Two-mode dynamics in different semiconductor laser structures

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ABSTRACT

We review three two-mode models for different semiconductor laser structures: Vertical-Cavity Surface-Emitting Lasers (VCSELs), Twin-Stripe Semiconductor Lasers (TSSL), and Semiconductor Ring Lasers (SRL). The VCSELs model and TSSL model display rich dynamic behavior when a saturable absorber is embedded in the cavity. VCSELs with saturable absorber showed polarization chaos, which found applications in encoded communications; TSSLs with saturable absorber show coherent locked states as well as chaotic behavior; and SRLs show a complex two-mode dynamics giving rise to bidirectional operation, alternate oscillations and spontaneous symmetry breaking toward quasi-unidirectional bistable solutions, with potential applications to all-optical switching.

1. INTRODUCTION

The Paper is organized as follows. In the subsection Polarization dynamics in Vertical-Cavity Surface-Emitting Lasers, we discussed the synchronization properties of the vectorial chaos generated by a chaotic VCSEL. A main difference of these devices with respect to conventional edge-emitting lasers is that, due to the two-mode polarization degree of freedom, chaos in both intensity and polarization can be obtained without any external perturbation or feedback scheme. We find that two identical systems can synchronize when using a Continuous Control Scheme in the receiver. Moreover, such a chaos can completely synchronize and partially desynchronize in a state where the two systems share the same total chaotic intensity, while they show very different polarization fluctuations. These two states can be exploited in a novel encryption scheme, where the information is encoded in the phase variables rather than in the intensity of the carrier light beam. This encoding scheme has two major advantages as compared to traditional ones. On the one hand, as the information is added in the phase, the average total intensity remains unaffected, a guarantee against unwanted eavesdropper attacks. On the other hand, our synchronization scheme is shown to be very fast, with a synchronization time scale of few picoseconds.

In the subsection Two - mode dynamics in Twin-Stripe Semiconductor Lasers, we review the analysis of a model for the fields and carrier dynamics two laterally coupled EELs, each containing an un-pumped region acting as a saturable absorber. From analytical and numerical analysis, we demonstrated that coherent self-sustained pulsations with different relative phase relationships between the electric field in the two lasers are possible (self-pulsating in-phase or out-of-phase supermodes) for a wide range of parameters of the considered device. We show the emergence of two coherent regimes: stable CW in-phase and out-of-phase supermodes, in-phase and out-of-phase pulsating super-modes, where the intensity of the superposition of the two fields is four times the intensity of the single field. This could represent a promising result in view of the possibility of synchronizing...

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a many-element array of pulsating lasers. Besides, we have found and discussed two different routes to optical chaos.

In the subsection Counterpropagating mode dynamics in Semiconductor Ring Lasers, we review the analysis of the bifurcation from bidirectional stable operation to oscillatory behavior in Semiconductor ring Lasers. Such scenario was described through a two-mode approach yielding analytical expressions for the oscillations onset and frequency near the bifurcation. The calculated results and and the whole $L - I$ curve is well reproduced by experiments.

2. POLARIZATION DYNAMICS IN VERTICAL-CAVITY SURFACE-EMITTING LASERS

Systems with vectorial degrees of freedom, as well as those described by scalar variables, show interesting dynamical regimes like frequency locking and chaos. In the field of optics, the vectorial character of the light is given by the two independent polarization components of the electric field. The dynamics of these polarization components has been investigated for many optical systems, including non-linear devices, gas and semiconductor lasers. The synchronization properties of chaotic systems have received much attention in the last decade motivated by the potential application in secure optical communications systems. To this purpose, also the polarization of the laser light revealed to be useful. Vertical-cavity surface-emitting lasers (VCSELs) are semiconductor lasers characterized by light emission orthogonal to the active layer, and showing many practical advantages in comparison to the more conventional Edge-Emitting Lasers (EELs), for example their compactness, circular beam shape, down to sub-milliampere threshold currents, and high efficiencies. From the non-linear dynamics and laser physics point of view, VCSELs differ in one crucial aspect from standard EELs: the polarization state of the light emitted by a VCSEL is not fixed a priori by the device’s almost perfect cylindrical symmetry. Therefore, a rich dynamical polarization behavior is found in these devices. Also, self-pulsations have been experimentally demonstrated in a solitary VCSEL. These self-pulsations in combination with the polarization degree of freedom allow, under certain operation conditions, for the existence of chaos without the need of any external perturbation or feedback scheme. As this chaos involves both intensity and polarization, we call it vectorial chaos. In the following we review some results concerning the synchronization properties of the vectorial chaos present in a VCSEL, showing that a particular transition from quasi-to fully synchronized chaotic states is possible. This transition can be exploited in secure communications applications, increasing the security level and the transmission velocity, and without the need of inducing chaos through external perturbations (e.g. feedback or injection).

We consider master and slave VCSEL where the active region is surrounded by a zone with saturable absorber. The VCSEL dynamics can be described in the framework of the standard Spin Flip Model for the polarization dynamics, combined with the well-known Yamada model for semiconductor laser in the presence of a saturable absorber. The rate equations describing the dynamical evolution of the slowly-varying complex amplitudes of the two circularly-polarized optical fields $F_{\pm M,S}$ in the Master and Slave lasers, respectively, can be written concisely as

$$
\dot{F}_{\pm M} = \mathcal{F}_{\pm M}(\gamma_{\pm M}, d_{\pm M}) F_{\pm M} + (\gamma_a + i \gamma_p) F_{\mp M} + f_{\pm M}(t), \quad (1)
$$

$$
\dot{F}_{\pm S} = \mathcal{F}_{\pm S}(\gamma_{\pm S}, d_{\pm S}) F_{\pm S} + (\gamma_a + i \gamma_p) F_{\mp S} + f_{\pm S}(t) + \Gamma (F_{\pm M} + H_0 e^{i \phi} F_{\pm S}). \quad (2)
$$

The field equations depend on the carrier populations and henceforth, Eqs (1) and (2) have to be complemented with carrier equations for $D_{1,2,M,S}$, the total carrier inversion between the conduction and valence bands (the indices 1 and 2 stand for the pumped and absorbing regions in each laser, consistently with the Yamada model). Furthermore, within the SFM description of VCSELs there is a dynamical dependence on $d_{1,2,M,S}$, the differences of the carrier populations with opposite spin orientations. The full set of equations can be found elsewhere. The optical fields $F_{\pm}$ in each laser are coupled through phase and amplitude anisotropies $\gamma_p$ and $\gamma_a$. Small values of $\gamma_a$ are usually encountered in VCSELs and, since our results only weakly depend on this parameter, we set $\gamma_a = 0$. The spontaneous emission noise terms $f_{\pm M,S}$ contain independent complex Gaussian random numbers with zero mean and delta-correlation. Finally, through the last term in Eq. (2) we describe unidirectional coupling of the Master laser to the Slave one, where $\Gamma$ is the overall injection attenuation, and $H_0$ and $\phi$ the
attenuation and de-phasing acquired in the feedback loop. All the numerical simulations were performed with a
given parameter set,\textsuperscript{13} except for the phase anisotropy which takes the value $\gamma_p = 25 \text{ ns}^{-1}$.

One of the dynamical regimes exhibited by the master laser is a region of chaotic behavior, subsequent to a
birefringence-induced Hopf bifurcation. In the rest of this section we concentrate on this particular regime. To
visualize the complex dynamics that arise in this region, we show in Fig. 1 the total intensity time trace together
with the time evolution of two of the projections of the vector electric field onto the Poincaré sphere. These
projections are shown in terms of the normalized Stokes parameters $S_1/S_0$, $S_2/S_0$ and $S_3/S_0$\textsuperscript{17}. It can be seen
that the electric field is changing its polarization state over a chaotic attractor in the Poincaré sphere, and its
amplitude of the electric-field vector evolves chaotically, showing that chaos is present in both polarization and
in the total emitted intensity. We have checked that the dynamics is indeed chaotic by computing the largest of
the Lyapounov exponents which is clearly positive for the parameters chosen.

We next consider the synchronization properties of the coupled system. The simplest coupling scheme between
two such systems is the direct coupling,\textsuperscript{4} i.e. $H_0 = 0$ in Eq.(2). However, we find that in such a standard scheme
only an intermittent synchronization can be achieved, that is not robust against introduction of noise and/or
parameter mismatches. Therefore, we use instead a Continuous Control Scheme coupling \textsuperscript{18} (CCS), that has
proven to yield more robust synchronization in EELs.\textsuperscript{19} For simplicity, we assume that the $M$ and $S$ systems
are twin systems (same values of all parameters). If we set the phase in the feedback loop to $\phi = \pi$ and increase
$\Gamma$, we find that after a certain critical value $\Gamma = \Gamma_c$ the trajectories of the $M$ and $S$ systems synchronize. To
quantitatively express the degree of synchronization we use the scalar mean relative error \textsuperscript{19} generalised to the
circular components of the field:

\begin{equation}
\sigma_\pm = \frac{< | F^M_\pm - F^S_\pm | >}{< | F^M_\pm | >}
\end{equation}

where $< \cdot >$ means temporal average. In Fig. 1 we show the transition of the coupled system from an unsynchro-
nized ($\sigma_\pm \sim 1$) to a synchronized state ($\sigma_\pm \ll 1$). We disregard the flight time $T_f$ from the Master to the Slave
laser, since the slave system dynamics remains invariant under a time translation $t - T_f$ by $t$. A high quality
synchronization can thus be reached using identical devices and under noisy conditions. The accuracy needed to
fulfil the condition $\phi = \pi$ has been already discussed in the literature\textsuperscript{19} and is found, in general, to be critical.
This is also the case for our VCSEL system. The difference $F^M_\pm - F^S_\pm$ requires coherent field superposition (with $\pi$
phase shift) at a beam combiner; therefore, a suitable active control of the path-length must be introduced in the
set-up, since, for efficient synchronization, the residual phase error must not exceed a few degrees. The accuracy
level needed for this practical requirement is of the same order as in coherent detection or interferometry.

The synchronization method just described can be exploited in a secure communication scheme, taking
advantage of the vectorial nature of the chaotic field. Similar to what has been demonstrated in a fiber ring
laser,\textsuperscript{8} the information can be encoded in the polarization state of the emitted light, leaving the average total
intensity unaffected. For this purpose, we introduce a polarization modulator at the output of the Master
laser. In the 'on' state for which we assume the bit "1", the polarization modulator changes the phase relation
between the two orthogonal polarizations, which is invisible in the total emitted intensity. The "off" state (bit
"0") leaves the emitted field unaffected. The demodulation scheme is a standard On-Off Chaos Shift Keying
(OOCSK)\textsuperscript{19,20}; the Slave synchronizes (de-synchronizes) to the received signal when a "0" ("1") bit is received.
It is interesting to note that, while the $x$ and $y$ polarizations (de)synchronize following the phase modulation,
the total intensity of the two lasers remain synchronized all the time except for small bursts (see Fig. 2).
Another interesting aspect is that the synchronization is much faster than the typical $ns$ time scale present
in conventional chaotic semiconductor lasers.\textsuperscript{19} To quantify the synchronization response we have calculated
the synchronization time, i.e., the time it takes the system to re-synchronize once the relative phase between
the two polarization components is set back to zero. The results are shown in Fig. 2, where it can be seen
that, for our parameters values, the mean synchronization time is about $1.35 \text{ ps}$, with a statistical distribution
originated by the chaotic nature of the fluctuations, when passing from a de-synchronized to a synchronized
state. For typical VCSELs parameters vectorial chaos synchronization could allow for encoding rates much
faster than any other traditional CSK scheme. Only if the total intensity synchronization is lost our system
acts like the traditional CSK, recovering the full synchronization with a characteristic time of the order of the
inverse of the relaxation oscillation frequency (about 3 GHz for our parameters). The very fast synchronization demonstrated here follows from the fact that our systems remains in a state of partial synchronization all the time: the total intensity remains synchronized, while the polarization components (de)synchronize following the phase modulation. Re-establishing the synchronization of the polarization components only involves the phases of the fields and not their intensities. Henceforth our polarization encoding is not limited by the relaxation oscillation frequency.

Figure 1. Left: The upper panels show the normalised Stokes parameters phase space, while the lower ones show the time evolution of the total intensity (left) and the Power spectrum (right). Parameter values are the same as in Ref., except for $\gamma_p = 25 \text{ ns}^{-1}$. Right: Synchronization error ($\sigma_+; \text{circles}, \sigma_-; \text{ dots}$) as a function of the coupling strength $\Gamma$. The transition from an unsynchronized ($\sigma_\pm \sim 1$) state to synchronization takes place at $\Gamma \sim 0.08$.

Figure 2. Left: Synchronization diagrams: (a) the total intensities of the Master and Slave sources ($T_1$ and $T_2$ respectively) keep synchronized all the time except for small bursts and a small variation in the cross-correlation between $T_1$ and $T_2$: 0.998 (0.996) when '0' ('1') is transmitted. The respective $x$ (b) and $y$ (c) polarizations (de)synchronize when a '0' ('1') is received. The units are dimensionless, $\Gamma = 0.9$, and the bit frequency is 3 Gbit/s. Right: Probability distribution of the synchronization time calculated as the time at which the difference between the two fields drops below the 10 per cent of the mean emitted power.

3. TWO - MODE DYNAMICS IN TWIN-STRIPE SEMICONDUCTOR LASERS

In this section, we review the analysis of a model to describe the temporal dynamics of the electric field and carrier populations in two Edge Emitting Lasers (EELs) – each given an un-pumped saturable absorption section – and laterally coupled through evanescent wave. Saturable absorption is a well-known method to achieve pulsating output for a laser source\textsuperscript{21,29} The structure resembles the well known Twin-Stripe,\textsuperscript{30} but here each stripe has two sections, an active one and an absorbing one. The inter-element field dynamics is the new feature with respect to the well-known self-pulsating EELs. Our description of the absorbing region relies on the standard one developed for the two-sections EEL.\textsuperscript{31} The inter-element optical coupling is modeled as in.\textsuperscript{32} Combining
the two approaches, we are able to investigate the dynamics of a two laterally coupled semiconductor lasers, each stripe including a saturable absorber. The case of two-coupled lasers is of particular interest since it allows a complete analytical study and provides a good physical insight in view of a more general study of a many-element array of pulsating lasers, because all the coherent behaviors (e.g. synchronized pulses) are likely to be present in the many-element case, as well.

3.1. The Model

We consider a laser structure consisting in two adjacent EELs, as the one schematically depicted in Fig. 3. Each EEL has a first pumped section providing gain (labelled as \( p \)), and a second region (labelled as \( a \)) acting as a saturable absorber. Physically, the two lasers are optically coupled due to diffraction, whereas in each laser the pumped and absorbing regions are coupled each other by carrier diffusion. We neglect further sources of coupling such as thermal ones, and cross-carrier diffusion between laser 1 and 2.

We assume that each unperturbed guide supports a single longitudinal mode, thus the total transverse electric field is written as a linear superposition of the unperturbed individual waveguide fields, and the residual radiation field is neglected. The analysis of diffraction induced cross-talk in terms of the coupling between modes of individual waveguides is formally exact, but the ability to analyze such problem is restricted to weakly guiding structures which are sufficiently well separated. The result of such analysis consists in a linear complex coupling term between the two fields, which is quantitatively individuated by the waveguide parameters (mainly the distance between the waveguides). Indeed, the actual values of the coupling parameters stem from the eigenvalue analysis of the coupled waveguides, and in general are found to be technologically- (or even device-) dependent. In our approach, we leave those coupling terms as free parameters of our model. The spatial wave propagation problem is therefore simplified in to rate equations for the modal amplitudes. The equations governing the optical and material variables read

\[
\frac{dE_i(t')}{dt'} = \frac{1 + j\alpha}{2} [g_p \xi_p (N_{p_i} - N_{t_p}) + g_a \xi_a (N_{a_i} - N_{t_a}) - \kappa] E_i - (k_d + jk_c) E_m + G_i, \tag{4}
\]

\[
\frac{dN_{p_i}(t')}{dt'} = \frac{J_i}{eV_p} - \gamma_{cp} N_{p_i} - \frac{N_{p_i} - N_{a_i}}{T_{pa}} - \frac{g_p \xi_p}{V_p} (N_{p_i} - N_{t_p}) |E_i|^2, \tag{5}
\]

\[
\frac{dN_{a_i}(t')}{dt'} = -\gamma_{ca} N_{a_i} - \frac{(N_{a_i} - N_{p_i})}{T_{ap}} - \frac{g_a \xi_a}{V_a} (N_{a_i} - N_{t_a}) |E_i|^2, \tag{6}
\]

where \( i = 1, 2 \) and \( m = 3 - i \), and \( j \) is the imaginary unit. \( E_i \) is the slowly-varying complex amplitude of the electric field of the optical mode supported by waveguide \( i \). \( N_{p_i} (N_{a_i}) \) is the carrier inversion density between the conduction and valence bands in the pumped (absorbing) regions of laser \( i \). The meaning and typical values of the different parameters in the model (4)-(6) is given in\(^{34}\) however two important points are worth mentioning. The dependence of the gain with the carrier density has been substituted by a two-linear approximation by taking different values of \( g_{p,a}, N_{tp} \) and \( N_{ta} \) depending whether the region \( p \) or \( a \) is considered. Secondly, we define a characteristic volume of the regions, \( V_p \equiv w d L_p \) and \( V_a \equiv w d L_a \), with \( w \) denoting the quantum well thickness, \( d \) the waveguide thickness and \( L_p \) and \( L_a \) are the lengths of the pumped regions and absorbers, respectively. The parameters \( \xi_{p,a} \) represents the fraction of the intensity of the electric field that lays in region \( p \) and \( a \).\(^{16}\) and whose values are given by the integrals of the spatial mode profile over regions \( p \) and \( a \). By choosing a standard expression for the spatial profile in the single longitudinal and transverse mode operation conditions (e.g. see\(^{27} \) p. 232), one finds that \( \xi_p/\xi_a \approx L_p/L_a \), under the approximation that \( L_{p,a} \gg \lambda \), being \( \lambda \) the laser wavelength. Normalization conditions impose \( \xi_p + \xi_a = 1 \). The two diffusion times vary according the volumes of the regions \( V_p/V_a \approx T_{pa}/T_{ap} \).\(^{25} \) The linewidth enhancement factor \( \alpha \) describes the phase-amplitude coupling mechanisms. As previously discussed, the coupling between the two self-pulsating lasers is described in terms of the parameters \( k_d \) and \( k_c \). The dissipative coupling term \( k_d \) represents additional optical losses in the region between the two lasers, where the two field modes interfere. The conservative (or coherent) coupling of the two localized field modes by optical diffraction is represented by \( k_c \). Finally, spontaneous emission processes are accounted through two independent Langevin noise sources \( \xi_{1,2}(t) \). We remark that in the model (4)-(6) the two lasers are individually described by the simplest model for a semiconductor laser including saturable

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absorptive effects: rate equations based on individual single-mode oscillations. Moreover, for the sake of completeness, a rigorous derivation of our model (4)-(6) would rely on a consistent treatment of diffusion and diffraction effects. The main approximations that make our model consistent with the full spatial approach are the same as discussed in.

### 3.2. Dimensionless model

For the sake of simplicity and numerical purposes, we rescale the dynamical variables by

\[
F = \frac{g_p \xi_p}{V_p \gamma_p \kappa} E,
\]

\[
D_p = \frac{g_p \xi_p}{\kappa} (N_p - N_{tp}),
\]

\[
D_a = \frac{g_a \xi_a}{\kappa} (N_a - N_{ta}).
\]

With these new variables, the equations (4)-(6) reduce to

\[
\dot{F}(t) = \frac{1}{2} \left[ 2D_{pi} + D_{ai} - 1 \right] F_i - (\varepsilon_d + j\varepsilon_c) F_m + f_i(t),
\]

\[
\dot{D}_{pi}(t) = \gamma_p [A_{pi} - D_{pi}(1 + |F_i|^2) + c_{pa} D_{ai}],
\]

\[
\dot{D}_{ai}(t) = \gamma_a [A_{ai} - D_{ai}(1 + r|F_i|^2) + c_{ap} D_{pi}],
\]

where

\[
\gamma_{a,p} \equiv \frac{1}{\kappa} \left( \gamma_{ca,p} + \frac{1}{T_{ap,pa}} \right),
\]

\[
c_{ap} \equiv \frac{g_a \xi_a}{g_p \xi_p} \frac{1}{\gamma_a \kappa T_{ap}}, \quad c_{pa} \equiv \frac{g_p \xi_p}{g_a \xi_a} \frac{1}{\gamma_p \kappa T_{pa}},
\]

\[
A_p \equiv \frac{g_p N_{tp} \xi_p}{\kappa} \left[ \frac{J}{J_t} - 1 + \frac{N_{ta}}{N_{tp} \kappa \gamma_p T_{pa}} \right],
\]

\[
A_a \equiv \frac{g_a N_{ta} \xi_a}{\kappa} \left[ -1 + \frac{N_{tp}}{N_{ta} \kappa \gamma_a T_{pa}} \right],
\]

\[
r \equiv \frac{g_a \xi_a \gamma_p}{g_p \xi_p \gamma_a},
\]

\[
\varepsilon_d \equiv k_d / \kappa, \quad \varepsilon_c \equiv k_c / \kappa.
\]

We have defined the transparency current as \( J_t \equiv e \kappa \gamma_p N_{tp} V_p \). The rescaling is the same for lasers 1 and 2, so we have dropped that index in Eqs. (7)-(9),(15) and (16). Equations (10)-(12) are written in a dimensionless form such that the dimensionless time \( t' = \kappa t \). The dot acting onto the dynamical variables means derivative with respect to \( t \). The rescaled dissipative (\( \varepsilon_d \)) and conservative (\( \varepsilon_c \)) are now the coupling parameters. The effective injection currents, with respect to the transparency value, are \( A_p \) and \( A_a \). Carrier diffusion is present in the equations through the coupling terms \( c_{pa} \) and \( c_{ap} \).

The Langevin noise sources \( f_i(t) \) can be approximated by

\[
f_i(t) = \sqrt{\beta \xi_p \gamma_p D_{pi} + \beta \xi_a \gamma_a D_{ai}} \xi_i(t),
\]

where \( \beta \xi_{a,p} \) represents the fraction of the spontaneously emitted photons that goes into the zone \( a \) or \( p \) of the lasing mode, \( \xi_{1,2} \) are two independent complex Gaussian random numbers, with zero mean \( \langle \xi_i(t) \rangle = 0 \) and correlation \( \langle \xi_i(t) \xi_j^*(t') \rangle = 2\delta_{ij} \delta(t - t') \).
3.3. Stationary Solutions

In the following we assume symmetric operation conditions \( A_{pi} = A_p, \) and \( A_{ai} = A_a \). The electric field, solution of Eqs. (10)-(12) is expressed as \( F_i(t) = Q_i e^{j(\omega t \pm \varphi)} \). We start our discussion looking at the symmetric Stationary Solutions (SS), i.e., \( Q_1 = Q, \omega_1 = \omega, \bar{D}_{ai} = \bar{D}_{ai} = 0 \). We find two types of SSs (resembling the super-modes in the Twin-Stripe\(^{32}\) structure): \( \varphi = 0, \) In-Phase (IP); and \( \varphi = \pi/2, \) Out-of-Phase (OP). The emission frequency (in \( \kappa \) units) is \( \omega = \pm (\varepsilon_d \alpha - \varepsilon_c) \), where the sign + (−) stands for a IP (OP) solution (labeled as IP (OP) supermodes in the following). By defining \( I = Q^2 \), the stationary carrier inversions are in turn given by

\[
\bar{D}_{pi} = \frac{A_p(1 + rI) + A_a c_{pa}}{(1 + I)(1 + rI) - c_{pa} c_{ap}}, \quad (20)
\]

\[
\bar{D}_{ai} = \frac{A_a(1 + I) + A_p c_{ap}}{(1 + I)(1 + rI) - c_{pa} c_{ap}}, \quad (21)
\]

by dropping for simplicity the index \( i \). \( \bar{D}_p \) and \( \bar{D}_a \) satisfy gain clamping condition \( \bar{D}_p + \bar{D}_a = \pi_d \), where \( \pi_d \equiv 1 \pm 2\varepsilon_d \) and the sign + (−) stands for a IP (OP) solution. This leads to a quadratic equation for \( I \) that reads

\[
r \pi_d I^2 + [(1 + r)\pi_d - rA_p - A_a]I + \pi_d(1 - c_{pa} c_{ap}) - A_p(1 + c_{ap}) - A_a(1 + c_{pa}) = 0. \quad (22)
\]

The laser first threshold is defined as the pump value for which the off-solution \( I = 0 \) loses stability. By taking \( I = 0 \) in Eqs. (20)-(21) and imposing the gain clamping condition we find that the threshold currents for the two solutions OP and IP are given by

\[
A_{p_{\text{Th}}}^{(IP, OP)} = \frac{(1 - c_{pa} c_{ap})\pi_d^{(IP, OP)} - A_a(1 + c_{pa})}{1 + c_{ap}}. \quad (23)
\]

Eq. (23) can be interpreted as follows: there is an increase in threshold current due to the absorption \( (A_a < 0) \) and due to the carrier diffusion from region 1 to region 2 \( (c_{pa}) \), while the threshold decreases if the inverse carrier flux is favored \( (c_{ap}) \). Either for \( \pi_d^{(OP)} \) or \( \pi_d^{(IP)} \), Eq. (22) has only one physical solution (positive root) for \( A_p > A_{p_{\text{Th}}} \). For sufficiently intense absorbing conditions, three solutions appear in a narrow range of currents \( A_p \lesssim A_{p_{\text{Th}}} \) producing a hysteresis cycle\(^{26,35}\): stable solutions with \( I = 0 \); high-power solutions, and intermediate-power solutions that result to be unstable states.

3.4. Linear Analysis

A linear stability analysis of the two SSs of the system (10)-(12) can be carried out by introducing a small perturbation, yielding

\[
F_i(t) = (Q + \eta_i(t)) e^{j(\omega t \pm \varphi)},
\]

\[
D_{pi}(t) = \bar{D}_p + \Delta_{pi}(t),
\]

\[
D_{ai}(t) = \bar{D}_a + \Delta_{ai}(t),
\]

where \( \eta_i \) is a complex perturbation of the field amplitude, and \( \Delta_{pi} \) and \( \Delta_{ai} \) are real-value perturbations of the carrier variables. Upon substituting (24)-(26) in the equations of the model (10)-(12) and linearizing, we obtain a set of coupled linear differential equations for the perturbations which, written for the variables \( S \equiv \eta_1 + \eta_2, \)

\( T_p = \Delta_{p1} + \Delta_{p2}, T_a = \Delta_{a1} + \Delta_{a2} \) and \( R \equiv \eta_1 - \eta_2, \tau_p = \Delta_{p1} - \Delta_{p2}, \tau_a = \Delta_{a1} - \Delta_{a2} \) decouples into two subsets. The first subset \((S, T_p, T_a)\) determines the stability of the intensities of the two supermodes, whereas the second subset \((R, \tau_p, \tau_a)\) describes the stability of the relative phase, as discussed below.
3.5. Intensity stability of the supermodes and self-pulsations conditions

The actual dimension of the subset \((S,T_p,T_a)\) is \(3 \times 3\), due to the presence of an invariant global phase, that implies the constant presence of a zero eigenvalue. Therefore, by introducing \(P = S + S^*\), we have

\[
\begin{align*}
\dot{P}(t) &= Q(T_p + T_a), \\
\dot{T}_p(t) &= \gamma_p \left[ -Q\dot{D}_p P - (1 + Q^2)T_p + c_{pa}T_a \right], \\
\dot{T}_a(t) &= \gamma_a \left[ -rQ\dot{D}_a P - (1 + rQ^2)T_a + c_{pa}T_p \right],
\end{align*}
\]

which determines the stability of the intensities (see discussion below) of each supermode. The characteristic equation for the eigenvalues \(s\) of the linearized system Eqs. (27)-(29) obey a third-order polynomial

\[s^3 + C_2s^2 + C_1s + C_0 = 0.\]

In this expression,

\[
\begin{align*}
C_0 &= \gamma_p\gamma_a I \left[ \dot{D}_p (1 + rI + c_{ap}) + \dot{D}_a (1 + I + c_{pa}) \right], \\
C_1 &= \gamma_p\gamma_a (1 + I)(1 + rI) - \gamma_p\gamma_a c_{pa} c_{ap} + \gamma_p I \dot{D}_p + \gamma_a r \dot{D}_a, \\
C_2 &= \gamma_a (1 + rI) + \gamma_p (1 + I).
\end{align*}
\]

By applying the Routh-Hurwitz criterion, when the condition \(C_2C_1 = C_0\) is fulfilled, the total intensity loses stability through a Hopf bifurcation, giving rise to intensity pulsations at frequency \(\Omega = \sqrt{C_1}\). By taking advantage of the small values of \(\gamma_p\) and \(\gamma_a\), as in \(\text{Ref.} 39\), the condition \(C_2C_1 = C_0\) is well approximated by

\[\left(1 + I\right)\dot{D}_p + \frac{\gamma_a^2}{\gamma_p^2} (1 + rI)\dot{D}_a - \frac{\gamma_a}{\gamma_p} (r\dot{D}_a c_{pa} + \dot{D}_p c_{ap}) = 0.\]

Substituting the solutions for \(\bar{D}_{a,p}\) reported in Eqs. (20)-(21) and solving Eq. (22) for \(I\) as function of \(A_p\), Eq. (34) gives two values of \(A_p\) \((A_p^{IP),(OP)}\) that bound the Hopf-bifurcation locus, for the IP and OP supermode, respectively. Notice that the location of Hopf bifurcation does not depend on the imaginary part of the coupling \(\varepsilon_c\). It is also independent of the \(\alpha\)-factor. The study \(\text{Ref.} 39\) reported coefficients of the secular determinant describing the intensity linear stability in a single self-pulsating EELs, including non-linear recombination effects. Considering only one laser, our coefficients (31)-(33) revert to those ones of Ref. \(\text{Ref.} 39\) when non-linear recombination effects are neglected. The subset Eqs. (27)-(29) turns out to be very similar to the single self-pulsating laser case. The difference relies in the fact that our case presents two stationary symmetric solutions (IP and OP supermodes) and this is due to the coupling between the two sources.

The stability boundaries of the intensity subset (27)-(29) are shown in Fig. 3, as function of the dimensionless real part of the coupling and scaled pumping. The corresponding values for the real part of physical coupling \(k_d\) can be calculated using (18) while the physical pumping is given by (15). In region 1 both supermodes are below threshold, there is no stimulated emission. In region 2 the OP supermode is above threshold while IP is below threshold. However the OP intensity is unstable, therefore a pulsating emission of the OP supermode is expected. In region 3 is the other way around and a pulsating IP emission is expected. In region 4 both IP and OP supermodes are above threshold and allowed to pulsate. In region 5 (6) the OP (IP) supermode pulsates, whereas the IP (OP) supermode reaches a steady state (CW operation). In region 7 both supermodes reach a steady state. The self-pulsation frequency calculated above is shown in Fig. 4 for the two supermodes.

We observe that the role of the dissipative coupling coefficient \(\varepsilon_d\) is to give different losses to the two supermodes IP and OP, therefore splitting the lines describing the oscillation onset and absorber saturation of each solution. The complete information about the supermode selection and stability is given by the merging of the stability properties of the intensity and of the phase subset, which is the object of the next subsection.
3.6. Phase stability of the supermodes

The second linear subset represents the effect of a perturbation enhancing the difference between the dynamic variables of the two lasers, thus gives informations about the phase stability of the supermodes. It reads

\[
\dot{R}(t) = \frac{1}{2}(1 + i\alpha)Q(\tau_p + \tau_a) \pm 2(\varepsilon_d + i\varepsilon_c)R
\]

(35)

\[
\dot{R}^*(t) = \frac{1}{2}(1 - i\alpha)Q(\tau_p + \tau_a) \pm 2(\varepsilon_d - i\varepsilon_c)R^*
\]

(36)

\[
\dot{\tau}_p(t) = \gamma_p \left[ c_{pa}\tau_a - Q\dot{D}_p(R + R^*) - (1 + Q^2)\tau_p \right],
\]

(37)

\[
\dot{\tau}_a(t) = \gamma_a \left[ c_{ap}\tau_p - rQ\dot{D}_a(R + R^*) - (1 + rQ^2)\tau_a \right],
\]

(38)

where \(+(-)\) stands for the stability of the IP (OP) supermode. The phase stability depends explicitly on both the real and the imaginary part of the coupling as well as on the \(\alpha\)-factor. In fact it is the interplay between the coupling coefficients \((\varepsilon_{d,c})\) and the \(\alpha\)-factor that determines the selection of the phase relationship between the two individual fields, and consequently the supermode selection. The subset (35)-(38) yields a fourth-order characteristic polynomial, that we analyze by numerical methods. We compute the phase stability boundaries for both IP and OP supermodes in the plane \(A_p\) versus \(\varepsilon_c\). We first consider the case \(\varepsilon_d < 0\), and \(\varepsilon_c > 0\). The OP supermode turns out to be always unstable. Fig. 5 shows the stability diagram for the IP supermode in the \((A_p, \varepsilon_c)\) plane. There are four regions in which the dynamics is qualitatively different. In region A both subsystems (27)-(29), and (35)-(38) are stable. Therefore the system shows stable CW-In-Phase fields (stable IP supermode), the intensity of the electric field in each laser \(\vert F_i \vert^2\) reaches a steady state, (see Fig. 5) and the relative phase of the two fields goes to zero after a transient. The carriers reach the stationary value given by (20) and (21). In region B the absorber is no longer saturated, the subsystem (27)-(29) is unstable and pulsating output takes place. Formally, the self-pulsating solution arises as consequence of a homoclinic bifurcation at threshold, which leads to the onset of a closed loop in the phase space, physically accounting for the field-medium energy exchange during the pulse. Increasing the pump current \(A_p\), e.g. moving from region B toward region A of Fig. 5, the limit cycle shrinks and disappears through a Hopf bifurcation. As discussed in the previous subsection the Hopf bifurcation locus does not depend on \(\varepsilon_c\) and it is displayed as a horizontal solid line in Fig. 5. The subsystem (35)-(38) is stable, so that the intensity of the two lasers pulsates synchronously, namely, both lasers emit intensity pulses at the same time. Furthermore both lasers emit coherently, with the same electric field phase. Fig. 6 shows the pulsating IP supermode: the intensity of the electric field in each laser \(\vert F_i \vert^2\) reaches a stationary pulsating regime, and the relative phase of the two fields goes to zero. We remark that regimes in regions A and B are coherent regimes. Therefore the intensity of the superposition of the two fields \(\vert F_1 + F_2 \vert^2\) is four times the intensity of the single source.

In region C and D the phase instability associated to the unstable eigenvalues of the subsystem (35)-(38) leads to the emergence of a complex non-linear dynamics, in which chaotic behaviors take place (see Fig. 7). This is explained in the section 3.7. In region E the IP solution is below threshold \(A_p < A^{\text{IP}}_{\text{rTh}}\) of Eq. (23), and the output intensity drops to zero.

Changing the precise value of \(\varepsilon_d\) does not change qualitatively the stability diagram. For smaller values of \(|\varepsilon_d|\) the instability regions widens, whereas it shrinks for larger values of \(|\varepsilon_d|\).

For \(\varepsilon_d > 0\) and \(\varepsilon_c < 0\) the IP supermode is always unstable. The stability diagram of the OP supermode has different regions whose shape is the same as in figure 5 but changing \(\varepsilon_c \rightarrow -\varepsilon_c\). In this case region A corresponds to stable CW out-of phase fields: the intensity of each laser reaches a steady state while the relative phase goes to \(\pi\) so that the total intensity vanishes. Region B corresponds to a regime in which the electric field of each laser reaches a stationary pulsating regime in which both lasers emit intensity pulses synchronously. However the relative phase goes to \(\pi\), so while the intensity of each individual laser is self-pulsating and pulses are synchronous, the total intensity vanishes.

For \(\varepsilon_d > 0\) and \(\varepsilon_c > 0\) the OP supermode is always stable while the IP supermode is always unstable. For \(\varepsilon_d < 0\) and \(\varepsilon_c < 0\) is the other way around, the OP supermode is always unstable while the IP supermode is always stable. Thus if \(\varepsilon_d\) and \(\varepsilon_c\) have the same sign no chaotic behavior is found, and only coherent regimes are displayed.
3.7. Chaotic Behavior

Chaotic attractors arise in semiconductor lasers due to phase-instabilities related to extra degrees of freedom, such as external injection, feedback, or mutual coupling like in the present case. However, the specific route to chaos depends on the specific structure. In our system chaos originates from the interplay of the α-factor and the imaginary part of the coupling $\varepsilon_c$. Indeed, when either $\alpha=0$ or $\varepsilon_c=0$, no chaotic behavior occurs. Physically, the $\alpha$-factor reverts the intensity pulsations to phase-pulsations, while $\varepsilon_c$ induces supplementary phase oscillations. In our system there are two different chaotic attractors, depending whether the absorber is saturated or not. Indeed, in the unstable regions $C$ and $D$ of Fig. 5, two different routes to chaos are found.

In region $C$ the two lasers are pulsating with an irregular amplitude and phase. When crossing from region $B$ to $C$, the IP supermode has an instability coming from the relative phase subset (35)-(38). Therefore the instability is in a direction transverse to the subspace embedding the intensity pulses. This leads to a regime where both lasers emit chaotic intensity pulses which are de-synchronized. The total intensity reflects the sum of the two incoherent chaotic dynamics (see Fig. 7) showing large pulses separated by practically vanishing intensity.

When crossing from region $A$ to $D$ the IP supermode has also an instability coming from the relative phase subset (35)-(38). In this case, since no pulses were present in region $A$, the emerging regime is characterized by large amplitude excursions in the direction transverse to the intensity subspace followed by decaying oscillations over the stable subspace of the IP solution (see Fig. 7).

Regions $C$ and $D$ are separated by the Hopf bifurcation locus shown as a horizontal solid line at $A_p = 5.6$ in Fig. 5. So intra-pulse oscillations decay toward a fixed point in region $D$, whereas they approach a limit cycle (the one generated by the Hopf bifurcation) in region $C$. Both scenarios (in region $C$ and $D$) lead to the emergence of a chaotic behavior with different attractors (see Fig. 8). Notice that for the chaotic attractor of region $C$ the total intensity shows large excursions departing from zero while in region $D$ the total intensity never vanishes, on the contrary excursions are departing from the previously stable stationary solution.
4. COUNTERPROPAGATING MODE DYNAMICS IN SEMICONDUCTOR RING LASERS

The two-mode dynamics in ring laser systems has been the subject of a huge amount of experimental and theoretical investigations. From the fundamental point of view, the analysis of the peculiar symmetry properties of the ring laser provides insight for non-linear dynamics studies, enhancing the knowledge of bifurcation theory applied to symmetry groups. Besides, the development of the ring laser gyroscope has raised a great practical interest, enhancing the efforts toward a full understanding of the ring laser physics. Semiconductor Ring Lasers (SRLs) are receiving growing interest due to their interesting features: they do not require cleaved facets or gratings for optical feedback and thus monolithic integration is easily achievable. They are promising candidates for wavelength filtering, multiplexing/demultiplexing applications, electrical and all-optical switching and bistables for optical memories. Hereby we review some analytical expressions for the oscillation threshold and frequency of oscillatory bidirectional regime in SRLs. The $L-I$ curves numerically reproduced using a two-mode model. A quantitative agreement between theory and measurements made over a large number of devices is obtained.

SRLs with a 2 $\mu$m wide single transverse mode ridge waveguide were fabricated in a Double Quantum Well GaAs/AlGaAs structure with 1 $\mu$m ring-radius; the latter value was chosen to minimize bending losses. An output straight waveguide with the same structure is directionally coupled to the ring, providing cross power transmission around 10%. The output waveguide has a 5 degrees tilt angle with respect to cleaved chip facets to minimize backreflections, and it is terminated at both ends with two reverse-biased contacts acting as integrated photodetectors for mode 1 (counter-clockwise) and mode 2 (clockwise). The reverse-biased contacts further reduce the optical feedback from end facets. Experimental $L-I$ curves measured at 25°C from the two integrated photodiodes PD1 and PD2 as a function of DC current injected into the ring contact. The photocurrents from
The equations read:

\[ \dot{E}_{1,2} = (1 + i\alpha) \left[ N(1 - s|E_{1,2}|^2 - c|E_{2,1}|^2) - 1 \right] E_{1,2} - (k_d + i k_c)E_{2,1}, \]  

(39)
where $\alpha$ accounts for phase-amplitude coupling, the self and cross saturation coefficients are given by $s$ and $c$ respectively; the parameters $k_d$ and $k_c$ represent the dissipative and conservative components of the backscattering, respectively.\(^5\) The carrier density $N$ obeys the usual rate equation for semiconductor lasers,

$$\dot{N} = \gamma (\mu - N - N(1 - s|E_1|^2 - c|E_2|^2)|E_1|^2 - N(1 - s|E_2|^2 - c|E_1|^2)|E_2|^2) \text{ ,}$$  \hspace{1cm} (40)

where $\mu$ is the dimensionless pump ($\mu = 1$ at laser threshold). In the set (39)-(40) the dimensionless time is rescaled by the photon lifetime $\tau_p$. The parameter $\gamma$ is the ratio of the carrier lifetime $\tau_s$ over $\tau_p$. The set (39)-(40) written for the intensities $S_{1,2} = |E_{1,2}|^2$ reverts to a previous model,\(^5\) except for the backscattering terms. In a real system, the ideal invariance symmetry along the ring is never met due to many effects (waveguide imperfections, output coupler, scattering centers). Any breaking of the invariance symmetry along the ring translates in a source of coupling between the two counterpropagating fields.\(^5\) Thus, backscattering terms have to be considered. In our approach, $k_d$ and $k_c$ are fitting parameters, since their actual values are, in principle, technology-dependent. According to a previous analysis,\(^6\) the saturation parameters $c$ and $s$ in a SRL fulfil the condition $c/s > 1$.

By substituting the general solution $E_{1,2} = Q_{1,2}(t) \exp(i\omega_{\pm}t + i\phi_{1,2}(t))$ in Eqs (39)-(40), the the steady state solutions are found by setting all the derivatives to zero. We find a symmetric ($Q_1 = Q_2 = Q$) steady state SS. By introducing $\psi = \phi_2 - \phi_1$, and $I = 2Q^2$, the SS = $(\omega_+, \psi, I_0, N_0)$ is given by $\omega_+ = \omega_- = -\alpha k_d + k_c$, $\psi_0 = \pi$,

$$I_0 = \frac{N_0 - 1 + k_d}{\eta N_0} \text{ ,}$$  \hspace{1cm} (41)

$$N_0 = \frac{\mu}{1 + I_0 - \eta I_0^2} \text{ .}$$  \hspace{1cm} (42)

where $\eta = (c + s)/2$. After linearization of the perturbations defined by $E_{1,2} = (\sqrt{I_0/2} + a_{1,2}) \exp(i\omega_{\pm}t + i\phi_{1,2})$, $N = N_0 + \Delta$, a linear stability analysis of the solutions (41)-(42) is performed; $a_{1,2}$ are complex perturbations of the field amplitudes, and $\Delta$ is a real perturbation of the carrier variables. To the first order in the perturbations, the system (39)-(40) decouples into two subsets. The first subset contains the variables $S = a_2 + a_1$ and $\Delta$, and accounts for the total intensity stability, and is always stable. The second subset contains the complex variable $R = a_2 - a_1$, and describes the stability of one field with respect to the other:

$$\dot{R} = \frac{1}{2} (1 + i\alpha)N_0I_0(c - s)(R + R^*) - 2(k_d + ik_c) \text{ ,}$$  \hspace{1cm} (43)

$$\dot{R}^* = \frac{1}{2} (1 - i\alpha)N_0I_0(c - s)(R + R^*) - 2(k_d - ik_c) \text{ .}$$  \hspace{1cm} (44)

The calculation of the eigenvalues associated to the set (43)-(44), shows that near threshold the SS is stable. Increasing the pump, the SS loses stability through a Hopf bifurcation when

$$4k_d = N_0I_0(c - s) \text{ ,}$$  \hspace{1cm} (45)

exhibiting pulsating behavior at the frequency

$$\Omega = 2\sqrt{k_d^2 + k_c^2 + \frac{N_0I_0}{2}(k_d + \alpha k_c)} \text{ .}$$  \hspace{1cm} (46)

We remark that an in-phase solution ($\psi = 0$) also exists, but it is always unstable and we do not consider it here. By expressing $I_0$ and $N_0$ as function of $\mu$, a particular value of $\mu = \mu_H$ satisfies Eq. (45). Thus, for $1 < \mu < \mu_H$, bidirectional stable operation is predicted; when $\mu > \mu_H$ oscillatory behavior takes place. This oscillation represents a limit cycle in the variables $R$ and $R^*$ and reverts to Alternate Oscillations of the two modes intensities $|E_{1,2}|^2$. The AO angular frequency at onset is obtained by introducing Eq. (45) in Eq. (46), yielding

$$\Omega_H = 2\sqrt{k_d^2 + k_c^2 + 2\alpha k_d k_c} \text{ .}$$  \hspace{1cm} (47)

Increasing further the pump, the oscillation frequency decreases, until oscillations disappear when two new quasi-unidirectional solutions become stable. In the latter regime the output power is
mainly concentrated in one direction, and no pulsation occurs. The analytical description of this transition is rather involved and will be the subject of further investigations.

Numerical results were obtained by integrating Eqs (39)-(40) through a standard Runge-Kutta algorithm. We find that the pump interval for the AO regime widens when $k_c$ is increased, while it shrinks when $k_d$ is increased. Due to the strong cross-gain saturation, the semiconductor medium tends to select unidirectional operation. However, due to backscattering, the pure unidirectional state is not a solution in the ring cavity, and bidirectional regimes are favored. The tendency to unidirectional behavior is recovered at higher pump level, where non-linear gain imposes a stronger mode selection.

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REFERENCES