A GEOMETRIC HAMILTON-JACOBI THEORY FOR CLASSICAL FIELD THEORIES

MANUEL DE LEÓN, JUAN CARLOS MARRERO, AND DAVID MARTÍN DE DIEGO

ABSTRACT. In this paper we extend the geometric formalism of the Hamilton-Jacobi theory for hamiltonian mechanics to the case of classical field theories in the framework of multisymplectic geometry and Ehresmann connections.

CONTENTS

1. Introduction 1
2. A geometric Hamilton-Jacobi theory for Hamiltonian mechanics 2
3. The multisymplectic formalism 3
3.1. Multisymplectic bundles 3
3.2. Ehresmann Connections in the fibration \( \pi_1 : J^1 \pi^* \longrightarrow M \) 5
3.3. Hamiltonian sections 5
3.4. The field equations 5
4. The Hamilton-Jacobi theory 6
5. Time-dependent mechanics 9
References 10

1. INTRODUCTION

The standard formulation of the Hamilton-Jacobi problem is to find a function \( S(t, q^A) \) (called the principal function) such that

\[
\frac{\partial S}{\partial t} + H(q^A, \frac{\partial S}{\partial q^A}) = 0.
\] (1.1)

To Prof. Demeter Krupka in his 65th birthday
2000 Mathematics Subject Classification. 70S05, 49L99.
Key words and phrases. Multisymplectic field theory, Hamilton-Jacobi equations.

This work has been partially supported by MEC (Spain) Grants MTM 2006-03322, MTM 2007-62478, project “Ingenio Mathematica” (i-MATH) No. CSD 2006-00032 (Consolider-Ingenio 2010) and S-0505/ESP/0158 of the CAM.
If we put
\[ S(t, q^A) = W(q^A) - tE, \]
where \( E \) is a constant, then \( W \) satisfies
\[ H(q^A, \frac{\partial W}{\partial q^A}) = E; \quad (1.2) \]
\( W \) is called the **characteristic function**.

Equations (1.1) and (1.2) are indistinctly referred as the **Hamilton-Jacobi equation**.

There are some recent attempts to extend this theory for classical field theories in the framework of the so-called multisymplectic formalism \[15, 16\]. For a classical field theory the hamiltonian is a function
\[ H(x^\mu, y^i, p^\mu_i), \]
where \((x^\mu)\) are coordinates in the space-time, \((y^i)\) represent the field coordinates, and \((p^\mu_i)\) are the conjugate momenta.

In this context, the Hamilton-Jacobi equation is \[17\]
\[ \frac{\partial S^\mu}{\partial x^\nu} + H(x^\nu, y^i, \frac{\partial S^\mu}{\partial y^i}) = 0 \quad (1.3) \]
where \( S^\mu = S^\mu(x^\nu, y^j) \).

In this paper we introduce a geometric version for the Hamilton-Jacobi theory based in two facts: (1) the recent geometric description for Hamiltonian mechanics developed in \[6\] (see \[8\] for the case of nonholonomic mechanics); (2) the multisymplectic formalism for classical field theories \[3, 4, 5, 7\] in terms of Ehresmann connections \[9, 10, 11, 12\].

We shall also adopt the convention that a repeated index implies summation over the range of the index.

## 2. A GEOMETRIC HAMILTON-JACOBI THEORY FOR HAMILTONIAN MECHANICS

First of all, we give a geometric version of the standard Hamilton-Jacobi theory which will be useful in the sequel.

Let \( Q \) be the configuration manifold, and \( T^*Q \) its cotangent bundle equipped with the canonical symplectic form
\[ \omega_Q = dq^A \wedge dp_A \]
where \((q^A)\) are coordinates in \( Q \) and \((q^A, p_A)\) are the induced ones in \( T^*Q \).

Let \( H : T^*Q \rightarrow \mathbb{R} \) a hamiltonian function and \( X_H \) the corresponding hamiltonian vector field:
\[ i_{X_H} \omega_Q = dH \]

The integral curves of \( X_H \), \((q^A(t), p_A(t))\), satisfy the Hamilton equations:
\[ \frac{dq^A}{dt} = \frac{\partial H}{\partial p_A}, \quad \frac{dp_A}{dt} = -\frac{\partial H}{\partial q^A} \]
Theorem 2.1 (Hamilton-Jacobi Theorem). Let $\lambda$ be a closed 1-form on $Q$ (that is, $d\lambda = 0$ and, locally $\lambda = dW$). Then, the following conditions are equivalent:

(i) If $\sigma : I \rightarrow Q$ satisfies the equation

$$\frac{dq^A}{dt} = \frac{\partial H}{\partial p_A}$$

then $\lambda \circ \sigma$ is a solution of the Hamilton equations;

(ii) $d(H \circ \lambda) = 0$.

To go further in this analysis, define a vector field on $Q$:

$$X^\lambda_H = T\pi_Q \circ X_H \circ \lambda$$

as we can see in the following diagram:

Notice that the following conditions are equivalent:

(i) If $\sigma : I \rightarrow Q$ satisfies the equation

$$\frac{dq^A}{dt} = \frac{\partial H}{\partial p_A}$$

then $\lambda \circ \sigma$ is a solution of the Hamilton equations;

(i)' If $\sigma : I \rightarrow Q$ is an integral curve of $X^\lambda_H$, then $\lambda \circ \sigma$ is an integral curve of $X_H$;

(i)" $X_H$ and $X^\lambda_H$ are $\lambda$-related, i.e.

$$T\lambda(X^\lambda_H) = X_H \circ \lambda$$

so that the above theorem can be stated as follows:

Theorem 2.2 (Hamilton-Jacobi Theorem). Let $\lambda$ be a closed 1-form on $Q$. Then, the following conditions are equivalent:

(i) $X^\lambda_H$ and $X_H$ are $\lambda$-related;

(ii) $d(H \circ \lambda) = 0$.

3. The multisymplectic formalism

3.1. Multisymplectic bundles. The configuration manifold in Mechanics is substituted by a fibred manifold

$$\pi : E \longrightarrow M$$
such that

(i) \( \dim M = n, \dim E = n + m \)
(ii) \( M \) is endowed with a volume form \( \eta \).

We can choose fibred coordinates \((x^\mu, y^i)\) such that

\[
\eta = dx^1 \wedge \cdots \wedge dx^n.
\]

We will use the following useful notations:

\[
d^n x = dx^1 \wedge \cdots \wedge dx^n
\]
\[
d^{n-1} x^\mu = i_{\frac{\partial}{\partial x^\mu}} d^n x.
\]

Denote by \( V\pi = \ker T\pi \) the vertical bundle of \( \pi \), that is, their elements are the tangent vectors to \( E \) which are \( \pi \)-vertical.

Denote by \( \Pi : \Lambda^n E \rightarrow E \)
the vector bundle of \( n \)-forms on \( E \).

The total space \( \Lambda^n E \) is equipped with a canonical \( n \)-form \( \Theta \):

\[
\Theta(\alpha)(X_1, \ldots, X_n) = \alpha(e)(T\Pi(X_1), \ldots, T\Pi(X_n))
\]

where \( X_1, \ldots, X_n \in T_e(\Lambda^n E) \) and \( \alpha \) is an \( n \)-form at \( e \in E \).

The \((n+1)\)-form

\[
\Omega = -d\Theta,
\]

is called the canonical multisymplectic form on \( \Lambda^n E \).

Denote by \( \Lambda^n_r E \) the bundle of \( r \)-semibasic \( n \)-forms on \( E \), say

\[
\Lambda^n_r E = \{ \alpha \in \Lambda^n E \mid i_{v_1 \wedge \cdots \wedge v_r} \alpha = 0, \text{ whenever } v_1, \ldots, v_r \text{ are } \pi \text{-vertical} \}
\]

Since \( \Lambda^n_r E \) is a submanifold of \( \Lambda^n E \) it is equipped with a multisymplectic form \( \Omega_r \), which is just the restriction of \( \Omega \).

Two bundles of semibasic forms play an special role: \( \Lambda^n_1 E \) and \( \Lambda^n_2 E \). The elements of these spaces have the following local expressions:

\[
\Lambda^n_1 E : p_0 d^n x
\]
\[
\Lambda^n_2 E : p_0 d^n x + p_\mu^i dy^i \wedge d^{n-1} x^\mu.
\]

which permits to introduce local coordinates \((x^\mu, y^i, p_0)\) and \((x^\mu, y^i, p_0, p_\mu^i)\) in \( \Lambda^n_1 E \) and \( \Lambda^n_2 E \), respectively.

Since \( \Lambda^n_1 E \) is a vector subbundle of \( \Lambda^n_2 E \) over \( E \), we can obtain the quotient vector space denoted by \( J^1\pi^* \) which completes the following exact sequence of vector bundles:

\[
0 \rightarrow \Lambda^n_1 E \rightarrow \Lambda^n_2 E \rightarrow J^1\pi^* \rightarrow 0.
\]

We denote by \( \pi_{1,0} : J^1\pi^* \rightarrow E \) and \( \pi_1 : J^1\pi^* \rightarrow M \) the induced fibrations.
3.2. Ehresmann Connections in the fibration \( \pi_1 : J^1 \pi_* \longrightarrow M \).

A **connection** (in the sense of Ehresmann) in \( \pi_1 \) is a horizontal subbundle \( \mathbf{H} \) which is complementary to \( V \pi_1 \); namely,

\[
T(J^1 \pi^*) = \mathbf{H} \oplus V \pi_1
\]

where \( V \pi_1 = \ker T \pi_1 \) is the vertical bundle of \( \pi_1 \). Thus, we have:

(i) there exists a (unique) horizontal lift of every tangent vector to \( M \);

(ii) in fibred coordinates \( (x^\mu, y^i, p^\mu_i) \) on \( J^1 \pi^* \), then

\[
V \pi_1 = \text{span} \left\{ \frac{\partial}{\partial y^i}, \frac{\partial}{\partial p^\mu_i} \right\}, \quad \mathbf{H} = \text{span} \left\{ \mathbf{H}_\mu \right\} ,
\]

where \( \mathbf{H}_\mu \) is the horizontal lift of \( \frac{\partial}{\partial x^\mu} \).

(iii) there is a horizontal projector \( h : T J^* \pi \longrightarrow \mathbf{H} \).

3.3. Hamiltonian sections. Consider a hamiltonian section

\[
h : J^1 \pi^* \longrightarrow \Lambda_2^\ast \mathcal{E}
\]

of the canonical projection \( \mu : \Lambda_2^\ast \mathcal{E} \longrightarrow J^1 \pi^* \) which in local coordinates read as

\[
h(x^\mu, y^i, p^\mu_i) = (x^\mu, y^i, -H(x, y, p), p^\mu_i) .
\]

Denote by \( \Omega_h = h^* \Omega_2 \), where \( \Omega_2 \) is the multisymplectic form on \( \Lambda_2^\ast \mathcal{E} \).

The field equations can be written as follows:

\[
i_h \Omega_h = (n - 1) \Omega_h , \quad (3.1)
\]

where \( h \) denotes the horizontal projection of an Ehresmann connection in the fibred manifold \( \pi_1 : J^1 \pi^* \longrightarrow M \).

The local expressions of \( \Omega_2 \) and \( \Omega_h \) are:

\[
\begin{align*}
\Omega_2 &= -d(p_0 d^\nu x + p^\mu_i dy^i \wedge d^{n-1}x^\mu), \\
\Omega_h &= -d(-H d^\nu x + p^\mu_i dy^i \wedge d^{n-1}x^\mu) .
\end{align*}
\]

3.4. The field equations. Next, we go back to the Equation (3.1).

The horizontal subspaces are locally spanned by the local vector fields

\[
H_\mu = h \left( \frac{\partial}{\partial x^\mu} \right) = \frac{\partial}{\partial x^\mu} + \Gamma^i_\mu \frac{\partial}{\partial y^i} + (\Gamma_\mu)^\nu_j \frac{\partial}{\partial p^\nu_j} ,
\]

where \( \Gamma^i_\mu \) and \( (\Gamma_\mu)^\nu_j \) are the Christoffel components of the connection.

Assume that \( \tau \) is an integral section of \( h \); this means that \( \tau : M \longrightarrow J^1 \pi^* \) is a local section of the canonical projection \( \pi_1 : J^1 \pi^* \longrightarrow M \) such that \( T \tau(x)(T_x M) = \mathbf{H}_{\tau(x)} \), for all \( x \in M \).

If \( \tau(x^\mu) = (x^\mu, \tau^i(x), \tau^\mu_i(x)) \) then the above conditions becomes

\[
\begin{align*}
\frac{\partial \tau^i}{\partial x^\mu} &= \frac{\partial H}{\partial p^\mu_i} , \quad \frac{\partial \tau^\mu_i}{\partial x^\mu} = -\frac{\partial H}{\partial y^i} .
\end{align*}
\]
which are the Hamilton equations.

4. The Hamilton-Jacobi Theory

Let $\lambda$ be a 2-semibasic $n$-form on $E$; in local coordinates we have

$$\lambda = \lambda_0(x, y) \, d^n x + \lambda^\mu_i(x, y) \, dy^i \wedge d^{n-1} x^\mu.$$ 

Alternatively, we can see it as a section $\lambda : E \to \Lambda^2_E$, and then we have

$$\lambda(x^\mu, y^i) = (x^\mu, y^i, \lambda_0(x, y), \lambda^\mu_i(x, y)).$$

A direct computation shows that

$$d\lambda = \left( \frac{\partial \lambda_0}{\partial y^i} - \frac{\partial \lambda^\mu_i}{\partial x^\mu} \right) \, dy^i \wedge d^n x + \frac{\partial \lambda^\mu_i}{\partial y^j} \, dy^j \wedge dy^i \wedge d^{n-1} x^\mu.$$ 

Therefore, $d\lambda = 0$ if and only if

$$\frac{\partial \lambda_0}{\partial y^i} = \frac{\partial \lambda^\mu_i}{\partial x^\mu} \quad (4.1)$$

$$\frac{\partial \lambda^\mu_i}{\partial y^j} = \frac{\partial \lambda^\mu_j}{\partial y^i}. \quad (4.2)$$

Using $\lambda$ and $h$ we construct an induced connection in the fibred manifold $\pi : E \to M$ by defining its horizontal projector as follows:

$$\tilde{h}_e : T_e E \to T_e E$$

$$\tilde{h}_e(X) = T\pi_{1,0} \circ h_{(\mu \circ \lambda)(e)} \circ \epsilon(X)$$

where $\epsilon(X) \in T_{(\mu \circ \lambda)(e)}(J^1 \pi^*)$ is an arbitrary tangent vector which projects onto $X$.

From the above definition we immediately proves that

(i) $\tilde{h}$ is a well-defined connection in the fibration $\pi : E \to M$.

(ii) The corresponding horizontal subspaces are locally spanned by

$$\tilde{H}_\mu = \tilde{h}(\frac{\partial}{\partial x^\mu}) = \frac{\partial}{\partial x^\mu} + \Gamma^i_\mu((\mu \circ \lambda)(x, y)) \frac{\partial}{\partial y^i}.$$ 

The following theorem is the main result of this paper.

**Theorem 4.1.** Assume that $\lambda$ is a closed 2-semibasic form on $E$ and that $\tilde{h}$ is a flat connection on $\pi : E \to M$. Then the following conditions are equivalent:

(i) If $\sigma$ is an integral section of $\tilde{h}$ then $\mu \circ \lambda \circ \sigma$ is a solution of the Hamilton equations.

(ii) The $n$-form $h \circ \mu \circ \lambda$ is closed.
Before to begin with the proof, let us consider some preliminary results.

We have

\[(h \circ \mu \circ \lambda)(x^\mu, y^i) = (x^\mu, y^i, -H(x^\mu, y^i, \lambda_i^\mu(x, y)), \lambda_i^\mu(x, y)),\]

that is

\[h \circ \mu \circ \lambda = -H(x^\mu, y^i, \lambda_i^\mu(x, y))\,dx^\mu + \lambda_i^\mu\,dy^i \wedge d^{n-1}x^\mu.\]

Notice that \(h \circ \mu \circ \lambda\) is again a 2-semibasic \(n\)-form on \(E\).

A direct computation shows that

\[d(h \circ \mu \circ \lambda) = -\left(\frac{\partial H}{\partial y^i} + \frac{\partial H}{\partial p_j^\nu} \frac{\partial \lambda^\nu_i}{\partial y^i} + \frac{\partial \lambda^\mu_i}{\partial x^\mu}\right)\,dy^i \wedge d^n x + \frac{\partial \lambda^\mu_i}{\partial y^i} dy^i \wedge dy^j \wedge d^{n-1}x^\mu.\]

Therefore, we have the following result.

**Lemma 4.2.** Assume \(d\lambda = 0\); then

\[d(h \circ \mu \circ \lambda) = 0\]

if and only if

\[\frac{\partial H}{\partial y^i} + \frac{\partial H}{\partial p_j^\nu} \frac{\partial \lambda^\nu_i}{\partial y^i} + \frac{\partial \lambda^\mu_i}{\partial x^\mu} = 0.\]

**Proof of the Theorem**

\((i) \Rightarrow (ii)\)

It should be remarked the meaning of \((i)\).

Assume that

\[\sigma(x^\mu) = (x^\mu, \sigma^i(x))\]

is an integral section of \(\tilde{h}\); then

\[\frac{\partial \sigma^i}{\partial x^\mu} = \frac{\partial H}{\partial p_i^\nu}.\]

\((i)\) states that in the above conditions,

\[(\mu \circ \lambda \circ \sigma)(x^\mu) = (x^\mu, \sigma^i(x), \sigma^\nu_j = \lambda^\nu_j(\sigma(x)))\]

is a solution of the Hamilton equations, that is,

\[\frac{\partial \sigma^i}{\partial x^\mu} = \frac{\partial \lambda^\nu_i}{\partial x^\mu} + \frac{\partial \lambda^\mu_i}{\partial y^i} \frac{\partial \sigma^i}{\partial x^\mu} = -\frac{\partial H}{\partial y^i}.\]
Assume (i). Then
\[
\frac{\partial H}{\partial y^i} + \frac{\partial H}{\partial p^j} \frac{\partial y^j}{\partial y^i} + \frac{\partial \lambda^\mu_i}{\partial x^\mu} = \frac{\partial H}{\partial y^i} + \frac{\partial \sigma^j}{\partial x^\nu} \frac{\partial y^j}{\partial y^\nu} + \frac{\partial \lambda^\mu_i}{\partial x^\mu},
\]
(since \( d\lambda = 0 \))
\[
= \frac{\partial H}{\partial y^i} + \frac{\partial \sigma^j}{\partial x^\nu} \frac{\partial y^j}{\partial y^\nu} + \frac{\partial \lambda^\mu_i}{\partial x^\mu},
\]
(since the first Hamilton equation)
\[
= 0 \quad (since (i))
\]
which implies (ii) by Lemma 4.2

(ii) ⇒ (i)

Assume that \( d(h \circ \mu \circ \lambda) = 0 \).

Since \( \tilde{h} \) is a flat connection, we may consider an integral section \( \sigma \) of \( \tilde{h} \). Suppose that
\[
\sigma(x^\mu) = (x^\mu, \sigma^i(x)).
\]
Then, we have that
\[
\frac{\partial \sigma^i}{\partial x^\mu} = \frac{\partial H}{\partial p^\mu_i}.
\]
Thus,
\[
\frac{\partial \sigma^\mu_i}{\partial x^\mu} = \frac{\partial \lambda^\mu_i}{\partial x^\mu} + \frac{\partial \lambda^\mu_j}{\partial y^j} \frac{\partial \sigma^i}{\partial x^\mu},
\]
(since \( d\lambda = 0 \))
\[
= \frac{\partial \lambda^\mu_i}{\partial x^\mu} + \frac{\partial \lambda^\mu_j}{\partial y^j} \frac{\partial \sigma^i}{\partial x^\mu},
\]
(since the first Hamilton equation)
\[
= -\frac{\partial H}{\partial y^i}, \quad (since (ii)). \square
\]

Assume that \( \lambda = dS \), where \( S \) is a 1-semibasic \((n - 1)\)-form, say
\[
S = S^\mu \ d^{n-1}x^\mu
\]
Therefore, we have
\[
\lambda_0 = \frac{\partial S^\mu}{\partial x^\mu}, \quad \lambda^\mu_i = \frac{\partial S^\mu}{\partial y^i}
\]
and the Hamilton-Jacobi equation has the form
\[
\frac{\partial}{\partial y^i} \left( \frac{\partial S^\mu}{\partial x^\mu} + H(x^\nu, y^i, \frac{\partial S^\mu}{\partial y^i}) \right) = 0.
\]
The above equations mean that
\[
\frac{\partial S^\mu}{\partial x^\mu} + H(x^\nu, y^i, \frac{\partial S^\mu}{\partial y^i}) = f(x^\mu)
\]
so that if we put \( \tilde{H} = H - f \) we deduce the standard form of the Hamilton-Jacobi equation (since \( H \) and \( \tilde{H} \) give the same Hamilton equations):

\[
\frac{\partial S^\mu}{\partial x^\mu} + \tilde{H}(x^\nu, y^i, \frac{\partial S^\mu}{\partial y^i}) = 0 .
\]

An alternative geometric approach of the Hamilton-Jacobi theory for Classical Field Theories in a multisymplectic setting was discussed in [15, 16].

5. Time-dependent mechanics

A hamiltonian time-dependent mechanical system corresponds to a classical field theory when the base is \( M = \mathbb{R} \).

We have the following identification \( \Lambda_1^1 E = T^* E \) and we have local coordinates \((t, y^i, p_0, p_i)\) and \((t, y^i, p_i)\) on \( T^* E \) and \( J^1 \pi^* \), respectively. The hamiltonian section is given by

\[
h(t, y^i, p_i) = (t, y^i, -H(t, y, p), p_i),
\]

and therefore we obtain

\[
\Omega_h = dH \wedge dt - dp_i \wedge dy^i .
\]

If we denote by \( \eta = dt \) the different pull-backs of \( dt \) to the fibred manifolds over \( M \), we have the following result.

The pair \((\Omega_h, dt)\) is a cosymplectic structure on \( E \), that is, \( \Omega_h \) and \( dt \) are closed forms and \( dt \wedge \Omega^n_h = dt \wedge \Omega_h \wedge \cdots \wedge \Omega_h \) is a volume form, where \( dimE = 2n + 1 \). The Reeb vector field \( R_h \) of the structure \((\Omega_h, dt)\) satisfies

\[
i_{R_h} \Omega_h = 0 , \ i_{R_h} dt = 1.
\]

The integral curves of \( R_h \) are just the solutions of the Hamilton equations for \( H \).

The relation with the multisymplectic approach is the following:

\[
h = R_h \otimes dt ,
\]

or, equivalently,

\[
h(\frac{\partial}{\partial t}) = R_h .
\]

A closed 1-form \( \lambda \) on \( E \) is locally represented by

\[
\lambda = \lambda_0 dt + \lambda_i dy^i .
\]

Using \( \lambda \) we obtain a vector field on \( E \):

\[
(R_h)_\lambda = T_{\pi_{1,0}} \circ R_h \circ \mu \circ \lambda
\]

such that the induced connection is

\[
h = (R_h)_\lambda \otimes dt
\]
Therefore, we have the following result.

**Theorem 5.1.** The following conditions are equivalent:

(i) \((R_h)^\lambda\) and \(R_h\) are \((\mu \circ \lambda)\)-related.

(ii) The 1-form \(h \circ \mu \circ \lambda\) is closed.

**Remark 5.2.** An equivalent result to Theorem 5.1 was proved in [14] (see Corollary 5 in [14]).

Now, if

\[
\lambda = dS = \frac{\partial S}{\partial t} dt + \frac{\partial S}{\partial y^i} dy^i,
\]

then we obtain the Hamilton-Jacobi equation

\[
\frac{\partial}{\partial y^i} \left( \frac{\partial S}{\partial t} + H(t, y^i, \frac{\partial S}{\partial y^i}) \right) = 0.
\]

**REFERENCES**


**Instituto de Ciencias Matemáticas**, CSIC-UAM-UC3M-UCM, Serrano 123, 28006 Madrid, Spain

*E-mail address*: mdeleon@imaff.cfmac.csic.es

J.C. Marrero: Departamento de Matemática Fundamental, Universidad de La Laguna, La Laguna, Canary Islands, Spain

*E-mail address*: jcmarrer@ull.es

D. Martín de Diego: Instituto de Ciencias Matemáticas, CSIC-UAM-UC3M-UCM, Serrano 123, 28006 Madrid, Spain

*E-mail address*: d.martin@imaff.cfmac.csic.es