a Market Equilibrium satisfying A1 exists for any $b \in (0, p)$.

Proof in Appendix C.

The approach to identifying a Market Equilibrium is the same as for a replication. Pick some arbitrary value $\overline{w} \in (w_1, p)$ and compute a Candidate Replication satisfying the conditions given in Proposition 1. Now a replication requires $V^s(0) = \underline{V} = V_u$, while (17) determines V_u. Hence a replication and a Market Equilibrium implies $V^s(0)$ must satisfy

$$\delta V^s(0) = u(b) + \lambda \int_0^{\infty} [V^s(t) - V^s(0)] dF^s(t).$$

The proof in Appendix C establishes that a $\overline{w} \in (w_1, p)$ exists where the corresponding Candidate Replication satisfies this equilibrium criterion. This Candidate Replication and $V_u = \widetilde{V}^s(0|\overline{w})$ then describes a Market Equilibrium.

6 Appendix A

Proof of Theorem 1

The proof takes four steps. Step 1 establishes that a firm always offers a contract such that a worker never quits into unemployment, i.e., $T(\widehat{w}) = \infty$. Step 2 uses a backward induction argument to characterize how wages change optimally with tenure. Step 3 then identifies the appropriate transversality condition; that wages converge to $\overline{w}$ as tenure becomes large. Step 4 then computes the limiting solution as $dt \to 0$ and so establishes the Theorem.

Step 1. Fix $dt > 0$ satisfying $\lambda dt < 1$, an F satisfying A1 and $V_u \leq \underline{V}$. Now choose any $V_p \in [\underline{V}, \overline{V})$ and let $w^* = \{w^*(t|V_p)\}_{t=0}^{\infty}$ denote the optimal wage tenure contract. As V_p is fixed throughout this proof, simplify notation by subsuming reference to it in w^*.

Given w^*, the Bellman equation (3) determines $V(t; w^*)$. Let $\Pi(t; w^*)$ denote the firm's expected profit given a worker with tenure t employed with wage contract w^*. For t where $V(t + dt; w^*) \geq V_u$, standard recursive arguments imply

$$\Pi(t; w^*) = [p - w^*(t)]dt + (1 - \delta dt)[1 - \lambda q(t + dt; w^*)dt]\Pi(t + dt; w^*) \tag{18}$$

where conditional on an outside offer, $q(t + dt; w^*) \equiv 1 - F(V(t + dt; w^*))$ is the quit probability of a worker with tenure $t + dt$.

Now consider the optimal contract at tenure τ, where using an optimal quit strategy, the worker obtains expected payoff $V(\tau; w^*)$. Clearly optimality implies that the wage payments w^* specified for tenures $t \geq \tau$ must maximize the firm's continuation payoff $\Pi(\tau; \widehat{w})$ given $V = V(\tau; w^*)$. This insight implies the following Claim.

Claim A1 :

(i) For tenures $\tau \geq 0$ where $V(\tau; w^*) \leq \overline{V}$, optimality implies $V(\tau; w^*) \geq V_u$ and $\Pi(\tau; w^*) > 0$,

(ii) the constraint (5) binds; i.e., $V(0; w^*) = V_p$.

Proof is by contradiction arguments.

(i) Suppose first that $V(\tau; w^*) < V_u$, in which case the worker quits and obtains payoff V_u while the firm obtains a zero payoff. But this is dominated by offering contract $\widehat{w}$ where $w(t) = \overline{w}$ for all $t > \tau$ and $w(\tau) = w$ satisfying $[u(w) - u(\overline{w})]dt = [V_u - \overline{V}]$. In that case the worker obtains the same payoff V_u and so does not quit into unemployment, while $\overline{w} < p$ and $V_u \leq \overline{V}$ imply the firm makes strictly positive profit, which contradicts the optimality of the original wage contract.

Suppose now that $\Pi(\tau; w^*) \leq 0$. As $V(\tau; w^*) \leq \overline{V}$ by assumption, that supposedly optimal contract is dominated by specifying $w(t) = \overline{w}$ for all $t > \tau$ and $w(\tau) = w$ satisfying $[u(w) - u(\overline{w})]dt = [V(\tau; w^*) - \overline{V}]$ which makes strictly positive profit, and hence contradicts $\Pi(\tau; w^*) \leq 0$.

(ii) If (5) is not binding, i.e. $V(0; w^*) > V_p$, then offering the same wage contract but cutting the period zero wage to w where $[u(w) - u(w^*(0))]dt = [V_p - V(0; w^*)]$ is strictly profit increasing (it satisfies (5)), does not affect the worker's quit strategy and strictly reduces total wages which is the required contradiction. This completes the proof of Claim A1.

Claim A1(i) establishes two facts. First, $V(t; w^*) \geq V_u$ for all t and so $T(w^*) = \infty$; the worker never quits into unemployment. Second, the liquidity constraint is never binding; i.e. $w^*(t) > 0$ for all t (otherwise $dt > 0$ and $w^*(t) = 0$ implies $V(t; w^*) = -\infty < u(b)/\delta \leq V_u$).

Step 2. Assume an optimal contract w^* exists and fix some tenure date $\tau \geq 0$. Now consider an alternative wage contract, denoted $w^\tau = \{w(t)\}_{t=0}^{\infty}$, which is defined by:
(i) $w(t) = w^*(t)$ for all $t \neq \tau$, $\tau + dt$
(ii) $w(\tau) = w^*(\tau) + dx$ and $w(\tau + dt) = w^*(\tau + dt) + dy$
where (dx, dy) are arbitrarily small and satisfy

$$u'(w^*(\tau))dx + (1 - \delta dt)[1 - \lambda q(\tau + dt; w^*)dt]u'(w^*(\tau + dt))dy = 0. \tag{19}$$

Note, this perturbation implies w^τ changes wages at just two tenure dates $\tau, \tau + dt$. Also note that as the liquidity constraint is never binding on w^*, this variation is always feasible as long as (dx, dy) are small enough .

Define $dV(t) = V(t; w^\tau) - V(t; w^*)$ which is the increase in the employee's expected payoff at tenure t given this contract perturbation. For dy arbitrarily small, backward induction now implies

$$dV(\tau + dt) = u'(w^*(\tau + dt))dydt. \tag{20}$$

and $dV(t) = 0$ for all other t. This occurs because $w^\tau = w^*$ for all tenures $t > \tau + dt$ and so $dV(t) = 0$ for those t. As $V(\tau + 2dt; w^\tau) = V^*(\tau + 2dt)$, (20) then follows from the Bellman equation describing $V(\tau + dt; \widehat{w})$. Note that a wage increase $dy > 0$ at tenure date $\tau + dt$ implies the worker is better off at this tenure date. But (19) then compensates by choosing dx so that $V(\tau; w^\tau) = V(\tau; w^*)$ (use the Bellman equation for $V(\tau; \widehat{w})$ and the Envelope Theorem). Backward induction then implies $dV(t) = 0$ for $t \leq \tau$. Most importantly, note that $V(0; w^\tau) = V(0; w^*)$ and so w^τ satisfies (5). Hence a necessary condition for optimality of w^* is that any such contract variation cannot be profit increasing.

Claim A1 implies the firm's continuation payoff at tenure τ satisfies (18) above. Now let $d\Pi(\tau) = \Pi(\tau; w^\tau) - \Pi(\tau; w^*)$ denote the increase in the firm's continuation payoff implied by the perturbed contract w^τ. As $\Pi(\tau + 2dt; w^*) = \Pi(\tau + 2dt; w^\tau)$, it follows using (18) that for dy arbitrarily small, the first order effect of this contract perturbation is

$$\begin{aligned} d\Pi(\tau) &= -dxdt - (1 - \delta dt)[1 - \lambda q(\tau + dt; w^*)dt]dydt \\ &\quad +(1 - \delta dt)\lambda dt[-dq(\tau + dt)]\Pi(\tau + dt; w^*) \end{aligned}$$

where $dq(\tau + dt) = q(\tau + dt; w^\tau) - q(\tau + dt; w^*)$ denotes the change in the worker's quit probability at tenure $\tau + dt$. Note, the first two terms are the direct wage costs of this perturbation, while the last term is the increase in expected profit by changing the worker's quit probability at tenure $\tau + dt$.

We now solve for $d\Pi(\tau)$. The definition of dq implies

$$dq(\tau + dt) = [1 - F(V(\tau + dt; w^\tau)] - [1 - F(V(\tau + dt; w^*)].$$

Hence F differentiable and dy arbitrarily small imply

$$dq(\tau + dt) = -\lambda F'(V(\tau + dt; w^*))dV(\tau + dt),$$

where dV is given by (20). Using this and (19) to substitute out dx give:

$$\begin{aligned} d\Pi(\tau) &= dydt(1 - \delta dt)[[1 - \lambda q(\tau + dt; w^*)dt][\frac{u'(w^*(\tau + dt))}{u'(w^*(\tau))} - 1] \\ &\quad +\lambda dtF'(V(\tau + dt; w^*)u'(w^*(\tau + dt))\Pi(\tau + dt; w^*)] \end{aligned}$$

But $dy \gtreqless 0$ and so optimality of the wage contract requires $d\Pi(\tau) = 0$. Hence a necessary condition for optimality is

$$\begin{aligned} & (1 - \lambda q(\tau + dt; w^*)dt) \left[1 - \frac{u'(w^*(\tau + dt))}{u'(w^*(\tau))}\right] \\ = \quad & \lambda dt F'(V(\tau + dt; w^*))u'(w^*(\tau + dt)\Pi(\tau + dt; w^*) \end{aligned} \tag{21}$$

and we have established the following Claim.

Claim A2. Given $dt > 0$ satisfying $\lambda dt < 1$, the optimal wage contract satisfies (21) for all τ, which implies

(i) if $V(t + dt; w^*) < \overline{V}$ then $w^*(\tau) \leq w^*(\tau + dt)$

(ii) if $V(t + dt; w^*) > \overline{V}$ then $w^*(\tau) = w^*(\tau + dt)$.

Proof : Implications (i) and (ii) follow directly from (21) given assumption A1 and $\Pi > 0$ (Claim A1(i)).

Note, Claim A2 implies w^* increases with tenure, and so $V(\tau; w^*)$ is also increasing with tenure. This also implies $V(\tau; w^*) \geq V(0; w^*) = V_p \geq \underline{V} \geq V_u$ for all $\tau \geq 0$.

Step 3: We now use forward induction to obtain the appropriate transversality condition.

Claim A3. Optimality implies

$$\lim_{\tau\to\infty} w^*(\tau) = \overline{w}, \text{ and } \lim_{\tau\to\infty} V^*(\tau) = \overline{V}$$

Proof: Consider any $\tau \geq 0$ where $V(\tau; w^*) \leq \overline{V}$. A contradiction argument using Claim A2 implies $V(\tau + dt; w^*) \leq \overline{V}$.[6] As Claim A1(ii) implies $V(0; w^*) = V_p < \overline{V}$, forward induction now implies $V(\tau; w^*) \leq \overline{V}$ for all τ.

Monotonicity of w^* now requires $w^*(\tau) \leq \overline{w}$ for all τ [otherwise $V(t; w^*) > \overline{V}$ for t large enough, which contradicts the previous paragraph]. Hence w^* must converge to some limit point $w_\infty \leq \overline{w}$ as $t \to \infty$. Finally, a contradiction argument using A1 implies $w_\infty = \overline{w}$ which completes the proof of the Claim.[7]

[6] Suppose instead $V(\tau + dt; w^*) > \overline{V}$. But Claim 1 in the text then implies $w^*(t) = w_p$ for all $t \geq \tau + dt$, where $u(w_p) = \delta V(\tau + dt; w^*)$. Claim A2(ii) then implies $w^*(\tau) = w^*(\tau + dt)$ and so $V(\tau; w^*) = V(\tau + dt; w^*) > \overline{V}$ which is the required contradiction.

[7] Suppose not, and so $\lim_{t\to\infty} w^*(t) = w_\infty < \overline{w}$. Let $V_\infty = \lim_{t\to\infty} V(t; w^*)$, $\Pi_\infty = \lim_{t\to\infty} \Pi(t; w^*)$. Note that $w_\infty < \overline{w}$ implies $V_\infty < \overline{V}$, and Claim A1(i) then implies $\Pi_\infty > 0$. But assumption A1 implies $F' > 0$ at $V = V_\infty$. Hence given $dt > 0$, (21) implies $w^*(t) - w^*(t + dt)$ is bounded away from zero as $t \to \infty$, which is the required contradiction.

Step 4. Steps 1 and 2 characterize the optimal contract for any $dt > 0$ satisfying $\lambda dt < 1$. Note that (21) can be written as

$$\begin{aligned} & \frac{1-\lambda q(\tau+dt;w^*)dt}{u'(w^*(\tau))u'(w^*(\tau+dt)} \left[\frac{u'(w^*(\tau))-u'(w^*(\tau+dt))}{dt}\right] \\ = \ & \lambda F'(V(\tau+dt;w^*))\Pi(\tau+dt;w^*) \end{aligned}$$

Hence the limiting solution as $dt \to 0$ implies the differential equation given in the Theorem, where the differential equations for V and Π follow from (3) and (18). Claim A3 implies the boundary condition (a), and Claim A1(ii) establishes the initial condition (b).

This completes the proof of Theorem 1.

7 Appendix B.

Proof of Claim 3.

$V_u \leq \underline{V}$ implies all unemployed workers accept the first job offer they receive. Hence steady state unemployment U must satisfy $\delta[1-U] = \lambda U$, which implies $U = \delta/(\lambda+\delta)$. As $1-G^s(0)$ is the steady state measure of employed workers (with salary points $t \geq 0$) then $G^s(0) = U$.

Now pick any point $t > 0$ on the baseline salary scale. As F^s is continuous for all $t > 0$ (A1 implies it is differentiable) then over any arbitrarily small time interval ε, steady state implies

$$\delta\varepsilon(1-G^s(t)) = [G^s(t)-G^s(t-\varepsilon)] + G^s(t-\varepsilon)\lambda\varepsilon(1-F^s(t)) + 0(\varepsilon^2),$$

where the LHS is the flow out workers in the set of employed workers with salary point no lower than t, and the RHS is the flow in, which includes those whose tenure increases sufficiently over time period ε, those who receive outside offers with starting point no lower than t,and the $0(\varepsilon^2)$ term captures those who receive outside offers in the interval $[t-\varepsilon, t)$ and whose tenure increases sufficiently within this ε period that they rise above the t threshold. Letting $\varepsilon \to 0$ implies G^s is continuous, while dividing by ε and rearranging, this limit then implies the differential equation stated in the Claim.

Proof of Claim 4.

We first establish that a replication implies there is no mass point in F at $\overline{V}$. Suppose instead that a mass point exists in F at $\overline{V}$. Then Claim 1 implies there is also a mass point in G at $\overline{V}$ (as all such employees receive $\overline{V}$ forever), which we denote by $m > 0$. Hence for any $t < \infty$, $1 - G^s(t) \geq m$, and so $\lim_{t\to\infty} G^s(t) \leq 1 - m$. As the definition of the baseline salary scale implies $\lim_{t\to\infty} \Pi^s(t) = \overline{\Pi}$, then the constant profit condition, which requires $\overline{\theta} = \lambda G^s(t)\Pi^s(t)$ for all $t > 0$ implies $\overline{\theta} \leq \lambda[1 - m]\overline{\Pi}$. Of course $\overline{\theta} > 0$ in a replication requires $\overline{\Pi} > 0$ and $\overline{w} < p$.

Now consider the deviating contract $\widehat{w} = \{\overline{w} + \varepsilon\}_{t=0}^{\infty}$ where $\varepsilon > 0$. Claim 1 implies this contract offers starting payoff $V_p = u(\overline{w} + \varepsilon)/\delta > \overline{V}$ which implies hiring rate λ. Further Claim 1 implies per worker profit of $\overline{\Pi} - \varepsilon/\delta$, and so $\widehat{w}$ generates steady state flow profit $\lambda[\overline{\Pi} - \varepsilon/\delta]$. Clearly given $m > 0$, ε small enough implies this profit flow exceeds $\overline{\theta}$ which contradicts the definition of a replication.

No mass point in F at $\overline{V}$ and the definition of F^s now implies (a). (b) then follows from (a) and Claim 3 [with $\lim_{t\to\infty} dG^s/dt = 0$]. As F^s has connected support $[0, \infty)$, then $\overline{\theta} = \lim_{t\to\infty}[\lambda G^s(t)\Pi^s(t)]$ which implies (c).

Proof of Lemma A.

The proof is by construction - we fix some $\overline{w} < p$ and starting at the limiting point described in boundary condition (a), iterate the differential equations described in Proposition 1 backwards through time. But first simplify those conditions as follows.

Claim B1. The conditions of Proposition 1 imply

$$G^s(t) = \left[\frac{p - \overline{w}}{p - w^s(t)}\right]^{\frac{1}{2}}, \Pi^s(t) = \frac{p - \overline{w}}{\delta G^s(t)}.$$

Proof. Substitute out Π^s using the constant profit condition (15) in (13). Using (14) to substitute out dG^s/dt in the resulting expression implies the above solution for G^s. Π^s then follows from (15).

We can use Claim B1 to substitute out G^s, Π^s in the conditions described in Proposition 1 and so reduce that system to a differential equation system for $\{w^s, V^s, F^s\}$. However, notice that V^s described by (12) contains an integral term. To obtain a system of autonomous first order differential equations, define the surplus function

$$S(t) = \int_{\tau=t}^{\infty} [V^s(\tau) - V^s(t)] dF^s(\tau),$$

and note that it is the solution to the differential equation

$$\frac{dS}{dt} = -[1 - F^s]\frac{dV^s}{dt},$$

subject to the boundary condition $\lim_{t\to\infty} S(t) = 0$.

Proposition 2 (A Reduced Form Characterization of a Replication).

A Candidate Replication with $\overline{w} < p$ is described by a sextuple $\{w^s, V^s, \Pi^s, F^s, G^s, S\}$ where

(I) $\{w^s, V^s, F^s, S\}$ satisfy

$$\frac{dw^s}{dt} = \delta\left[\frac{p - w^s}{p - \overline{w}}\right]^{\frac{1}{2}} - [\delta + \lambda[1 - F^s]], \tag{22}$$

$$\frac{dV^s}{dt} = -u(w^s) - \lambda S + \delta V^s, \tag{23}$$

$$\frac{dF^s}{dt} = \frac{-u''(w^s)\delta}{u'(w^s)^2\lambda[p - \overline{w}]^{\frac{1}{2}}[p - w^s]^{\frac{1}{2}}}\frac{dw^s}{dt}\frac{dV^s}{dt}; \tag{24}$$

$$\frac{dS}{dt} = -[1 - F^s]\frac{dV^s}{dt}, \tag{25}$$

subject to the boundary conditions

(a) $\lim_{t\to\infty}(w^s(t), V^s(t), F^s(t), S(t)) = (\overline{w}, \overline{V}, 1, 0)$ where $\overline{V} = u(\overline{w})/\delta$,

(b) $[\frac{p-\overline{w}}{p-w^s(0)}]^{1/2} = \delta/(\lambda + \delta)$,

(II) $\{G^s, \Pi^s\}$ are given by Claim B1, and

(III) F^s, G^s are positive and strictly increasing for all $t \geq 0$.

Note Claim B1 implies that boundary condition (b) for $G^s(0)$ defined in Proposition 1 now defines a boundary condition for $w^s(0)$. Establishing Lemma A reduces to showing a solution exists to the conditions given in Proposition 2.

First consider boundary condition (b) in Proposition 2. Given w_1 defined in the Lemma, straightforward algebra establishes that $\overline{w} < w_1$ requires $w^s(0) < 0$ which cannot be part of a replication.

Now fix a $\overline{w} \in (w_1, p)$ and define $w_0 = w_0(\overline{w})$ where $p - w_0 = (\lambda+\delta)^2(p-\overline{w})/\delta^2$ (and note this implies $w_0 \in (0, \overline{w})$). Also note that boundary condition (b) in Proposition 2 is satisfied if and only if $w^s(0) = w_0$. We now use backward induction to show a solution exists to the conditions of Proposition 2.

First note that the limiting point $(\overline{w}, \overline{V}, 1, 0)$ is a stationary point of the system (22)-(25). Standard stability analysis implies this stationary point has one stable root, one unstable root and two degenerate roots. The eigenvectors for that system imply a convergent saddle path exists with limiting solution:

$$\begin{aligned}
\widetilde{w}^s &= \overline{w} + A_1 e^{-\gamma t} \\
\widetilde{V}^s &= \frac{u(\overline{w})}{\delta} + \frac{u'(\overline{w})[p-\overline{w}]}{\delta[p-\overline{w}+0.5]} A_1 e^{-\gamma t} \\
\widetilde{F}^s &= 1 \\
\widetilde{S} &= 0,
\end{aligned}$$

where $\gamma = 0.5\delta/[p-\overline{w}] > 0$. Note, the degenerate roots allow a continuum of potential steady states (which we are essentially indexing by $\overline{w}$).[8]

Given the limiting solution for this saddle path (with $A_1 < 0$, which implies $dw^s/dt > 0$ for t large enough), we now use backward induction on the system (22)-(25). Obviously as equations (22)-(25) are continuous in (w^s, V^s, F^s, S) (while $w^s > 0$) a solution always exists while $w^s > 0$. The issue is whether the saddle path that is traced out satisfies the conditions of Proposition 2.

Claim B2. The saddle path implied by the conditions of Proposition 2 implies $dw^s/dt > 0$ while $w^s(t) > 0$.

Proof by contradiction. As $dw^s/dt > 0$ for t large enough [along the saddle path], then if Claim B2 fails, there must exist some t_0 where $w^s(t_0) > 0$ and $dw^s/dt = 0$, and $dw^s/dt > 0$ for all $t > t_0$. Furthermore, as wages are ever increasing for $t > t_0$, this implies $dV^s/dt \geq 0$ for all $t \geq t_0$ and $V^s(t_0) > u(w^s(t_0))/\delta$. We now argue to a contradiction.

Differentiating (22) with respect to t and using (24) implies,

$$\frac{\ddot{w}^s}{\dot{w}^s} = \frac{\delta}{(p-\overline{w})^{1/2}(p-w^s)^{1/2}}\left[-\frac{1}{2} + \left(\frac{-u''(w^s)}{[u'(w^s)]^2}\right)\dot{V}^s\right]$$

which can be integrated as

$$\begin{aligned}
\ln w^s(t_0) &= \ln w^s(T) \\
&\quad + \int_{t_0}^{T} \frac{\delta}{(p-\overline{w})^{1/2}(p-w^s)^{1/2}}\left[\frac{1}{2} - \frac{-u''(w^s)}{[u'(w^s)]^2}\frac{dV^s}{dt}\right] dt
\end{aligned}$$

Now given $w^s(t_0) > 0$, let

$$B_0 = \max_{w^s(t_0) \leq x \leq \overline{w}} \left(\frac{-u''(x)}{[u'(x)]^2}\right)$$

[8] The second degenerate root implies a root which is of order t and so is asymptotically unstable.

and note that $u(x)$ strictly increasing and twice differentiable for $x > 0$ implies $0 < B_0 < \infty$. The above now implies

$$\ln w^s(t_0) \geq \ln w^s(T) + \frac{0.5\delta(T - t_0)}{(p - \overline{w})^{1/2}(p - w^s(t_0))^{1/2}} - \frac{B_0[\delta\overline{V} - u(w^s(t_0))]}{(p - \overline{w})}$$

But choosing any T satisfying $t_0 < T < \infty$ implies the RHS is greater than -∞, and so $\dot{w}^s\,(t_0) > 0$ which is the required contradiction.

A simple contradiction argument also implies the saddle path must imply a solution to $w^s(t) = w_0$ exists. Suppose instead while iterating backwards, the saddle path converges to some $w^c \in (w_0, \overline{w})$; i.e. it never reaches $w_0 > 0$. But as $t \to -\infty$, the above inequality implies

$$\ln w^s(t) \geq \ln w^s(T) + \frac{0.5\delta(T - t)}{(p - \overline{w})^{1/2}(p - w^c)^{1/2}} - \frac{B_0[\delta\overline{V} - u(w^c)]}{(p - \overline{w})}$$

and so $w^s(t)$ becomes arbitrarily large, which is the required contradiction. Hence there exists some point in time where the saddle path implies $w^s = w_0$. By renormalizing time to $t = 0$ at that point we satisfy boundary condition (b).

All that remains to show is that this solution satisfies part (III) in Proposition 2. Now w^s is strictly increasing over time and converges to $\overline{w}$ [Claim B2] and so G^s given in Claim B1 implies G^s satisfies (III). Also $dw^s/dt > 0$ and $dV^s/dt > 0$ everywhere along the saddle path, and so (24) implies F^s is also strictly increasing. Further at $t = 0$, (22) and $w^s(0) = w_0$ imply

$$F^s(0) = \frac{1}{\lambda}\frac{dw^s(0)}{dt} > 0.$$

and so F^s satisfies (III). Note, there is a mass point at $t = 0$. Hence, given $\overline{w} \in (w_1, p)$, a solution exists to the conditions of Proposition 2, which therefore describes a Candidate Replication.

Proof of Claim 5.

Claim 4 implies $\overline{\theta} = \lambda G^s(0)\Pi^s(0)$ in a replication, and so Claims 3 and 4 imply $\Pi^s(0) = (\lambda+\delta)[p-\overline{w}]/\delta^2$. $\overline{w} > w_1$ and the definition of w_1 now imply $\Pi^s(0) < p/(\lambda+\delta)$. Also note that a replication must also be a Candidate Replication, and the

proof of Lemma A establishes that $\overline{w} > w_1$ implies $w^s(0) > 0$.

Given the above facts, we now prove Claim 5 using a contradiction argument. Suppose Claim 5 is not true, and so Claim 3 implies $V_u < \underline{V}$. Now consider the deviating contract $w^\varepsilon = \{w(t)\}_{t=0}^{\infty}$ where for some $\varepsilon > 0$ (small), w^ε makes the following wage payments:

$$\begin{aligned} w(t) &= w_0 \text{ for all } 0 \leq t < \varepsilon \\ w(t) &= w^s(t-\varepsilon) \text{ for all } t \geq \varepsilon \end{aligned}$$

where w_0 satisfies $0 < w_0 < \min[w^s(0), p/(\lambda+\delta) - \Pi^s(0)]$. Note the first paragraph implies such a w_0 exists.

Clearly at tenure $\tau = \varepsilon$ this contract w^ε coincides with the baseline salary scale, and so $V(\varepsilon; w^\varepsilon) = V^s(0) = \underline{V}$ and $\Pi(\varepsilon; w^\varepsilon) = \Pi^s(0)$. Most importantly, $V^s(0) > V_u$, by assumption, and $w_0 > 0$ now guarantee $V(0; w^\varepsilon) \geq V_u$ for ε small enough. Hence contract w^ε will continue to attract the unemployed for ε small enough, though during the early low wage phase workers will quit at rate λ [as $w_0 < w^s(0)$].

Note that the firm's continuation payoff $\Pi(t; w^\varepsilon)$ satisfies (8) in Theorem 1. As $\Pi(\varepsilon; w^\varepsilon) = \Pi^s(0)$, then at $t = \varepsilon^-$;

$$\frac{d\Pi}{dt} = [\delta + \lambda]\Pi^s(0) - [p - w_0].$$

But the above choice of w_0 implies $d\Pi/dt < 0$ and so for ε small enough $\Pi(0; w^\varepsilon) > \Pi^s(0)$. However, as this contract implies the same hiring rate as one offering $V_p = V^s(0)$ (both only attract the unemployed) this deviating contract generates greater steady state profit which contradicts the definition of a replication.

Proof of Theorem 2.

Establishing the existence of a replication requires finding a $\overline{w} \in (w_1, p)$ and corresponding Candidate Replication which satisfies the additional boundary condition $\widetilde{V}^s(0|\overline{w}) = V_u$.

Recall that the proof of Lemma A establishes that for any $\overline{w} \in (w_1, p)$, a Candidate Replication exists. Further, that solution corresponds to the saddle path to the differential equations described in proposition 2, where the saddle path implies $w = \widetilde{w}^s(t|\overline{w})$ is strictly increasing in t. Given that, it is now helpful to re-parameterize the saddle path as follows.

Given the baseline salary scale $w = \widetilde{w}^s(t|\overline{w})$ (which exists) define the inverse function $\tau(w|\overline{w}) = (\widetilde{w}^s)^{-1}(w|\overline{w})$ which is the salary point at which a worker is paid wage w on the baseline salary scale [in that Candidate Replication]. Given a worker is currently receiving wage w, a Candidate Replication then implies the worker's expected payoff, denoted $V^e(w|\overline{w})$,is given by

$$V^e(w|\overline{w}) = \widetilde{V}^s(\tau(w|\overline{w})|\overline{w}).$$

Similarly we can define $F^e(w|\overline{w}) = \widetilde{F}^s(\tau(.)|\overline{w})$, and $S^e(w|\overline{w}) = \widetilde{S}(\tau(.)|\overline{w})$.

This allows us to describe the saddle path in w-space. In particular, as a candidate replication implies $\widetilde{V}^s(t|.) \equiv V^e(\widetilde{w}^s(t|.)|.)$ we obtain

$$\frac{dV^e}{dw} = \frac{d\widetilde{V}^s/dt}{d\widetilde{w}^s/dt}$$

and similarly for $dF^e/dw, dS^e/dw$. Hence, we can rewrite the conditions of Proposition 2 as follows.

Proposition 3.

Given $\overline{w} \in (w_1, p)$, a Candidate Replication implies a baseline salary scale $\widetilde{w}^s(t|\overline{w})$ and a triple $\{V^e, F^e, S^e\}$ satisfying

$$\frac{dV^e}{dw} = \frac{-u(w) - \lambda S^e + \delta V^e}{\delta \left[\frac{p-w}{p-\overline{w}}\right]^{1/2} - [\delta + \lambda(1 - F^e)]}, \tag{26}$$

$$\frac{dF^e}{dw} = \left(\frac{-u''(w)}{u'(w)^2}\right) \frac{\delta[-u(w) - \lambda S^e + \delta V^e]}{\lambda[p - \overline{w}]^{1/2}[p - w]^{1/2}}; \tag{27}$$

$$\frac{dS^e}{dw} = -[1 - F^e]\frac{dV^e}{dw}, \tag{28}$$

and the boundary conditions

(a) $(V^e, F^e, S^e) = (\overline{V}, 1, 0)$ at $w = \overline{w}$, and

(b) $dV^e/dw = u'(\overline{w})[p - \overline{w}]/[\delta(p - \overline{w} + 0.5)]$ at $w = \overline{w}$.

Proof The differential equations follow directly from Proposition 2 and the definitions of $\{V^e, F^e, S^e\}$. However, note that the initial conditions imply (26) is not well defined at $w = \overline{w}$. Instead the value of dV^e/dw (which is given in (b)) is determined by the eigenvectors associated with the saddle path at the limiting steady state (as described in the Proof of Lemma A).

This Proposition transforms the dynamical system describing a Candidate Replication to an initial value problem. Given starting values $(\overline{V}, 1, 0)$ at $w = \overline{w}$, iterate these differential equations backwards with w, stopping at $w = w_0$ where $w_0 = w_0(\overline{w})$

as previously defined. Most importantly, we know that the Candidate Replication implies $\widetilde{w}^s(0|\overline{w}) = w_0$. Hence by definition of V^e, the boundary condition $\widetilde{V}^s(0|\overline{w}) = V_u$ is satisfied if and only if $V^e(w_0|\overline{w}) = V_u$. Establishing Theorem 2 simply requires establishing that a $\overline{w} \in (w_1, p)$ exists where $V^e(w_0|\overline{w}) = V_u$. The Candidate Replication with that particular value of $\overline{w}$ then defines a Replication.

Claim B3.

The conditions of Proposition 3 imply $V^e(w_0(\overline{w}) \mid \overline{w})$ is continuous in $\overline{w}$ for all $\overline{w} \in (w_1, p)$,and assumption A2 implies,

(i) $\lim_{\overline{w}\to w_1^+} V^e(w_0|\overline{w}) = -\infty$,and

(ii) $\lim_{\overline{w}\to p^-} V^e(w_0|\overline{w}) = u(p)/\delta$.

The proof is relegated to Appendix C which considers the properties of the differential equation system defined in Proposition 3 in detail. However given any $V_u < u(p)/\delta$, Claim B3 now implies there exists $\overline{w} \in (w_1, p)$ where $V^e(w_0|\overline{w}) = V_u$. Hence for that value of $\overline{w}$, the corresponding Candidate Replication defines a replication which completes the proof of Theorem 2.

8 Appendix C.

This Appendix focuses on the properties of the differential equation system defined in Proposition 3 in Appendix B, which we quickly restate.

Given $\overline{w} \in (w_1, p)$, a Candidate Replication implies a baseline salary scale $\widetilde{w}^s(t \mid \overline{w})$ and a triple $\{V^e, F^e, S^e\}$ satisfying

$$\frac{dV^e}{dw} = \frac{-u(w) - \lambda S^e + \delta V^e}{\delta\left[\frac{p-w}{p-\overline{w}}\right]^{1/2} - [\delta + \lambda(1 - F^e)]}, \tag{29}$$

$$\frac{dF^e}{dw} = \left(\frac{-u''(w)}{u'(w)^2}\right)\frac{\delta[-u(w) - \lambda S^e + \delta V^e]}{\lambda[p-\overline{w}]^{1/2}[p-w]^{1/2}}; \tag{30}$$

$$\frac{dS^e}{dw} = -[1 - F^e]\frac{dV^e}{dw}, \tag{31}$$

subject to the boundary conditions

(BC1) $(V^e, F^e, S^e) = (\overline{V}, 1, 0)$ at $w = \overline{w}$, and

(BC2) $dV^e/dw = u'(\overline{w})[p - \overline{w}]/[\delta(p - \overline{w} + 0.5)]$ at $w = \overline{w}$.

Note that $\widetilde{w}^s$ strictly increasing and (22) imply the denominator in (29) is strictly positive for all $w < \overline{w}$ (along the solution path).

Proof of Claim B3. We prove each part of Claim B3 in turn.

Step 1. To prove $V^e(w_0(\overline{w})|\overline{w})$ is continuous for all $\overline{w} \in (w_1, p)$ we first establish the following Claim.

Claim C1. For any $w \in (0, \overline{w})$, $V^e(w|\overline{w})$ is continuous in $\overline{w}$

Proof. Consider the Candidate Replication given $\overline{w}$. Then for $w = \overline{w} - \varepsilon$, where $\varepsilon > 0$ but small, the boundary conditions (BC1),(BC2) imply

$$V^e(\overline{w} - \varepsilon|\overline{w}) = \frac{u(\overline{w})}{\delta} - \varepsilon u'(\overline{w})[p - \overline{w}]/[\delta(p - \overline{w} + 0.5)] + o(\varepsilon)$$

$$F^e(\overline{w} - \varepsilon|\overline{w}) = 1 - o(\varepsilon), S^e(\overline{w} - \varepsilon|\overline{w}) = o(\varepsilon)$$

where $o(\varepsilon)$ describes the residual term which has the property $\lim_{\varepsilon\to 0}[o(\varepsilon)/\varepsilon] = 0$.

Now for $w \leq \overline{w} - \varepsilon$, define $\Delta V^\varepsilon(w) = V^e(w|\overline{w}) - V^e(w|\overline{w} - \varepsilon)$ which describes the distance between these two trajectories at any point w. Similarly define $\Delta F^\varepsilon(w)$, $\Delta S^\varepsilon(w)$. The above establishes that $\Delta V^\varepsilon(\overline{w} - \varepsilon) = 0(\varepsilon)$, while $\Delta F^\varepsilon(\overline{w} - \varepsilon) = o(\varepsilon)$, $\Delta S^\varepsilon(\overline{w} - \varepsilon) = o(\varepsilon)$, i.e., these trajectories are initially arbitrarily close to each other at $w = \overline{w} - \varepsilon$.

Now use backward iteration. Along the saddle path (in any Candidate Replication) Lemma A establishes that $dw^s/dt > 0$, and so (22) implies $\delta[(p-w)/(p-\overline{w})]^{1/2} - [\delta+\lambda]+\lambda F^e > 0$ for all $w < \overline{w}$. Hence, for any $w \in (0, \overline{w}-\varepsilon]$, the differential equation system (26)-(28) describing V^e, F^e, S^e is continuously differentiable in $\overline{w}, V^e, F^e, S^e$. Hence starting at $w = \overline{w} - \varepsilon$ implies

$$\frac{d}{dw}[\Delta V^\varepsilon] = \frac{dV^e(w|\overline{w})}{dw} - \frac{dV^e(w|\overline{w} - \varepsilon)}{dw} = 0(\varepsilon);$$

and similarly for $\Delta F^\varepsilon, \Delta S^\varepsilon$; i.e. the trajectories separate very slowly with w. Backward iteration and integration now imply $\Delta V^\varepsilon(w), \Delta F^\varepsilon(w), \Delta S^\varepsilon(w)$ remain $0(\varepsilon)$ while $w > 0$ (i.e., the paths remain close together over any bounded interval $[w, \overline{w} - \varepsilon]$). Of course, $\Delta V^\varepsilon(w) = 0(\varepsilon)$ establishes Claim C1.

The following corollary establishes Step 1.

Corollary of Claim C1. $V^e(w_0(\overline{w})|\overline{w})$ is continuous in $\overline{w}$ for all $\overline{w} \in (w_1, p)$.

Proof. The definition of w_0 implies $w_0(\overline{w} - \varepsilon) = w_0(\overline{w}) - \varepsilon(\lambda + \delta)^2/\lambda^2 > 0$. As the differential equations in Proposition 3 are continuously differentiable in w for $w > 0$

[along the saddle path], then for ε small enough

$$V^e(w_0(\overline{w}-\varepsilon)|\overline{w}-\varepsilon) = V^e(w_0(\overline{w})|\overline{w}-\varepsilon) + 0(\varepsilon)$$

and Claim C1 now establishes the corollary.

Step 2.

$$\lim_{\overline{w}\to p} V^e(w_0|\overline{w}) = u(p)/\delta.$$

Proof As this limit also implies $w_0 \to p$, then $w^s(t) \in [w_0, p]$ for all t in any Candidate Replication implies all wages paid are arbitrarily close to p.

Step 3.

$$\lim_{\overline{w}\to w_1} V^e(w_0|\overline{w}) = -\infty.$$

Proof Fix a $\overline{w} \in (w_1, p)$ and define

$$\phi(w|\overline{w}) = \delta V^e(w|\overline{w}) - u(w) - \lambda S^e(w|\overline{w}). \tag{32}$$

Differentiating (32) with respect to w and using (26) and (28) implies:

$$r(w|\overline{w})\phi - \frac{d\phi}{dw} = u'(w), \tag{33}$$

where

$$r(w|\overline{w}) = \frac{\delta + \lambda(1-F^e)}{\delta\left[\frac{p-w}{p-\overline{w}}\right]^{1/2} - [\delta + \lambda(1-F^e)]}.$$

As the definition of ϕ implies $\phi = 0$ at $w = \overline{w}$, integrating (33) implies

$$\phi(w|\overline{w}) = \int_w^{\overline{w}} e^{-\int_w^y r(x|\overline{w})dx} u'(y)dy.$$

Furthermore, as $\int_w^{\overline{w}} r(x|\overline{w})dx = \infty$,[9] integration by parts implies

$$\phi(w|\overline{w}) = \int_w^{\overline{w}} r(y|\overline{w})e^{-\int_w^y r(x|\overline{w})dx}[u(y) - u(w)]dy.$$

[9] Define transformation $x = w^s(t)$ and note that

$$\int_w^{\overline{w}} r(x \mid \overline{w})dx = \int_\tau^\infty [\delta + \lambda(1 - F^s(t))]dt.$$

Also note this implies

$$\int_w^{\overline{w}} r(y \mid \overline{w})e^{-\int_w^y r(x|\overline{w})dx}dy = 1.$$

Hence putting $w = w_0$, (32) implies:

$$\begin{aligned}\delta V^e(w_0|\overline{w}) &= \int_{w_0}^{\overline{w}} r(w|\overline{w}) e^{-\int_{w_0}^{w} r(x|\overline{w})dx}[u(w) - u(w_0)]dw \\ &\quad + u(w_0) + \lambda S^e(w_0|\overline{w})\end{aligned}$$

and as $\int_{w_0}^{\overline{w}} r(w|\overline{w}) e^{-\int_{w_0}^{w} r(x|\overline{w})dx} dw = 1$, this reduces to

$$\delta V^e(w_0|\overline{w}) = \int_{w_0}^{\overline{w}} r(w|\overline{w}) e^{-\int_{w_0}^{w} r(x|\overline{w})dx} u(w)dw + \lambda S^e(w_0|\overline{w}) \tag{34}$$

Now the definition of S in the text [and transform $w = \widetilde{w}^s(t \mid \overline{w})$] implies

$$\begin{aligned}S^e(w_0|\overline{w}) &= \int_{w_0}^{\overline{w}} [V^e(w|\overline{w}) - V^e(w_0|\overline{w})] dF^e(w|\overline{w}) \\ &\leq [\overline{V} - V^e(w_0|\overline{w})],\end{aligned}$$

and so using this equation in (34) and re-arranging gives

$$(\lambda + \delta)[V^e(w_0|\overline{w}) - \overline{V}] \leq \int_{w_0}^{\overline{w}} r(w|\overline{w}) e^{-\int_{w_0}^{w} r(x|\overline{w})dx}[u(w) - u(\overline{w})]dw.$$

Note that the integral on the RHS is a weighted average of $[u(w) - u(\overline{w})]$, where these weights necessarily add up to one. Further, this integral increases as we reduce $r(w|.)$ for any $w < \overline{w}$ (reducing $r(w|.)$ reduces the weight on $u(w)$ and increases it on all $u(w') > u(w)$). Furthermore, the definition of r implies

$$r \geq \frac{\delta}{\delta[\frac{p-w_0}{p-\overline{w}}]^{1/2} - \delta}$$

and the definition of w_0 now implies $r(w \mid \overline{w}) \geq \delta/\lambda$. Hence

$$\begin{aligned}(\lambda + \delta)[V^e(w_0|\overline{w}) - \overline{V}] &\leq \int_{w_0}^{\overline{w}} \frac{\delta}{\lambda} e^{-\frac{\delta}{\lambda}[w - w_0]}[u(w) - u(\overline{w})]dw \\ &\leq \frac{\delta}{\lambda} e^{-\frac{\delta[\overline{w} - w_0]}{\lambda}} \int_{w_0}^{\overline{w}} [u(w) - u(\overline{w})]dw\end{aligned}$$

Finally, assumption A2, that

$$\int_0^a u(w)dw = -\infty \text{ for any } a > 0$$

now implies $V^e(w_0 \mid \overline{w}) \to -\infty$ as $\overline{w} \to w_1, w_0 \to 0$.

This completes the proof of Claim B3.

Proof of Theorem 3. Identifying a Market Equilibrium requires finding a $\overline{w} \in (w_1, p)$ and a Candidate Replication so that (17) is satisfied. Given the definitions of V^e, F^e, S^e in the proof of Theorem 2, such a fixed point requires finding a $\overline{w}$ where

$$\delta V_u = u(b) + \lambda \int_{V_u}^{\overline{V}} [V^e(w|\overline{w}) - V_u] dF^e(w|\overline{w}). \tag{35}$$

Claim B4. A Market Equilibrium exists if and only if a $\overline{w} \in (w_1, p)$ exists which satisfies

$$u(b) = \int_{w_0}^{\overline{w}} r(w|\overline{w}) e^{-\int_{w_0}^{w} r(x|\overline{w})dx} u(w) dw \tag{36}$$

where r is defined in the Proof of Claim B3.

Proof. As a replication requires $V^e(w_0|\overline{w}) = V_u$, (35) for a Market Equilibrium implies

$$\begin{aligned}
\delta V^e(w_0|\overline{w}) &= u(b) + \lambda \int_{V^e(w_0|\overline{w})}^{\overline{V}} [V^e(w|\overline{w}) - V^e(w_0|\overline{w})] dF^e(w|\overline{w}) \\
&= u(b) + \lambda S^e(w_0|\overline{w})
\end{aligned}$$

by definition of S. Claim B4 now follows from (34) in the Proof of Claim B3.

Given this condition for $\overline{w}$, we now use the arguments demonstrated in the proof of claim B3. In particular, the proof of claim B3 (Step 3) establishes that the RHS of (36) is a weighted average of $u(w)$, where those weights integrate up to one, and which shift as $\overline{w}$ changes value. The proof of Claim B3 (Step 1) implies this integral is a continuous function of $\overline{w}$. The proof of claim B3 (Step 2) implies this integral limits to $u(p)$ as $\overline{w} \to p^-$ (as this also implies $w_0 \to p^-$). Conversely $\overline{w} \to w_1^+$ implies $w_0 \to 0^+$ and given assumption A2, the proof of Claim B3 (Step 3) implies this integral goes to -∞. Hence for any $b \in (0, p)$, a $\overline{w} \in (w_1, p)$ exists which satisfies (36). Given that value for $\overline{w}$, a Market Equilibrium then exists with the corresponding Candidate Replication and $V_u = V^e(w_0|\overline{w})$.

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