Analytical and Numerical Studies of Noise-induced Synchronization of Chaotic Systems

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We study the effect that the injection of a common source of noise has on chaotic systems. The paper aims to clarify some recent results on the subject of whether identical systems can become synchronized if subjected to the same source of random noise. We present particular examples of 1-d maps and the Lorenz system, both in the chaotic region, and give numerical evidence that the addition of a common random noise, of large enough intensity, to different trajectories which start from different initial conditions, leads eventually to the perfect synchronization of the trajectories. When synchronization occurs due to the presence of the noise terms, the largest Lyapunov exponent becomes negative. For a simple map we are able to present an analytical calculation that gives useful bounds on the Lyapunov exponent for a system under the effect of noise. Finally, we study the structural stability of the phenomenon and conclude that small differences in the parameters of the chaotic systems lead to small errors in the trajectory synchronization.

The synchronization of chaotic systems has received a lot of attention in the last years. The use of a chaotic signal as a carrier has been suggested as a way of masking the information contained in a message. Amongst the many possible methods, synchronization using noise can offer an additional degree of difficulty for an intruder who tries to recover the message. There have been some contradictory results in the literature on whether chaotic systems can indeed be synchronized using a common source of noise and the issue has began to be clarified only very recently. In this paper we give explicit examples of chaotic systems that can become synchronized by the addition of, Gaussian, white noise. We also analyze the structural stability of the phenomenon, namely, the robustness of the synchronization against a small mismatch in the parameters of the chaotic sender and receiver.

I. INTRODUCTION

One of the most surprising results of the last decades in the field of stochastic processes has been the discovering that fluctuation terms (loosely called *noise*) can actually induce some degree of order in a large variety of non-linear systems. The first example of such an effect is that of *stochastic resonance* [1,2] by which a bistable system responds better to an external signal (not necessarily periodic) under the presence of fluctuations, either in the intrinsic dynamics or in the external input. This phenomenon has been shown to be relevant for some physical and biological systems described by a nonlinear dynamics [3–5]. In purely temporal dynamical systems, other examples include phenomena such as noise-induced transitions [6], noise-induced transport [7], coherence resonance [8–11], etc. In extended systems, noise is known to induce a large variety or ordering effects [12], such as pattern formation [13,14], phase transitions [15–18], phase separation [19,20], spatiotemporal stochastic resonance [21,22], noise-sustained structures [23,24], doubly stochastic resonance [25], amongst many others. All these examples have in common that some sort of *order* appears only in the presence of the right amount of noise.

There has been also some recent interest on the interplay between chaotic and random dynamics. Some counterintuitive effects such as coherence resonance, or the appearance of a quasi–periodic behavior, in a chaotic system in the presence of noise, have been found recently [26]. The role of noise in standard synchronization of chaotic systems has been considered in [27,28], as well as the role of noise in synchronizing non–chaotic systems [29]. In this paper we address the different issue of synchronization of chaotic systems by common random noise sources, a topic that has attracted much attention recently. The accepted result is that, for some chaotic systems, the introduction of the same noise in independent copies of the systems could lead (for large enough noise intensity) to a common collapse onto the same trajectory, independently of the initial condition assigned to each of the copies. This synchronization of chaotic systems by the addition of random terms is a remarkable and counterintuitive effect of noise and although some clarifying papers have appeared recently, still some contradictory results exist for the existence of this phenomenon of noise–induced synchronization. It is the purpose of this paper to give further analytical and numerical evidence that chaotic systems can synchronize under such circumstances and to analyze the structural stability of the phenomenon. Moreover, the examples presented here open the possibility to obtain such a synchronization in electronic circuits, hence suggesting that noise-induced synchronization of chaotic circuits can indeed be used for encryption purposes.

The issue of which is the effect of noise in chaotic systems was considered already at the beginning of the 80's by Matsumoto and Tsuda [30] who concluded that the introduction of noise could actually make a system less chaotic. Later, Yu, Ott and Chen [31] studied the transition from chaos to non–chaos induced by noise. Synchronization induced by noise was considered by Fahy and Hamman [32] who showed that particles in an external potential, when driven by the same random forces, tend to collapse onto the same trajectory, a behavior interpreted as a transition from chaotic to non–chaotic behaviors. The same system has been studied numerically and analytically [33–35]. A paper that generated a lot of controversy was that of Maritan and Banavar [36]. These authors analyzed the logistic map in the presence of noise:

$$x_{n+1} = 4x_n(1 - x_n) + \xi_n \tag{1}$$

where ξ_n is the noise term, considered to be uniformly distributed in a symmetric interval $[-W, +W]$. They showed that, if W was large enough (i.e. for a large noise intensity) two different trajectories which started with different initial conditions but used otherwise the same sequence of random numbers, would eventually coincide into the same trajectory. The authors showed a similar result for the Lorenz system (see section III). This result was heavily criticized by Pikovsky [37] who argued that two systems can synchronize only if the largest Lyapunov exponent is negative. He then shows that the largest Lyapunov exponent of the logistic map in the presence of noise is always positive and concludes that the synchronization is, in fact, a numerical effect of lack of precision of the calculation. The analysis of Pikovsky is confirmed by Longa *et al.* [38] who study the logistic map with arbitrary numerical precision. The criterion of negative Lyapunov exponent has also been shown to hold for other types of synchronization of chaotic systems and Zhou and Lai [39] showed that previous results by Shuai, Wong and Cheng [40] showing synchronization with a positive Lyapunov exponent were again an artifact of the limited precision of the calculation.

Besides the above criticisms, Herzel and Freund [41] and Malescio [42] pointed out that the noise used to simulate Eq.(1) and the Lorenz system in [36] is not really symmetric. While the noise in the Lorenz system is non–symmetric by construction, in the case of the map, the non–zero mean arises because the requirement $x_n \in (0, 1)$, $\forall n$, actually leads to discard those values for the random number ξ_n, which do not fulfill such condition. The average value of the random numbers which have been accepted is different from zero, hence producing an effective *biased* noise, i.e. one which does not have zero mean. The introduction of a non-zero mean noise means that we are altering essentially the properties of the deterministic map. Furthermore, Gade and Bassu [43] argue that the synchronization observed by Maritan and Banavar is due to the fact that the bias of the noise leads the system to a non–chaotic fixed point and conclude that a zero–mean noise can never lead to synchronization in the Lorenz system. The same conclusion is reached by Sánchez *et al.* [44] who study experimentally a Chua circuit and conclude that synchronization by noise only occurs if the noise does not have a zero mean. The same conclusion is obtained studying numerically [45] a single and an array of Lorenz models and experimentally an array of Chua circuits [46] with multiplicative colored noise. Therefore a widespread belief existed that it is not possible to synchronize two chaotic systems by injecting the same noisy unbiased, zero–mean, signal to both of them.

Contrary to these negative results that imply that synchronization of chaotic systems by injection of common noise is only possible when the noise does not have a zero mean, Lai and Zhou [47] have shown that some chaotic maps can indeed become synchronized by additive zero–mean noise. A similar result has been obtained by Loreto *et al.* [48], and by Minai and Anand [49,50], in the case where the noise appears multiplicatively in one of the parameters of the map, and the implications to secure digital communications have been considered in [51,52]. An equivalent result about the synchronization of Lorenz systems using a common additive noise has been shown by the authors of the present paper in [53]. The actual mechanism that leads to synchronization has been explained by Zhai and Lou [47], see also [54]. As Pikovsky [37] required, synchronization can only be achieved if the Lyapunov exponent is negative. The presence of noise allows the system to spend more time in the "convergence region" where the local Lyapunov exponent is negative, hence yielding a global negative Lyapunov exponent. This argument will be developed in more detail in section II, where an explicit calculation in a simple map will confirm the analysis. The results of Lai and Zhou have been extended to the case of coupled map lattices [55] where Pikovsky's criterion has been extended for spatially extended systems.

In this paper we give further evidence that it is possible to synchronize two chaotic systems by the addition of a common noise which is Gaussian distributed and not biased. We analyze specifically a 1-d map and the Lorenz system, both in the chaotic region. The necessary criterion introduced in ref. [37] and the general arguments of [47] are fully confirmed and some heuristic arguments are given about the general validity of our results.

The organization of the paper is as follows. In section II we present numerical and analytical results for some 1-d maps, while section III studies numerically the Lorenz system. In section IV we analyze the structural stability of the phenomenon, i.e. the dependance of the synchronization time on the parameter mismatch. Finally, in section V we present the conclusions as well as some open questions relating the general validity of our results.

II. RESULTS ON MAPS

The first example is that of the map:

$$x_{n+1} = F(x_n) = f(x_n) + \epsilon \xi_n \tag{2}$$

where ξ_n is a set of uncorrelated Gaussian variables of zero mean and variance 1. We use explicitly

$$f(x) = \exp\left[-\left(\frac{x-0.5}{\omega}\right)^2\right] \tag{3}$$

We plot in Fig.(1) the bifurcation diagram of this map. In the noiseless case, we can see the typical windows in which the system behaves chaotically. The associated Lyapunov exponent, λ, in these regions is positive. For instance, for $\omega = 0.3$ (the case we will be considering throughout the paper) it is $\lambda \approx 0.53$. In Fig.(2) we observe that the Lyapunov exponent becomes negative for most values of ω for large enough noise level ϵ. Again for $\omega = 0.3$ and now for $\epsilon = 0.2$ it is $\lambda = -0.17$. For the noiseless case, it is $\lambda > 0$ and trajectories starting with differential initial conditions, obviously, remain different for all the iteration steps. The corresponding synchronization diagram shows a uniform distribution of points (see Fig.(3a)). However, when moderated levels of noise ($\epsilon \gtrsim 0.2$) are used, λ becomes negative and trajectories starting with different initial conditions, but using the same sequence of random numbers, synchronize perfectly, see the synchronization diagram in Fig.(3b).

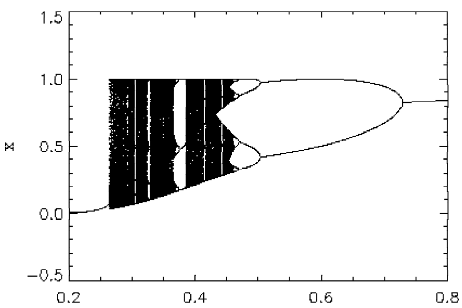

FIG. 1. Bifurcation diagram of the map given by Eqs.(2) and (3) in the absence of noise terms.

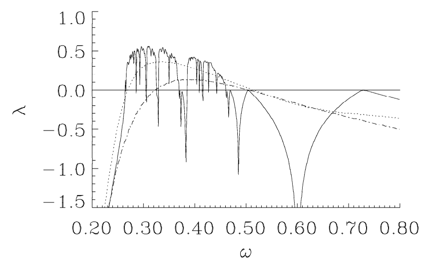

FIG. 2. Lyapunov exponent for the noiseless map ($\epsilon = 0$, continuous line) and the map with a noise intensity $\epsilon = 0.1$ (dotted line) and $\epsilon = 0.2$ (dot-dashed line).

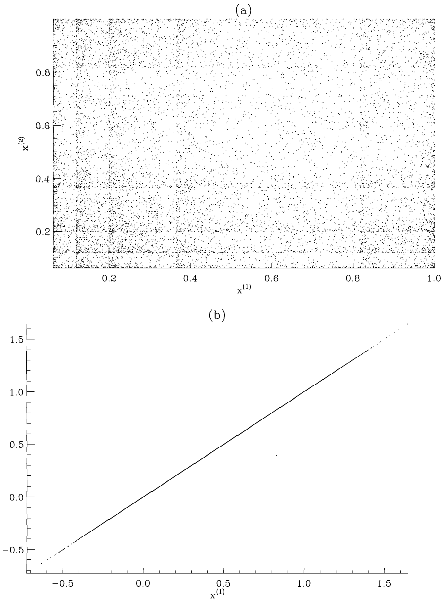

FIG. 3. Plot of two realizations $x^{(1)}$, $x^{(2)}$ of of the map given by Eqs. (2) and (3). Each realization consists of 10,000 points which have been obtained by iteration of the map starting in each case from a different initial condition (100,000 initial iterations have been discarded and are not shown). In figure (a) there is no noise, $\epsilon = 0$ and the trajectories are independent of each other. In figure (b) we have use a level of noise $\epsilon = 0.2$ producing a perfect synchronization (after discarding some initial iterations).

According to [37], convergence of trajectories to the same one, or loss of memory of the initial condition, can be stated as *negativity of the Lyapunov exponent*. The Lyapunov exponent of the map (2) is defined as

$$\lambda = \lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^{N} \ln |F'(x_i)| \tag{4}$$

It is the average of (the logarithm of the absolute value of) the successive slopes F' found by the trajectory. Slopes in $[-1, 1]$ contribute to λ with negative values, indicating trajectory convergence. Larger or smaller slopes contribute with positive values, indicating trajectory divergence. Since the deterministic and noisy maps satisfy $F' = f'$ one is tempted to conclude that the Lyapunov exponent is not modified by the presence of noise. However, there is noise-dependence through the trajectory values x_i, $i = 1, ..., N$. In the absence of noise, λ is positive, indicating trajectory separation. When synchronization is observed, the Lyapunov exponent becomes negative, as required by the argument in [37].

By using the definition of the *invariant measure on the attractor*, or *stationary probability distribution* $P_{st}(x)$, the Lyapunov exponent can be calculated also as

$$\lambda = \langle \log |F'(x)| \rangle = \langle \log |f'(x)| \rangle \equiv \int P_{st}(x) \log |f'(x)| dx \tag{5}$$

Here we see clearly the two contributions to the Lyapunov exponent: although the derivative $f'(x)$ does not change when including noise in the trajectory, the stationary probability does change (see Fig.4), thus producing the observed change in the Lyapunov exponents. Synchronization, then, can be a general feature in maps which have a large region in which the derivative $|f'(x)|$ is smaller than one. Noise will be able, then , to explore that region and yield, on the average, a negative Lyapunov exponent. This is, basically, the argument developed in [47].

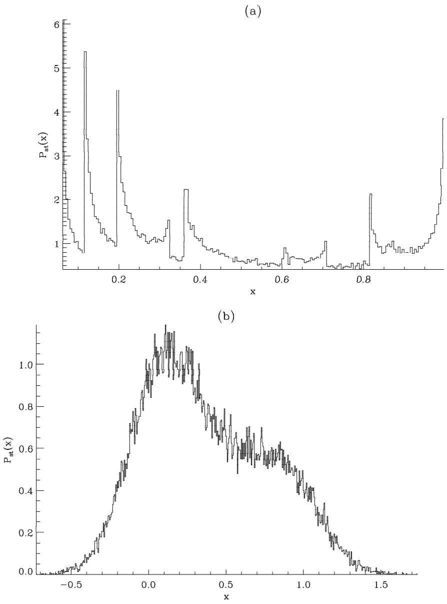

FIG. 4. Plot of the stationary distribution for the map given by Eqs.(2) and (3) in the (a) deterministic case $\epsilon = 0$, and (b) the case with noise along the trajectory, $\epsilon = 0.2$.

In order to make some analytical calculation that can obtain in a rigorous way the transition from a positive to negative Lyapunov exponent, let us consider the map:

$$x_{n+1} = F(x_n) = f(x_n) + \xi_n \tag{6}$$

ξ_n is a sequence of independent identically distributed random numbers with zero mean, and

$$f(x) = \begin{cases} a(1-\exp(1+x)) & \text{if } x < -1 \\ -2-2x & \text{if } x \in (-1,-.5) \\ 2x & \text{if } x \in (-.5,.5) \\ 2-2x & \text{if } x \in (.5,1) \\ a(-1+\exp(1-x)) & \text{if } x > 1 \end{cases} \tag{7}$$

with $0 < a < 1$. This particular map, based in the tent map [56], has been chosen just for convenience. The following arguments would apply to any other map that in the absence of noise takes values in the region with the highest slopes. In the case of (7), the values given by the deterministic part of the map, after one iteration from arbitrary initial conditions, fall always in the interval $(-1,1)$. This is the region with the highest slope $|F'| = 2$. In the presence of noise the map can take values outside this interval and, since the slopes encountered are smaller, the Lyapunov exponent can only be reduced from the deterministic value. To formally substantiate this point, it is enough to recall the definition of Lyapunov exponent (4): An upper bound for $|F(x)'|$ is 2, so that a bound for λ is immediately obtained: $\lambda \leq \ln 2$. Equality is obtained for zero noise.

The interesting point about the map (7) and similar ones is that one can demonstrate analytically that λ can be made negative. The intuitive idea is that it is enough to decrease a in order to give arbitrarily small values to the slopes encountered outside $(-1,1)$, a region accessible only thanks to noise. To begin with, let us note that $|F(x)'| = 2$ if $x \in (-1,1)$, and $|F(x)'| < a$ if $|x| > 1$, so that an upper bound to (5) can be written as

$$\begin{aligned} \lambda \;\leq\; & \lim_{N\to\infty} \left(\tfrac{N_I}{N}\ln 2 + \tfrac{N_O}{N}\ln a\right) = \\ & p_I \ln 2 + p_O \ln a = \ln 2 - p_O \ln(2/a). \end{aligned} \tag{8}$$

N_I/N and N_O/N are the proportion of values of the map inside $I = (0,1)$ and outside this interval, respectively, and we have used that as $N \to \infty$ they converge to p_I and p_O, the invariant measure associated to I and to the rest of the real line, respectively ($p_I + p_O = 1$). A sufficient condition for $x_{n+1} = f(x_n) + \xi_n$ to fall outside I is that $|\xi_n| > 2$. Thus $\mathrm{Prob}(|x_{n+1}| > 1) > \mathrm{Prob}(|\xi_n| > 2) \equiv T$, where Prob means "probability". We can take now the limit $n \to \infty$ to allow the left hand side of the inequality to approach the invariant measure p_O. T is independent on x_n and on n (for example, if noise is Gaussian with variance σ^2, $T = \mathrm{erfc}(\sqrt{2}/\sigma)$), so that one obtains $p_O > T$. In consequence, from (8) one finds

$$\lambda \leq \ln 2 - T \ln(2/a). \tag{9}$$

The important point is that T is also independent on the map parameters, in particular on a. Thus, by decreasing a, (9) implies than the value of λ can be made as low as desired. If $T > \ln 2/\ln(2/a)$, λ will be certainly negative. In the case of Gaussian noise, increasing σ increases T. Thus strong enough noise will always induce "noise-induced synchronization".

III. THE LORENZ SYSTEM

In this section we give yet another example of noise–induced synchronization. We consider the well known Lorenz [57] model with additional random terms of the form [36]:

$$\begin{aligned} \dot{x} &= p(y-x) \\ \dot{y} &= -xz + rx - y + \epsilon\xi \\ \dot{z} &= xy - bz \end{aligned} \tag{10}$$

ξ now is white noise: a Gaussian random process of mean zero and delta correlated, $\langle\xi(t)\xi(t')\rangle = \delta(t-t')$. We have used $p = 10$, $b = 8/3$ and $r = 28$ which, in the deterministic case, $\epsilon = 0$ are known to lead to a chaotic behavior (the largest Lyapunov exponent is $\lambda \approx 0.9 > 0$). As stated in the introduction, previous results seem to imply that synchronization is only observed for a noise with a non–zero mean. However, our results show otherwise.

We have integrated numerically the above equations using the stochastic Euler method [58] with a time step $\Delta t = 0.001$. For the deterministic case, trajectories starting with different initial conditions are completely uncorrelated, see Fig. (5a). This is also the situations for small values of ϵ. However, when using a noise intensity $\epsilon = 40$ the noise is strong enough to induce synchronization of the trajectories. Again the presence of the noise terms makes the largest Lyapunov exponent become negative (for $\epsilon = 40$ it is $\lambda \approx -0.2$). As in the example of the map, after some transient time, two different evolutions which have started in completely different initial conditions synchronize towards the same value of the three variables (see Fig. (5b) for the z coordinate). Therefore, these results prove that synchronization by common noise in the chaotic Lorenz system does occur for sufficiently large noise intensity. Notice that although the noise intensity is large, the basic structure of the "butterfly" Lorenz attractor remains present as shown in Fig. (6).

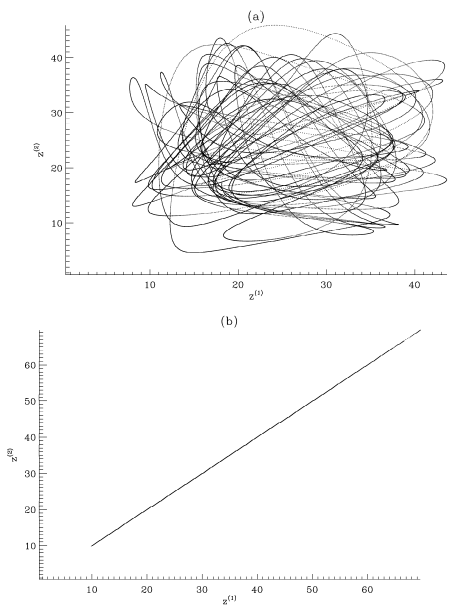

FIG. 5. Same than figure (1) for the z variable of the Lorenz system, Eqs.(10) in the (a) deterministic case $\epsilon = 0$ and (b) $\epsilon = 40$. Notice the perfect synchronization in case (b).

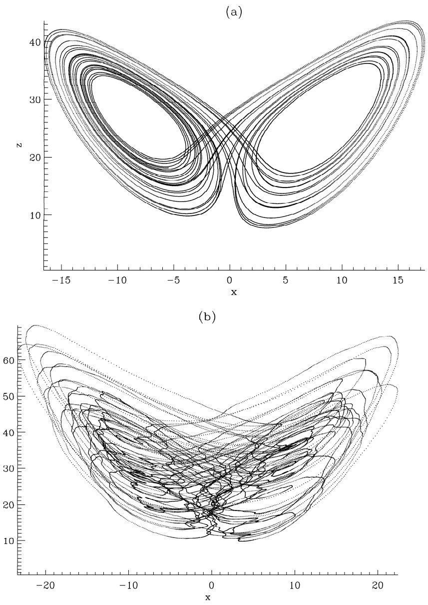

FIG. 6. "Butterfly" attractor of the Lorenz system in the cases (a) of no noise $\epsilon = 0$ and (b) $\epsilon = 40$.

IV. STRUCTURAL STABILITY

An important issue concerns the structural stability of this phenomenon, in particular how robust is noise synchronization to small differences between the two systems one is trying to synchronize. If the synchronization by common noises observed in the Lorenz system (or in any other chaotic system) can be observed in the laboratory, depends on whether the phenomenon is robust when allowing the two Lorenz systems to be not exactly equal (as they can not be in a real experiment). It is obvious that any practical realization of this system can not produce two *identical* samples. If one wants to use this kind of stochastic synchronization in electronic emitters and receivers (for instance, as a means of encryption) one should be able to determine the allowed discrepancy between circuits before the lack of synchronization becomes unacceptable. In order to study this important issue, we have first analytically considered the simple case of a noise-forced map, and then performed computer simulations with Lorenz systems.

We consider the following two maps forced by the same noise:

$$x_{n+1} = f(x_n) + \xi_n \tag{11}$$

$$y_{n+1} = g(y_n) + \xi_n \tag{12}$$

Linearizing in the trajectory difference $u_n = y_n - x_n$, assumed to be small, we obtain

$$u_{n+1} = g'(x_n)u_n + g(x_n) - f(x_n) \equiv g'(x_n)u_n + \Delta(x_n) \tag{13}$$

We have defined $\Delta(x) \equiv g(x) - f(x)$, and we are interested in the situation in which the two systems are just slightly different, for example, because of a small parameter mismatch, so that Δ will be small in some sense specified below.

Iteration of (13) leads to the formal solution:

$$u_n = M(n-1,0)u_0 + \sum_{m=0}^{n-1} M(n-1,m+1)\Delta(x_m) \quad (14)$$

We have defined $M(j,i) = \prod_{k=i}^{j} g'(x_k)$, and $M(i-1,i) \equiv 1$. An upper bound on (14) can be obtained:

$$|u_n|^2 \leq |M(n-1,0)|^2 |u_0|^2 + \sum_{m=0}^{n-1} |M(n-1,m+1)|^2 |\Delta(x_m)|^2 \quad (15)$$

The first term in the r.h.s. is what would be obtained for identical dynamical systems. We know that $M(n-1,0) \to e^{\lambda n}$ as $n \to \infty$, where λ is the largest Lyapunov exponent associated to (12). We are interested in the situation in which $\lambda < 0$, for which this term vanishes at long times. By using the definition of $M(i,j)$ we can bound the last term in (15) by

$$|M(n-1,0)|^2 \sum_{m=0}^{n-1} |M(m,0)|^{-2} |\Delta(x_m)|^2 \ . \quad (16)$$

Further analysis is done first for the case in which $\Delta(x)$ is a bounded function (or x is a bounded trajectory with Δ continuous). In this situation, there is a real number μ such that $|\Delta(x_m)| < \mu$. For $\lambda < 0$, the sum is dominated by the largest values of m, and we can approximate (16) at large n by

$$\mu^2 e^{2\lambda n} \sum_{m=0}^{n-1} e^{-2\lambda(m+1)} = \mu^2 \frac{1-e^{2\lambda n}}{1-e^{2\lambda}} \ . \quad (17)$$

Thus, at large n:

$$|u_n|^2 \lesssim \mu^2 (1-e^{2\lambda})^{-1} \quad (18)$$

The difference between trajectories is bounded by a quantity which is independent of n, so that for small μ both trajectories remain close (although not perfectly synchronized) at all times. Small μ is also required to allow the linearization leading to (13) to be justified at all times. The quality of synchronization worsens when λ approaches zero. The analogous relationship for slightly different differential equations systems forced by the noise would be (if the modulus of its difference remains bounded by μ): $|u_n|^2 \lesssim \mu^2/(2|\lambda|)$

The situation is different when there is not upper bound to $|\Delta(x)|$. For commonly used noise statistics (such as Gaussian noise), bounds can be still obtained, for example, for the variance of u_n. But there is a small finite probability that $|\Delta(x_n)|$ takes arbitrarily large values at some times. At these times the linearization leading to (13) would be invalid. We expect that the trajectory difference will be still most of the time below a value similar to (18), with μ^2 replaced by the variance $\langle \Delta(x_n)^2 \rangle$ over the invariant measure of (12), but eventual large departures may be present.

We have performed simulations in which two Lorenz systems as in (10), with parameters differing by a small amount, are forced by the same noise. In order to discern the effect of each parameter separately, we have varied independently each one of the three parameters, (p, b, r), while keeping constant the other two. The results are plotted in Fig. 7. In this figure we plot the percentage of time in which the two Lorenz systems are still synchronized with a tolerance of 5%. This means that trajectories are considered synchronized if they differ in less than 5%. Since the parameters multiply the variables x, y, and z, the difference between the two dynamical systems is not a vector of bounded norm, and we expect departures from approximate synchronization from time to time. They are in fact observed, but from Fig. 7 we conclude that small variations (of the order of 1%) still yield a synchronization time of more than 85%.

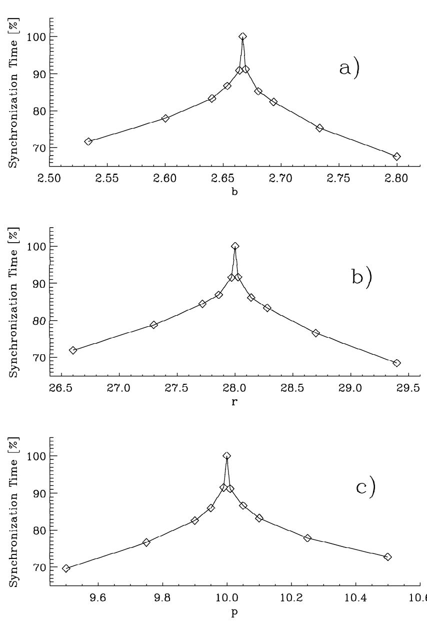

FIG. 7. Percentage of time that two Lorenz systems synchronize (up to a 5% level) when the parameters of both are not exactly equal. While one of the two systems has the typical values $p = 10$, $b = 8/3$, $r = 26$ the other varies the value of one of the three parameters. Notice that the percentage of synchronization time is still higher that 85% if the difference in each parameter is less than 1%.

V. CONCLUSIONS AND OPEN QUESTIONS

In this paper we have addressed the issue of synchronization of chaotic systems by the addition of common random noises. We have considered two explicit examples: a 1-d map and the Lorenz system under the addition of zero–mean, Gaussian, white noise. While the example of the map confirms previous results in similar maps, the synchronization observed in the Lorenz system contradicts some previous results in the literature. We have also proven analytically in a simple example that noise can change the sign of the Lyapunov exponent. Finally, we have analyzed the structural stability of the observed synchronization in the Lorenz system, with the conclusion that synchronization times larger than 85% can still be achieved if the parameters of the system are allowed to change in less that 1%.

It is important to point out that noise induced synchronization in between identical systems subjected to a common noise is equivalent to noise induced order, in the sense that the Lyapunov exponent defined in (4) becomes negative, in a single system subjected to noise. One can ask whether the state with negative Lyapunov exponent induced by noise may be still be called 'chaotic' or not. This is just a matter of definition: if one defines chaos as exponential sensibility to initial conditions, and one considers this for *for a fixed noise realization*, then the definition of Lyapunov exponent implies that trajectories are not longer chaotic in this sense. But one can also consider an extended dynamical system containing the forced one *and* the noise generator (for example, in numerical computations, it would be the computer random number generator algorithm). For this *extended system* there is strong sensibility to initial conditions in the sense that small differences in noise generator seed leads to exponential divergence of trajectories. In fact, this divergence is at a rate given by the Lyapunov exponent of the noise generator [59], which approaches infinity for a true Gaussian white process. Trajectories in the noise-synchronized state are in fact more irregular than in the absence of noise, and attempts to calculate the Lyapunov exponent just from the observation of the time series will lead to a positive and very large value, since it is the *extended* dynamical system the one which is observed when analyzing the time series [59] (typically such attempts will fail because the high dimensionality of good noise generators, ideally infinity, would put them out of the reach of standard algorithms for Lyapunov exponent calculations). Again, whether or not to call such irregular trajectories with just partial sensibility to initial conditions 'chaotic' is just a matter of definition.

There remain still many open questions in this field. They involve the development of a general theory, probably based in the invariant measure, that could give us a criterion to determine the range of parameters (including noise levels) for which the Lyapunov exponent becomes negative, thus allowing synchronization. In this work and similar, the word synchronization is used in a very restricted sense, namely: the coincidence of asymptotic trajectories. This contrasts with the case of interacting periodic oscillations where a more general theory of synchronization exists to explain the phenomenon of non trivial phase locking between oscillators that individually display very different dynamics. Indications of the existence of analogue non trivial phase locking have been reported for chaotic attractors [60]. There a "phase" with a chaotic trajectory defined in terms of a Hilbert transform is shown to be synchronizable by external perturbations in a similar way as it happens with periodic oscillators. Whether or not this kind of generalized synchronization can be induced by noise is, however, a completely open question. Last, but not least, it would be also interesting to explore whether analogous of the recently reported synchronization of spatio-temporal chaos [61] may be induced by noise.

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