A PROOF OF BRST INVARIANCE

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Abstract

Introducing a geometric normal ordering, we give a proof of BRST invariance of the states associated to an arbitrary genus Riemann surface with a puncture, in the operator formalism.

1 Introduction

In the last years the interest in formulating conformal field theories on higher genus Riemann surfaces has been the central motivation of many works. This interest is motivated mainly by the necessity to calculate multiloop string scattering amplitudes.

The operator formalism ([1],[2],[3]) is one of the most fruitful approaches. In the operator formalism, we associate a state on some Hilbert space to a punctured Riemann surface on which we have defined a conformal field theory. The state is defined as the state annihilated by some conserved charges of the theory. These conserved charges correspond to invariances of the path integral under shifts of the fields. They result in annihilation operators in relation to the Bogoliubov transformation of the ordinary vacuum. If there are several fields in the theory, then we can take the tensor product of the states associated to each field. In string theory we thus take the tensor product of the states associated to each $X^a$ (matter) field and the state associated to the conformal ghost fields $b$ and $c$.

This formalism has many advantages. Nevertheless, some statements are proved in terms of the sewing of surfaces and the "charge transport argument" ([1]), by making use of well-known results for the simplest cases (one, two and three punctured spheres). They are still lacking a more general proof.
larly there is no such a proof for the BRST invariance of the states associated in
this formalism to arbitrary genus punctured Riemann surfaces.

In this paper that proof is given for one punctured Riemann surfaces (bosonic
string). The generalization to more punctures and super-Riemann surfaces
(fermionic string) seems straightforward. In the last section we will make some
comments on these subjects.

We first introduce in section 2 dual bases of vectors and quadratic differentials
in a local coordinate around the puncture. These bases contain information
about the global properties of the Riemann surface: they include the Teichmüller
deformations and the vectors and quadratic differentials which are meromorphic
at the puncture but extend holomorphically to the rest of the surface. The string
path integral is invariant under shifts of b and c by precisely the latter elements
of the bases. The associated conserved charges are the b and c Fourier modes

\[ \{Q_{BRST}, b(z)\} = T^{Tot}(z) \]

where \( T^{Tot}(z) \) is the geometrically normal ordered total energy-momentum
tensor. This allows us to claim the BRST invariance of \( \phi > \) and has the properties of a good BRST charge. It is
nilpotent (in 26 dimensions). It is remarkable that its anticommutation relation
with the b ghost field is:

\[ \{Q_{BRST}, b(z)\} = T^{Tot}(z) \]

2) a basis of all vectors which are holomorphic on D but do not extend
holomorphically off P.

3) a basis of all holomorphic vectors on D which do not extend holomorphically off P.

respectively to \( \Sigma - D \).

Respectively:

1) a basis of all quadratic differentials which are holomorphic in D but do not
extend holomorphically to the rest of \( \Sigma \).

2) a basis for all holomorphic vectors on D which do not extend holomorphically off P.

3) a basis for all quadratic differentials which are meromorphic to \( \Sigma - D \).

An immediate consequence is

\[ \oint_C S_{K^1} V_{L^2} = B^2_{K^1 L^2} \]

where for later convenience all indices k+1 and all indices with superindex 2
are dropped.

We can complete those bases to obtain bases for every quadratic differential
on the surface.

In section 4, we briefly study the algebra of the geometrically normal ordered
energy-momentum tensor Fourier modes, and the conformal anomaly cancellation.

The geometrically normal ordered BRST charge, introduced in section 5, triv-
ially annihilates \( \phi > \) and has the properties of a good BRST charge. It is

1) a basis of all quadratic differentials which are meromorphic to \( \Sigma - D \).

2) a basis of all vectors which are holomorphic to \( \Sigma - D \).

3) a basis for all holomorphic vectors

2 Dual basis

Let \( K \) be the canonical line bundle over a generic point \( P \in \Sigma \). Consider
the space of meromorphic sections of \( K \) (quadratic differentials) and \( K^{-1} \) (vacuum expectation values).

According to the Weierstrass gap theorem (see again the appendix B of ref. [1]),
we can write

\[ S_{K^1} = (z^{K^1} + \sum_{m^2} z^{m^2}) \]

\[ V_{L^2} = (z^{-L^2+1} + \cdots) \]

as we can see by pulling off the surface the contour C.

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with the b ghost field is:

\[ \{Q_{BRST}, b(z)\} = T^{Tot}(z) \]

This does not depend on the particular choice of coordinate
around P. This is a nontrivial feature of \( Q_{BRST} \) because \( Q_{BRST} \) is not at first
sight conformally invariant: if you want \( T^{Tot} \) to transform as a honest quadratic
differential, then, the combination \( T(X) + \frac{1}{2}T^{bc} \) which enters in the definition of
\( Q_{BRST} \), does not do it.
As a consequence of (2.1), the basis \( \{S_K\} \) and \( \{V_L\} \) are dual with the scalar product
\[
<S_K|V_L> = \oint_{C_P} S_K V_L = \delta_{KL}
\]
where \( C_P \) is a contour on \( D \) with index +1 in respect of \( P \).

To be more specific, are dual between them: (1) the vectors which extend holomorphically "off \( P \)" and the quadratic differentials which are holomorphic on \( D \) but do not extend and vice versa, and (2) the globally defined quadratic differentials and the vectors which are neither holomorphic on \( D \) nor on \( \Sigma - D \). These later vectors correspond to the Teichmüller deformations and induce the moduli changes.

It is also straightforward that acting on vectors written as kets and quadratic differentials written as bras one has the relation
\[
\sum_K |V_K><S_K| = 1
\]
(6)

We can see this by explicitly decomposing these in Fourier modes:
\[
B_L = <b|V_L> = \oint_{C_P} b V_L
\]
(7)
\[
C_K = <S_K|c> = \oint_{C_P} S_K c
\]
and summing
\[
\sum_L B_L S_L = \sum_L <b|V_L><S_L| = b
\]
\[
\sum_K C_K V_K = \sum_K |V_K><S_K|c >= c
\]
(8)
The relation with the usual decomposition
\[
b = \sum_n b_n z^{-2} (dz)^2
\]
\footnote{In general there is neither globally defined holomorphic vectors nor quadratic differentials that are neither holomorphic on \( D \) nor on \( \Sigma - D \). The only exception is the sphere. We focus our attention mainly in the \( g \geq 1 \) case, but the usual sphere and torus cases can be treated on the same foot.}

\[c = \sum_n c_n \]
is
\[B_K^1 = b_K^1\]
\[B_K^2 = b_K^2\]
\[C_K^1 = c_{K^1}\]
\[C_K^2 = c_{K^2}\]

We will use these decompositions in the momentum tensor of them and the matter case we have
\[
\{C_K, B_L\} = 0
\]
\[
\{C_K, C_L\} = 0
\]

Recall now that the ghost part of the one-punctured genus \( g \) Riemann surface \( |\chi> \) is defined as the state annihilated by the \( B_K^2 \) and \( C_L^1 \) operators [1]. Such a state \( |\chi> \) can be obtained from the ghost vacuum if we previously shift the Dirac sea level \( k = 3g - 3 \) units. This shift is necessary to take into account the ghost number violation by this quantity in genus \( g \) Riemann surfaces. We call this vacuum state \( |k> \) and it is defined by
\[
c_n |k> = 0 \quad n
\]
\[
b_n |k> = 0 \quad n
\]
\[
<k|k> = 0
\]
\[
g(B) = exp(-\sum_K b_K^1 c_{K^1})
\]
\[
|\chi> = g(B)|k>
\]
Note that
\[ g(B) c_l g(B)^{-1} = C_l \]
and then we see that the condition defining \(|\chi>\) :
\[ B_{K^2}|\chi> = 0 \]
\[ C_{L^1}|\chi> = 0 \]
is simply the generalization via Bogoliubov transformation of the condition which defines \(|k>\).

We can also think of a Bogoliubov transformation as a different form of filling the negative energy states [1]. The creation operators are invariant under Bogoliubov transformations, and the annihilation operators transform into the annihilation operators of the Bogoliubov-transformed vacuum. These observations have general validity.

Finally note that the vectors verify the Lie algebra
\[ [V_M, V_N] = \sum_L C_{MN}^L V_L \]
\[ C_{MN}^L = \oint [V_M, V_N] \]
where \([,]\) is the Lie bracket
\[ [f(z)\partial_z, g(z)\partial_z] = (f(z)\partial_zg(z) - g(z)\partial zf(z))\partial_z \]
and the structure constants of the form \(C_{M^2 N^2}^{L^1}\) vanish
\[ C_{M^2 N^2}^{L^1} = 0 \]

3 Geometric normal ordering

We are now ready to introduce a natural normal ordering for anomalous dimension \(h = 2, -1\) conformal fields. Recall first an elementary fact in quantum field theory: the normal ordering is done always place on the right the modes to the remainder (the creation modes). The normal ordering results in the subtraction of expectation values and finally of the terms in which the divergences appear.

The most natural thing we can do is to generalize the usual normal ordering in the ghost case, and \(|0>\) in the now as reference for the normal ordering of the Bogoliubov transformed vacuum. The properties of the Riemann surface we are.

Consequently we define a geometric * as follows: the Bogoliubov-transformed modes annihilating \(|φ>\) are to be placed to the right and the remainder, (the creation modes, untouched by the Bogoliubov transformation) are to be placed to the left.

In order to complete the parallelism, we introduce a state \(|-k>\) of ghost number opposite to the ghost number of \(|k>\), and defined by
\[ c_n |-k> = 0 \quad n \geq k + 2 \]
\[ b_n |-k> = 0 \quad n \geq -(k + 1) \]
\[ <-k|-k> = 0 \]
\[ <-k|+k> = 1 \]
which verifies the property
\[ <-k|C_{K^2}^L = 0 \]

Using this state tensored with the matter
\[ \Theta \]
field in expectation values on the surface has to make in expectation values on the surface has to make
\[ \Theta \]
it is now clear that our definition of the geometric normal ordering for the Fourier modes of the operators is complete.

\[ \Theta \]

The state of the ghost system tensored with the state corresponding to the \(\phi\) fields.
the \( <\Theta|...|\phi> \) expectation values, which probably contain divergencies. In higher genus Riemann surfaces the divergent terms in vacuum expectation values are the same as in the \( g=0,1 \) cases. We are doing local field theory. But in addition there are finite terms which depends on the genus of the surface. It is necessary also to subtract these finite "global" terms.) Thus

\[
\star A(z)B(z)^\star = \{A(z)B(w) - <\Theta|A(z)B(w)|\phi>\}_{w \rightarrow z}
\]  

(22)

The analog of the Wick theorem is valid and

\[
<\Theta|\star O(z)\star|\phi> = 0
\]  

(23)

for any geometrically normal ordered operator \( \star O(z) \star \).

It is a nice exercise to calculate now the bc two point function by using the basis \( \{S_K\} \) and \( \{V_L\} \), the expansion (2.7), the definition of \( |\phi> \) and (3.2).

## 4 Virasoro algebra

The energy-momentum tensor Fourier modes are

\[
\mathcal{L}_{bc}^M \overset{\text{def}}{=} \oint_{C_B} dz V_M(z) \star T^{bc} \star
\]

\[
\star T^{bc} \star = \star (2\partial_c b + c\partial_c b) \star
\]  

(24)

\[
\mathcal{L}_M^{(X)} \overset{\text{def}}{=} \oint_{C_B} dz V_M(z) \star T^{(X)} \star
\]

\[
\star T^{(X)} \star = -\frac{1}{2} \star (X \partial_c X) \star
\]

They are the direct generalization of the usual \( L_m \) via \( g(B) \). Their algebra is

\[
[\mathcal{L}_M^{(X)},\mathcal{L}_N^{(X)}] = \sum_L C_{MN}^L \mathcal{L}_L^{(X)} + A_{MN}^{(X)}
\]  

(25)

\[
[\mathcal{L}_M^{bc},\mathcal{L}_N^{bc}] = \sum_L C_{MN}^L \mathcal{L}_L^{(X)} + A_{MN}^{bc}
\]  

(26)

\[
[\mathcal{L}_M^{bc},\mathcal{L}_N^{bc}] = \sum_L C_{MN}^L \mathcal{L}_L^{(X)} + A_{MN}^{bc}
\]  

(27)

where

\[
A_{MN}^{(X)} = \oint_{C_B} dw \oint_{C_W} dz V_M(z) V_N(w) \sigma \partial_{c} \partial_{b} \partial_{c} \partial_{b}
\]

\[
A_{MN}^{bc} = \oint_{C_B} dw \oint_{C_W} dz V_M(z) V_N(w) \sigma \partial_{c} \partial_{b} \partial_{c} \partial_{b}
\]

(Here \( <...> = <\Theta|...|\phi> \).

Observe first that now

\[
\mathcal{L}_{MN}^{bc} = \oint_{C_B} dw \oint_{C_W} dz V_M(z) V_N(w) \sigma \partial_{c} \partial_{b} \partial_{c} \partial_{b}
\]

is a simple consequence of the definition of normal ordering and (2.16).

Calculating now the leading terms in the integrands of expressions (4.3), we have as expected

\[
A_{MN}^{(X)} = 0
\]

and the conformal anomaly cancels in local fields.

## 5 BRST charge

The generalization of the BRST charge

\[
Q_{BRST} \overset{\text{def}}{=} \sum_M \mathcal{L}_M \frac{Q}{Q}
\]

\[
= \sum_M \mathcal{L}_M \frac{Q}{Q}
\]

\[
= \oint_{C_B} dz \star T^{(X)}
\]

This charge verifies

\[
\{Q_{BRST},c(z)\} = \star c(z) \star T^{(X)}
\]

\[
\{Q_{BRST},b(z)\} = \star T^{(X)}
\]
and, if the total conformal anomaly vanishes

\[ Q_{BRST}^2 = \frac{1}{2} \{ Q, Q \} = 0 \]  \hspace{1cm} (35)

As we have explained in section 1, this property, combined with (5.3), ensures \( Q_{BRST} \) is well defined when we change of coordinate patch.

\[ |\phi> \text{ verifies} \]

\[ Q_{BRST}|\phi> = 0 \]  \hspace{1cm} (36)

6 Conclusion

We can conclude that the geometric normal ordering we have introduced in this paper is the natural extension of the usual normal ordering on the sphere. The generalization to more punctures seems to be straightforward. The two punctures case involves the Krichever-Novikov algebras [5]. The point is the different Hilbert spaces associated to each puncture.

The generalization to super-Riemann surfaces seems also straightforward because there are analogs of the Riemann-Roch [4] and Weierstrass gap [3] theorems.

Work is in progress on these subjects.

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References


