Transformations of continuously self-focusing and continuously self-defocusing dissipative solitons

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Abstract: Dissipative media admit the existence of two types of stationary self-organized beams: continuously self-focused and continuously self-defocused. Each beam is stable inside of a certain region of its existence. Beyond these two regions, beams loose their stability, and new dynamical behaviors appear. We present several types of instabilities related to each beam configuration and give examples of beam dynamics in the areas adjacent to the two regions. We observed that, in one case beams loose the radial symmetry while in the other one the radial symmetry is conserved during complicated beam transformations.

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References and links

1. Introduction

The notion of dissipative solitons [1] is a useful concept that allows us to describe, in general terms, a variety of phenomena in physics, chemistry, biology and medicine [2, 3, 4]. Some specific features of these formations are common for all of them, independent of the problem that we are solving and the model we are using. Many problems in optics involving gain and loss can be formulated in terms of a single “master equation” that is, in one or another form, of the complex cubic-quintic Ginzburg-Landau equation (CGLE) type [5, 6]. This class of problems includes among others, passively mode-locked laser systems [7, 8, 9], optical parametric amplifiers [10], wide aperture lasers [11], spatial dissipative solitons [12] and three-wave mixing in optical fibers [13]. Many of these problems can be (1+1) dimensional and they deal with the evolution in time of spatially self localized one-dimensional fields [14]. More complicated are the (2+1) dimensional problems, when the optical field evolving in time is two-dimensional. Examples of two-dimensional dissipative solitons in optics are cavity solitons (CS) [15] and solitons in wide aperture lasers [11]. The variety of transverse profiles of two dimensional solitons can be enormous. These profiles can evolve in time as well as can be stationary and stable. An interesting observation is that the majority of these stable stationary profiles are not radially symmetric.

As in conservative homogeneous 2-D systems, radially symmetric node-less beams should be considered as the ground-state modes of its corresponding 2-D nonlinear dynamical system. Ground state modes are stable in certain regions of the parameter space. At the edges of the stability regions, we can expect the appearance of either pulsating radially symmetric solutions or stable radially asymmetric solutions. Transformations happen at the points of bifurcation which in a multi-parameter space can be generalized to surfaces of bifurcations. Other types of solutions can also appear as a result of such bifurcations. Thus, it is natural to find, as a first step, the set of radially symmetric solutions and then extend it to more complicated cases. However, it is occurred that dissipative systems in contrast to conservative ones admit more than one class of radially symmetric solutions.

Namely, we have shown, in a previous work [16], that there are two major types of radially symmetric dissipative solitons of the two-dimensional CGLE. Both have bell-shaped amplitude profiles without nodes in their radial field distribution. The qualitative difference between them is their either concave or convex phase profiles, making them either continuously self-focusing...
or continuously self-defocusing beams. Both are robust in their corresponding regions of stability. However, at the edges of the stability regions, their behaviors differ significantly. The study of this difference can shed light on the properties of the stationary solitons as well. Thus, our aim in this work is to progress in this direction.

In particular, we found that in (1+1)-D configuration, dissipative solitons beyond their region of stability can be transformed into pulsating solutions and these into exploding solitons [17]. The latter stay as stationary solitons for certain distance of propagation and intermittently explode into many pieces [18]. In the (2+1)-D case, this is the only instability we were able to find around continuously self-defocusing beams (dissipative antisolitons). On the other hand, continuously self-focusing beams have a rich structure of bifurcations around the boundaries of the region of their stable existence. Radially symmetric beams can be transformed into beams of elliptic shape, beams of completely asymmetric shape, pulsating beams etc., when we change the parameters of the system. The study of these transformations is one of the interesting aspects of the theory of dissipative solitons. Thus, in this work we give (2+1)-D solutions with distinctive features which appear close to the boundaries of the regions of existence of continuously self-focusing and continuously self-defocusing solitons.

2. Statement of the problem

Our present study is based on an extended complex Ginzburg-Landau equation [16], that includes cubic and quintic nonlinear terms. In normalized form, this propagation equation reads:

\[ i\psi_z + D\nabla^2 \psi + |\psi|^2\psi + \nu|\psi|^4\psi = i\delta\psi + i\varepsilon|\psi|^2\psi + i\beta\nabla^2_\perp \psi + i\mu|\psi|^4\psi. \]  

(1)

where \( \psi = \psi(x,y,z) \) is the normalized envelope of the field,

\[ \nabla^2_\perp = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \]

is the transverse Laplacian, \( z \) is the propagation distance and \( (x,y) \) are the transversal coordinates, \( D \) is the diffraction coefficient that without loss of generality can be set to 1, \( \nu \) is the saturation coefficient of the Kerr nonlinearity, \( \delta \) represents linear losses, \( \varepsilon \) is the nonlinear gain coefficient, \( \beta \) stands for angular spectral filtering of the cavity, and \( \mu \) characterizes the saturation of the nonlinear gain.

Stationary solutions of Eq. (1) can be found in wide regions of the space of the equation parameters [16]. They are usually radially symmetric. In this case, the regions of their existence can be found semi-analytically using approximate methods such as in Ref. [16], with good agreement with the numerically obtained ones by solving directly Eq. (1). Around the boundaries of existence of these radially symmetric stationary solutions, more complicated objects can be observed. Then numerics appears to be the main tool to study them. They can be detected when we start from the region of stationary solutions and continuously change one or two of the equation parameters fixing all the others. At some point, the radially symmetric solution ceases to be stable, and a new solution appears.

The main parameter of the solution that we monitor in simulations is the beam power, \( Q \):

\[ Q(z) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |\psi(x,y,z)|^2 dx dy, \]  

(2)

The value of \( Q \) for a localized solution is finite and changes smoothly while the solution stays within the region of existence of a certain type of solitons. The value of \( Q \) changes abruptly when there is a bifurcation and the solution jumps from a branch of solitons that become unstable to another branch of stable solitons. Thus, monitoring \( Q \) allows us to find bifurcations in
an easy way. Observing a finite $Q$ also reveals the stability of a solution. As soon as the solution becomes unstable, it diverges and the value of $Q$ either converges to another fixed value, vanishes or goes to infinity. In the case that the resulting solution is pulsating, instead of a fixed value of $Q$, we obtain a band of $Q$ values that correspond to the changing power of the soliton. As we are interested in bifurcations from radially symmetric solitons, we start simulations from those found in Ref.[16].

![Region I and Region II](image)

**Fig. 1.** Regions (in blue) in the parameter space with radially symmetric stable stationary beams. The plot on the left (a) shows the region for continuously self-focusing beams (region I) and on the right (b) the one for continuously self-defocusing beams (region II). The location of these two regions in the five-dimensional parameter space is such that they cannot be represented in the same plane. Soliton solutions beyond these regions are either non-stationary or lose the radial symmetry. Below, we give examples located at the points indicated here by green, yellow and red lines. The change in color corresponds to a bifurcation. The arrows indicate the direction in which the parameters have been changed while obtaining a particular bifurcation diagram.

Figure 1(a) shows a region in the parameter space (region I) where radially symmetric continuously self-focusing beams have been found in our previous work [16]. A region of existence of continuously self-defocusing beams (region II) is shown in Fig. 1(b). Our aim here is to study what happens around these two regions. A complete mapping is a difficult task because of the multiplicity of the different types of solutions and the myriad of bifurcations between them. Thus, we restrict ourselves to certain examples which on one hand show these difficulties and on the other hand represent the possible types of solutions. Any attempt of classification of the solutions at this stage would be premature.

We have chosen a few lines that cross the boundaries of the blue regions in Fig. 1 and studied the types of solutions that appear around them. When two types of solutions are stable simultaneously, the simulations produce the one which is closer to the initial condition. Moving step by step in small increments of $\varepsilon$ or $\beta$ we can use as the initial condition the solution obtained at the previous step. This technique allows us to follow the branch of the solution that was stable at the previous step. Moving first in one direction and then in the opposite direction provides us with the information about the types of solutions that exist on this line and the bifurcations between them. The bifurcation diagrams presented below are obtained in this way. Shifting the direction in the parameter space along which we move, changes the bifurcation diagram but keeps the major types of solutions obtained here intact.

As the blue region I in Fig. 1(a) represents the “ground state” solutions, there are no solitons
below it because the power pumped into the system is smaller for lower values of \( \varepsilon \). Solutions with higher power \( Q \) exist above the blue region I. The specific cases presented below are obtained on the two vertical lines in Fig.1(a) and on the upper left horizontal line in Fig.1(b).

Namely, Fig.2 corresponds to the right hand side vertical line with \( \mu = -0.03 \) at (a). Figs. 3, 4 and 5 are obtained for \( \mu = -0.08 \), i.e. on the left hand side vertical line in (a). The horizontal line in (b) for \( \varepsilon = 5 \) corresponds to the case shown in Fig. 12. Each color interval of the lines (green, yellow or red) corresponds to a particular type of solution. The arrows show the direction in which the parameters \( \varepsilon \) or \( \beta \) were changed when obtaining a bifurcation diagram. When there is no arrow, the bifurcation diagram does not depend on the direction of changing the parameter. The red solid circle in (a) represents the location of the case shown in Fig. 8.

3. Bifurcation diagrams

As explained in the introduction, we are interested in soliton dynamics at the parameter regions adjacent to the regions of stable stationary solitons. Firstly we concentrate on the regions close to the region of continuously self-focusing radially symmetric solitons (region I). The existence of boundaries manifests the instability of the stationary soliton and the corresponding bifurcation of the soliton into a different state. These boundaries are surfaces in a five-dimensional parameter space \( (\nu, \delta, \varepsilon, \beta, \mu) \). They can be visualized using two dimensional plots with three of the parameters being fixed. Transitions related to the loss of stability can take a multiplicity of different forms. Each one has to be studied individually. What can be done in the frame of one publication is to present the crossing of the boundary of stability at a single point and in a single direction. We may expect that moving this point along the boundary will leave the qualitative features of the transition being similar to the one that we present when the point is still in close proximity to the original one. At the same time, we may also expect that the major qualitative difference in the type of transition will happen when we switch from the boundary of solitons to the boundary of anti-solitons. This was explained, roughly, in Ref.[16].

One of the simplest bifurcations that we have found is the transition from a radially symmetric solution to a soliton of elliptic shape that is also stable and stationary. This transformation happens close to the blue region in Fig. 1(a) on the short green vertical line with \( \mu = -0.03 \). The bifurcation diagram for such transformation is shown in Fig. 2. The beam with radial symmetry loses its stability after the bifurcation (blue dashed curve in the diagram). This can be considered as an example of symmetry breaking bifurcation in the case of 2-D beams. Due to the symmetry of the equation 1 the long axis of the elliptic soliton can have any direction in the \((x,y)\)-plane, what causes the solution to rotate. This elliptic soliton is the only stationary,
although rotating, solution different from the radially symmetric one that we were able to find for \( \mu = -0.03 \). It exists in a stripe (marked in red) above the upper boundary of the blue region (region I). Depending on the initial conditions, the beam may oscillate around the configuration with radial symmetry before finally converging to the stationary beam of elliptic shape. An example of such oscillations for different set of the equation parameters is given in section 5. Although in that case the final stable solution is different.

Figure 3(a) shows, in red dotted line, the beam power of the solutions obtained when we increase \( \varepsilon \), starting from a radially symmetric solution at \( \varepsilon = 0.2 \). The power \( Q \) increases continuously until an abrupt transition occurs at \( \varepsilon \approx 0.52 \). The latter is shown by the black arrow pointing up. It indicates the transformation of the single soliton solution into a stable rotating double-beam complex, which exists in the range of \( \varepsilon \) from 0.47 to 0.6. Thus, from here, we can either increase or decrease \( \varepsilon \) and stay at the same branch of solitons. Reaching further \( \varepsilon = 0.6 \), we observe a second bifurcation where \( Q \) takes a continuous interval of values. It corresponds to pulsating and simultaneously rotating double beams with an oscillating value of \( Q \).

Alternatively, when we decrease the value of \( \varepsilon \), \( Q \) follows the blue line. Two solutions can be observed in a certain interval of \( \varepsilon \) values, manifesting a hysteresis phenomenon. Moreover, we can observe additional bifurcations in this region which can be clearly seen if we magnify properly this region of \( \varepsilon \) (see Fig. 3(b)). When we move from right to left, three main transitions can be clearly seen: i) transition from a double-beam complex (black solid line) to an asymmetric comma-shaped solution (blue dashed line), both being stationary, ii) transition to pulsating solutions (vertical lines indicating that \( Q \) takes a continuous set of values) and iii) transition to elliptically shaped beams along the red dotted line. The red and blue vertical arrows indicate the values of \( \varepsilon \) for which the solution is plotted in Figs. 4(a) and 4(b) respectively. These 3D-plots illustrate the type of solutions that we obtain in the ranges between the bifurcations. Fig. 4(a) shows the solution with an elliptic profile while Fig. 4(b) shows the solution that looks as composed of two unequal beams closely attached to each other. The profile would look more like a comma-shaped on a contour plot.
Figure 3 shows that soliton solutions are highly sensitive to the change of $\varepsilon$. Indeed, if we further magnify the scale of $\varepsilon$ in Fig. 3(b) we will be able to see more bifurcations. These are shown in Fig. 5. For the sake of clarity, we just show here the maximal and minimal values of $Q(z)$ denoted as $Q_M$ and $Q_m$ respectively. The green line at the left hand side of this plot corresponds to rotating solitons with fixed elliptic profile. Their fixed, although rotating, shape results in their fixed values of $Q$. The green line at the right hand side of the plot corresponds to highly asymmetric solitons with the fixed shape of a comma that are also rotating. Typical examples of the two shapes are shown in Fig. 4. Several bifurcations appear in between these two regions. The first one, when reducing $\varepsilon$ appears as a transition from stationary solution to pulsating soliton with a single period of pulsations. The resulting $Q$ oscillates between the red
and blue curves that correspond to the maximal and minimal values of $Q$ of the pulsations. The oscillations of $Q$ are very close to being harmonic for pulsating solutions with a single period. The pulsations turn into period doubled ones at the second bifurcation in Fig. 5. As a result, the two curves split into four at this point. Further reduction in $\varepsilon$ results in the soliton evolution that looks very much like chaotic before finally turning into an elliptically shaped stationary solitons. Generally speaking, a myriad of other bifurcations may occur at this short interval.

4. Period-1 pulsating solitons

Pulsating solutions appear in dissipative systems as naturally as stationary ones as they represent limit cycles of the infinite-dimensional dynamical system [18]. For 1-D systems they are usually located near the stationary solutions in the parameter space. The transformation of stationary solutions into pulsating ones occurs in the form of an Andronof-Hopf bifurcation. Due to the relative simplicity of the soliton profile in the 1-D case, this bifurcation can be studied using a trial function approximation in combination with the method of moments [22]. In the 2-D case, the pulsating soliton profile can be much more complicated. In fact, in most of the cases it lacks the radial symmetry, thus making the approximation with a trial functions difficult. Radially symmetric pulsating solutions exist only in the vicinity of the region of continuously self-defocusing beams (region II). Near the region I, the shape of pulsating solitons look similar to the examples shown in Fig. 6. The soliton profile changes continuously upon propagation. The profiles given in Fig. 6 are taken when the value of $Q$ takes its maximal or minimal value inside each period of pulsations. As we can see, in each case, the beam lacks the radial symmetry.

![Pulsating 2-D soliton profiles](image)

**Fig. 6.** Pulsating 2-D soliton profiles when the oscillating power $Q$ takes its (a) maximal and minimal values (dashed and dotted vertical lines in Fig. 7 respectively).

The beam power $Q$ in this case is a periodic and almost harmonic function of the propagation distance $z$. This function calculated for a fixed set of equation parameters is shown in Fig. 7(a)). The green region in this plot shows one period of evolution of the soliton. The dashed vertical lines determine the value of $z$ for which the profiles in Fig. 6 are taken. Namely, the red dashed line corresponds to a maximum of power $Q$ while the blue dotted line corresponds to a minimum of $Q$. The beam rotates, continuously, changing its profile at the same time. The animation of the beam evolution presented in Fig. 7(b) shows clearly that the beam is at the verge of splitting into two separate beams. Thus, the dissipative pulsating soliton in this example should be considered as a two-soliton complex with continuous unsuccessful attempts to split into two independent beams. In the particular case that is shown above, one of the beams comprising the complex is weaker than the other one making the structure asymmetric. In addition, the whole structure is rotating. In fact, it is the asymmetry of the soliton that pushes it into a spiraling motion. The
periodicity itself has nothing to do with its rotation. The angular speed of rotation is not related to the frequency of pulsations. Thus, the rotation does not influence the periodicity in evolution of $Q$. From the movie, we can also notice the average transverse motion of the whole structure. This is not surprising because the momentum as well as the power $Q$ are not conserved in dissipative systems. The asymmetry of the beam causes its motion because of the asymmetry in the energy flow across the soliton. The asymmetry of the soliton changes with the speed different from the angular speed of its rotation thus resulting in the average transverse motion.

![fig7](image)

Fig. 7. (a) Periodic evolution of $Q$ versus $z$ for a pulsating soliton with a single period. The red and blue vertical lines show a maximum and a minimum of $Q$. i The corresponding profiles are shown in Fig. 6(a) and Fig. 6(b) respectively. The actual evolution during a period (marked in green in (a)) is shown in (Media 1) (b).

5. Elliptic beam oscillations

Soliton pulsations in the example of the previous section do not have any radial symmetry. The question arises whether pulsations that keep the radial symmetry are at all possible around the region I. When shifting to a different point (e.g., $\mu = -0.05$) along the upper boundary of region I in Fig.1a we can find symmetric beam oscillations which, after some propagation distance, end up being transformed into an elliptic shape solution. These are shown in Fig. 8. The main plot shows the evolution of $Q$ with $z$. Simulations started from a slightly elongated beam at $z = 0$, namely

$$\psi(x,y,0) = 5 \exp \left[ -\left( \frac{x}{1.2} \right)^2 - \left( \frac{y}{1.1} \right)^2 \right]$$

As the parameters are chosen at the margin of stability of stationary solutions, the beam cannot converge to the radially symmetric solution of Eq. (1). Instead, the small initial asymmetry increases and after an initial transition, the beam converges to a pulsating state without radial symmetry. The beam still has radially symmetric profile when $Q$ takes its maximum value but elongates alternatively either in $x$ or in $y$ directions when $Q$ takes its minimal value. Three consecutive color coded contour plots for the intensity profiles at the extremal points of $Q(z)$ are shown in the lower left inset panels of the figure. The orientation of the elongated profile changes from one consecutive minimum to another one, as shown in the inset. Oscillations of $Q$ are harmonic with high accuracy as shown in the upper left panel of the figure which is a magnified version of the $Q(z)$-plot between $z = 100$ and $z = 110$. These oscillations are persistent and last from $z \approx 60$ till $z \approx 260$. However, the pulsating solution at the chosen set
of parameters is not completely stable and it is transformed into another stable solution at the end of this interval. Namely, it is transformed into another elongated structure which can be considered as a two-soliton complex. The color contour plot for the intensity profile is shown in the upper right panel of the figure. This beam is also rotating around the central point of its symmetry. Thus, we tend to think that pulsations that keep the radial symmetry at any \( z \) do not exist around the region I. Additional studies are needed to give a more definite answer to the question posed in the beginning of this section.

6. More complicated pulsations

One of the distinctive features of the simple pulsations presented in the two previous sections is the almost harmonic evolution of \( Q(z) \). This feature disappears when we move into the visibly chaotic region in the center of the diagram in Fig. 5. The beam evolution is still periodic but far from being harmonic. Three examples of complicated evolution of \( Q(z) \) are shown in Fig. 9(a). The three curves (blue, green and red) have a few common features. The power of the beam experiences periodic bursts with preliminary oscillations of smaller amplitude. The latter are preceded by intervals of almost constant \( Q \). These complicated changes of \( Q \) appear due to the involved evolution of the soliton profile itself. The animation of the beam evolution is shown in Fig. 9(b). In the intervals with relatively small changes of \( Q \), the profile takes an elongated asymmetric shape. Due to the asymmetry, the beam center rotates in the transverse plane. Subsequent oscillations of \( Q \) correspond to the appearance of two maxima in the beam profile and a tendency to split the solution into two beams with radiation waves shed around them. The beam profile at this stage has the familiar comma shape. The bursts of power appear when the two beams are separated by the maximal distance inside each period. The beam itself
wanders around with the center of mass moving in circles in the transverse direction.

Figure 9. (a) Periodic evolution of $Q$ versus $z$ for three values of epsilon. As epsilon decreases the maxima of $Q$ increases as well as the separation between “bursts”. The red curve is horizontal for large interval of values of $z$ during which the solution is of elliptic shape. (b) (Media 2) showing the evolution of the beam profile.

Figure 10(a) shows the trajectory of the main maximum of the optical field in one full period of evolution. The curve is plotted for the value of $\epsilon = 0.4446$ that corresponds to the blue curve in Fig.9a. Fig.10b reproduces more clearly one period of evolution of $Q$ for $\epsilon = 0.4446$. The “quiet” stage of evolution with almost constant $Q$ is shown in Fig. 10(a) in blue line while the “turbulent” stage of evolution, when $Q$ changes substantially, is shown in green line. The latter corresponds to the interval in Fig. 10(b) enclosed inside the green rectangle. The beam moves intermittently from one position to another one inside this green interval. The beam rotates but stays almost at the same position outside of the green box where $Q$ hardly changes. The blue trajectory in Fig. 10(a) is a circle rather than a point because the maximum of the field is shifted relative to the center of mass of the beam. To complete the description we should add that the evolution of $Q$ is strictly periodic rather than intermittent.

Fig. 10. Periodic evolution of the solution for $\epsilon = 0.4446$ (blue curve in Fig.9). The green trajectory in (a) corresponds to the green zone in (b).

Similar behavior can be observed in a small band of $\epsilon$ values from 0.4446 to 0.4451. Six consecutive periods of evolution of the beam for $\epsilon = 0.4449$ are shown in Fig. 11. Each of the
six periods is depicted in a different color. The evolution of the power $Q$ is identical in each of the periods. The trajectory of the maximum of the optical field in each period is similar to the one shown above. Each time, it converges to a circle whose center shifts to another position from one period to the next. The shifts are regular and all circles are located around a fixed point in space. So, in average, the beam stays at the same place in space. The fast part of the trajectory is sensitive to numerical errors. Thus, the boundaries of the used numerical grids are taken to be far away from the beams and calculations have been repeated with several grid sizes and step lengths to be sure that no numerical artifact are presented here.

Fig. 11. a) Trajectory of the peak intensity of the solutions in the (X,Y) plane and b) evolution of $Q$ versus $z$. Six periods are plotted, each with a different color. The position of the solution behaves somehow chaotic.

7. Beam evolution around the region of continuously self-defocusing beams

As the last example, we consider the soliton evolution next to the region of continuously self-defocusing beams (region II) shown in Fig. 1(b). The beam evolution is similar across the whole lower left boundary (black solid curve). Thus, we consider a single line of exit from this region at $\varepsilon = 5$ (tri-color arrow in the top of Fig. 1(b)). The beam is transformed into a pulsating one on the yellow part of the arrow and further into the exploding soliton at the red part of the arrow. However, our simulations show that upon these transformations the beams do not lose the radial symmetry.

A typical example of bifurcation diagram when moving along the tri-color line out of the region of stationary beams is shown in Fig. 12. The power $Q$ is finite and stays constant in $z$ at values of $\beta$ higher than 0.55. At $\beta \approx 0.55$, we observe an Andronov-Hopf bifurcation where the beam starts to pulsate. These pulsations are revealed with the splitting of the $Q$ value into a maximum, $Q_M$, and a minimum $Q_m$ values in the bifurcation diagram. This occurs at $\beta \approx 0.55$. An example of oscillating $Q(z)$ is shown in the inset of Fig. 12 by the magenta dashed curve. It is taken at $\beta = 0.42$. This value of $\beta$ is indicated by the vertical dashed line in the bifurcation diagram itself. As mentioned, the beam stays radially symmetric in these pulsations. Any perturbation with radial asymmetry vanishes on propagation.

At lower values of $\beta$, the evolution becomes chaotic. This happens below $\beta \approx 0.41$. Then instead of just two values of $Q$, we have a wide band of them corresponding to local maxima and minima of $Q$. The beam in this region becomes exploding. Its width increases dramatically to high values and the corresponding power also increases. The soliton explosions are similar
to those observed in the (1 + 1)-D case [17, 18]. They occur intermittently and each explosion is different from the previous one. The increase of $Q$ for one explosion is shown in the inset of Fig. 12 with the amber curve. The radial symmetry is lost at the explosion point but it is restored again in returning back to a beam with radial symmetry. These two types of evolution are the only ones that we were able to observe near the region of existence of self-defocusing beams.

Fig. 12. (a) Bifurcation diagram at the boundary of the region II. Parameters of the system are given inside the plot. An Andronov-Hopf bifurcation of a stationary self-defocusing beam into a pulsating one occurs at $\beta = 0.55$. The inset shows the power evolution with $z$ for two cases: $\beta = 0.42$ (magenta) and $\beta = 0.4$ (amber). In the interval $0.41 < \beta < 0.55$ the beam is pulsating. The $Q(z)$ curve is harmonic. The evolution is chaotic at the values of $\beta < 0.41$. The $Q(z)$-curve (amber) reveals the beam explosions. (Media 3) in (b) shows one period of evolution of the pulsating beam at $\beta = 0.42$.

8. Conclusion

In conclusion, we studied a variety of two-dimensional beams and their evolution in dissipative media beyond the two main regions of existence of stable stationary beams with radial symmetry (regions of ground state solitons). We have found a multiplicity of transverse shapes of solitons and a myriad of bifurcations between them. We have also found that the beam shapes and their evolution depend strongly on which of the two regions in the parameter space we are close to.

Near the region of stable stationary continuously self-focusing solitons (region I of ground state solitons) we have found several types of bifurcations. (1) Bifurcation into a stable stationary beam of elliptic shape. (2) Bifurcation into pulsating beams with harmonic pulsations or more complicated pulsations with highly involved shape evolution. Beams can be pulsating and rotating simultaneously. Some of the evolution scenarios are caused by the attempts of soliton of splitting into two separate beams. Generally speaking, any bifurcation around this region destroys the radial symmetry of the solution.

On the other hand, near the region of stable stationary continuously self-defocusing beams (region II of ground state solitons), the soliton keeps its radial symmetry after the bifurcation. Beams in this case are pulsating but stay radially symmetric. Pulsating beams can further be transformed into exploding ones, that loose the radial symmetry at the moment of the explosion but recover the original profile between the explosions.
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