The Supersymmetric Vistas of the Supergravity Landscape

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In the recent times a lot of effort has been devoted to improve our knowledge about the space of string theory vacua ("the landscape") in order to find statistical grounds to justify how and why the theory selects its vacuum. Particularly interesting are those vacua that preserve some supersymmetry, which are always supersymmetric solutions of some supergravity theory. We are going to review some recent results on the problem of finding all the supersymmetric solutions of a supergravity theory. We will also review some interesting solutions that have been discovered using these methods.

1 Introduction

The vacuum selection problem is the most important problem in Superstring Theory and similar unification schemes that include gravity. The vacuum state is the most important state of any Relativistic Quantum Field Theory: its symmetries (which are usually assumed to include the Poincaré group) determine all the kinematic properties: the possible conserved charges and the spectrum of allowed particles. In theories that include gravity the same is still true, the only difference being that different vacua can have different spacetime symmetries and the Poincaré group is not the only possible spacetime symmetry group. In Kaluza-Klein theories, for instance, one considers vacua whose spacetime symmetry group is the product of Poincaré (or Anti de Sitter (AdS)) and a semisimple Lie group which is interpreted as an internal symmetry and gives rise to an associated Yang-Mills sector in the theory. Thus, in Kaluza-Klein theories the vacuum also determines the interactions and the same happens in Superstring Theory.

The possibility of having different vacua that determine many of the properties of a theory constitutes an important conceptual advantage of these theories since, instead of having to explain many different arbitrary choices, one has to explain only one: the choice of vacuum. However, the energies of the vacua of these theories cannot be compared and it is not known how the vacuum is chosen, and, therefore, why our Universe is the way it is.

This is an old and very well known problem. It is also of crucial importance. And it is still unsolved. The failure to solve the vacuum selection problem through some dynamical mechanism has favored recently a purely statistical approach in which one first has to explore and chart ("classify") the space of vacua a.k.a. Landscape. In this approach, our Universe is the way it is because the probability of this kind of Universe is overwhelming. Of course, this way of thinking can be combined with different forms of the Anthropic Principle.

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1 One could enunciate a Kaluza-Klein principle as follows: the global symmetries of the vacuum become the local symmetries of the compactified theory.
Charting the superstring landscape is a very difficult problem and some simplifications have been suggested: for instance, one could consider all supersymmetric String Theory vacua, which correspond to different kinds of supergravities [1] or only the vacua with 4-dimensional Poincaré symmetry and a Calabi-Yau internal space, which correspond to $N = 1, d = 4$ supergravities and give Standard-Model-like theories [2]. One could also consider, as proposed by Van Proeyen [3], all possible supergravities, even if the stringy origin of many of them is unknown (the supergravity landscape).

In this talk, which is based on Refs. [4, 5, 6], we are going to review some recent general results on the classification of supersymmetric String Theory vacua and new techniques that can be used to find them, presenting some particular results on the classification of the supersymmetric vacua of the toroidally compactified Heterotic String Theory ($N = 4, d = 4$ SUGRA). First, we are going to define what is a supersymmetric configuration, describing some useful special identities that they satisfy (Killing spinor identities). Then we will move on to define the problem of finding all the supersymmetric configurations of a given supergravity theory (Tod’s problem) and we will explain the strategy to solve it in most (4-dimensional) cases. Finally, we will consider the case of $N = 4, d = 4$ supergravity.

2 Supersymmetric configurations and solutions

Supersymmetric configurations\(^2\) (a.k.a. configurations with residual or unbroken or preserved supersymmetry) are classical bosonic configurations of supergravity (SUGRA) theories which are invariant under some supersymmetry transformations. Let us see what this definition implies.

Generically, the supersymmetry transformations take, schematically, the form

\[ \delta_\epsilon \phi^b \sim \bar{\epsilon} \phi^f, \quad \delta_\epsilon \phi^f \sim \partial \epsilon + \phi^b \epsilon, \quad (2.1) \]

where $\phi^b$ stands for bosonic fields (or products of an even number of fermionic fields) and $\phi^f$ for the fermionic fields (or products of an odd number of fermionic fields) and $\epsilon$ are the infinitesimal, local, parameters of the supersymmetry transformations, which are fermionic.

Then, a bosonic configuration (i.e. a configuration with vanishing fermionic fields $\phi^f = 0$) will be invariant under the infinitesimal supersymmetry transformation generated by the parameter $\epsilon^\alpha(x)$ if it satisfies the Killing spinor equations (one equation for each $f$), which have the generic form

\[ \delta_\epsilon \phi^f \sim \partial \epsilon + \phi^b \epsilon = 0. \quad (2.2) \]

The concept of unbroken supersymmetry is a generalization of the concept of isometry, an infinitesimal general coordinate transformation generated by $\xi^\mu(x)$ that leaves the metric $g_{\mu\nu}$ invariant because it satisfies the Killing (vector) equation

\[ \delta_\xi g_{\mu\nu} = 2 \nabla_{(\mu} \xi_{\nu)} = 0. \quad (2.3) \]

As it is well known, in this case, to each bosonic symmetry we associate a generator

\[ \xi^\mu_{(I)}(x) \rightarrow P_I, \quad (2.4) \]

of a symmetry (lie) algebra

\[ [P_I, P_J] = f_{IJ}^K P_K, \quad \Leftrightarrow \ [\xi_{(I)}, \xi_{(J)}] = f_{IJ}^K \xi_{(K)}. \quad (2.5) \]

where the brackets in the right are Lie brackets of vector fields and $f_{IJ}^K$ are the structure constants.

\(^2\) It will be very important for our discussion to distinguish between general field configurations and (classical) solutions of a given theory. General field configurations may or may not satisfy the classical equations of motion and, therefore, may or may not be classical solutions. As we are going to see, supersymmetry does not ensure that the equations of motion are satisfied.
In our case, the unbroken supersymmetries are associated to the odd generators
\[ \epsilon_{(\alpha)}^{\alpha}(x) \to Q_n, \]
of a superalgebra
\[ [Q_n, P_I] = f_{nI}^{\ \ m} Q_m, \quad \{Q_n, Q_m\} = f_{nm}^\ I P_I. \] (2.7)

The calculation of these commutators and anticommutators is explained in detail in Refs. [7, 8]. According to the Kaluza-Klein principle we enunciated at the beginning, conveniently generalized to the supersymmetric case, this global supersymmetry algebra becomes the algebra of the local symmetries of the field theories constructed on this field configuration.

Of course, we do not want to construct field theories on just any field configuration but only on vacua of the theory. In general, for a field configuration to be considered a vacuum, we require that it is a classical solution of the equations of motion of the theory. Apart from this requirement, it is not clear what a priori characteristics a good vacuum must have except for classical and quantum stability, which are difficult to test in general, but which are, under certain conditions, guaranteed by the presence of unbroken supersymmetry. This is one of the reasons that makes supersymmetric vacua interesting. We also prefer highly symmetric vacua (such as Minkowski or AdS space) since, on them, we can define a large number of conserved quantities, but it is uncertain why Nature should have the same prejudices.

Sometimes, when a vacuum solution has a clear (possibly warped) product structure, we can distinguish internal and spacetime (super-) symmetries and, if we choose this vacuum, our choice implies spontaneous compactification.

3 Tod’s problem
This is the problem of finding all the supersymmetric bosonic field configurations, i.e. all the bosonic field configurations \( \phi^b \) for which a SUGRA’s Killing spinor equations
\[ \delta_\epsilon \phi^f|_{\phi^f=0} \sim \partial \epsilon + \phi^b \epsilon = 0, \] (3.1)
have a solution \( \epsilon \), which includes all the possible supersymmetric vacua and compactifications.

Observe that, as we announced, not all supersymmetric bosonic field configurations satisfy the classical bosonic equations of motion for which we use the notations\(^3\) \[ \frac{dS}{d\phi^f} |_{\phi^f=0} \equiv S_b|_{\phi^f=0} \equiv \mathcal{E}(\phi^b). \] Actually, the bosonic equations of motion of supersymmetric bosonic field configurations satisfy the so-called Killing spinor identities (KSI) [4, 5] that relate different equations of motion of a supersymmetric theory. These identities can be derived as follows: The supersymmetry invariance of the action implies, for arbitrary local supersymmetry parameters \( \epsilon \)
\[ \delta_\epsilon S = \int d^4x \ (S_b \delta_\epsilon \phi^b + S_f \delta_\epsilon \phi^f) = 0. \] (3.2)
Taking the functional derivative w.r.t. the fermions and setting them to zero
\[ \int d^4x \ [S_b \delta_\epsilon \phi^b + S_b (\delta_\epsilon \phi^b), f_1 + S_f \delta_\epsilon \phi^f + S_f (\delta_\epsilon \phi^f), f_1] \Big|_{\phi^f=0} = 0. \] (3.3)
The terms \( \delta_\epsilon \phi^b|_{\phi^f=0}, \ S_f|_{\phi^f=0}, \ (\delta_\epsilon \phi^f), f_1|_{\phi^f=0} \) vanish automatically because they are odd in fermion fields \( \phi^f \) and so we are left with
\[ \{S_b (\delta_\epsilon \phi^b), f_1 + S_f \delta_\epsilon \phi^f \} |_{\phi^f=0} = 0. \] (3.4)

\(^3\) Throughout this paper, we will mean by “equation of motion” the expression \( \mathcal{E}(\phi^b) \) that gives what is usually meant by equation of motion \( \mathcal{E}(\phi^b) = 0 \) when it is equaled to zero. This abuse of language is necessary for us and should lead to no confusion.
This is valid for any fields $\phi^b$ and any supersymmetry parameter $\epsilon$. For a supersymmetric field configuration $\epsilon$ is a Killing spinor $\delta_{\epsilon}\phi^b|_{\phi=0}$ and we obtain the KSI
\begin{equation}
E(\phi^b) (\delta_{\epsilon}\phi^b),_{\phi=0} = 0.
\end{equation}

These non-trivial identities are linear relations between the bosonic equations of motion and can be used to solve Tod’s problem, obtain Bogomol’nyi-Prasad-Sommerfeld (BPS) -type bounds etc. Let’s see some examples.

### 3.1 Example: $N = 1$, $d = 4$ Supergravity

This is the simplest supergravity theory. Its field content is $\{e^a_{\mu}, \psi_{\mu}\}$ (the Vierbein and the gravitino fields, respectively, whose quanta should be the graviton and gravitino). The bosonic action (Einstein-Hilbert’s) and the equations of motion (Einstein’s) are
\begin{equation}
S_{\psi_{\mu}=0} = \int d^4 x \sqrt{|g|} R, \quad \Rightarrow \quad E^a_{\mu}(e) \approx G^a_{\mu},
\end{equation}
where $g$ is the determinant of the metric, which is related to the Vierbeins by $g_{\mu\nu} = \eta_{ab} e^a_{\mu} e^b_{\nu}$, where $\eta_{ab}$ is the Minkowski metric, $R$ is the Ricci scalar for $g_{\mu\nu}$ and $G^a_{\mu} = \eta_{ab} e^b_{\nu} G^{ab}_{\mu}$, where $G^{ab}_{\mu}$ is the Einstein tensor for $g_{\mu\nu}$.

The supersymmetry transformations of the graviton and gravitino are
\begin{equation}
\delta_{\epsilon} e^a_{\mu} = -i \bar{\epsilon} \gamma^a \psi_{\mu}, \quad \delta_{\epsilon} \psi_{\mu} = \nabla_{\mu} \epsilon = \partial_{\mu} \epsilon - \frac{1}{4} \omega_{\mu}^{ab} \gamma_{ab} \epsilon.
\end{equation}

The KSI can be readily computed from the general formula Eq. (3.5) and simplified
\begin{equation}
-\bar{\epsilon} \gamma^a G^a_{\mu} = 0, \quad \Rightarrow \quad R = 0, \quad -\bar{\epsilon} \gamma^a R^a_{\mu} = 0.
\end{equation}

On the other hand, in trying to solve the Killing spinor equations (KSEs) which, here, take the form
\begin{equation}
[\nabla_{\mu}, \nabla_{\nu}] \epsilon = -\frac{1}{4} R_{\mu \nu}^{ab} \gamma_{ab} \epsilon = 0, \quad \Rightarrow \quad R^a_{\mu} \gamma^a \epsilon = 0.
\end{equation}

Thus, at least in this case, the KSI are contained in the integrability conditions. We will see later how to obtain more information from these identities.

### 3.2 Example: $N = 2$, $d = 4$ Supergravity

This is the next simplest supergravity theory, if we do not consider adding matter supermultiplets to the $N = 1$ theory. Its field content is $\{e^a_{\mu}, A_{\mu}, \psi_{\mu}\}$ (but now $\psi_{\mu}$ is a Dirac spinor, instead of a Majorana spinor as in the $N = 1$ case). The bosonic action (Einstein-Maxwell’s) and the equations of motion (Einstein’s and Maxwell’s) are
\begin{equation}
S_{\psi_{\mu}=0} = \int d^4 x \sqrt{|g|} \left[ R - \frac{1}{4} F^2 \right], \quad \Rightarrow \quad \begin{cases}
E^a_{\mu}(e) = -2 \{ G^a_{\mu} - \frac{1}{4} T^a_{\mu} \}, \\
E^a_{\mu}(A) = -\nabla_{\mu} F^{a\mu},
\end{cases}
\end{equation}
where $A_{\mu}$ is the Maxwell vector field, $F_{\mu\nu} = 2 \partial_{[\mu} A_{\nu]}$ is its field strength, $F^2 = F_{\mu\nu} F^{\mu\nu}$ and $T^a_{\mu} = \eta_{abc} b^{b\mu} T^{c\mu}$, where $T^{a\mu} = F_{\mu\rho} F^{\rho\nu} - \frac{1}{8} F^2$ is its energy-momentum tensor.

The supersymmetry transformations are
\begin{equation}
\delta_{\epsilon} e^a_{\mu} = -i \bar{\epsilon} \gamma^a \psi_{\mu} + c.c., \quad \delta_{\epsilon} A_{\mu} = -2i \bar{\epsilon} \psi_{\mu} + c.c., \quad \delta_{\epsilon} \psi_{\mu} = \nabla_{\mu} \epsilon - \frac{1}{8} F^{ab} \gamma_{ab} \epsilon \equiv \bar{D}_{\mu} \epsilon.
\end{equation}
Using the bosonic fields supersymmetry transformations, we find that the KSIs take the form
\[ \bar{\epsilon} \{ \mathcal{E}_a^{\mu} (e) \gamma^a + 2 \mathcal{E}^{\mu} (A) \} = 0. \] (3.12)

On the other hand, the integrability conditions of the KSEs \( \delta_{\epsilon} \psi_{\mu} = \tilde{D}_\mu \epsilon = 0 \) are
\[ [\tilde{D}_\mu, \tilde{D}_\nu] \epsilon = -\frac{1}{4} \left\{ \left[ R_{\mu\nu}^{\ ab} - e_{\ [\mu}^a T_{\nu]}^b \right] \gamma_{ab} + \nabla^a (F_{\mu\nu} + *F_{\mu\nu} \gamma_5) \gamma_a \right\} \epsilon = 0, \] (3.13)

\[ \Rightarrow \{ \mathcal{E}_a^{\mu} (e) \gamma^a + 2 \left[ \mathcal{E}^{\mu} (A) + B^{\mu} (A) \gamma_5 \right] \} \epsilon = 0. \]

In this case we get a more general formula from the integrability conditions, valid for the case in which the Bianchi identities are not satisfied. When they are satisfied we recover the KSIs, which is consistent since we have explicitly used the supersymmetry variations of the vector field in order to derive them, assuming, then, implicitly, that the Bianchi identities are satisfied.

The last formula (which we are also going to call KSI) has one important advantage over the original KSI: it is covariant under the \( U(1) \) group of electric-magnetic duality rotations of the Maxwell and Bianchi identities that act as chiral rotations of the spinors.

### 4 Solving Tod’s problem

As shown by Tod [9] in 1983, the problem in \( N = 2, d = 4 \) SUGRA could be completely solved using just integrability and consistency conditions. However, he used the Newman-Penrose formalism, unfamiliar to most particle physicists and suited only for \( d = 4 \). Thus, there were no further results until 1995, when Tod, using again the same methods, solved partially the problem in \( N = 4, d = 4 \) SUGRA [10]. Then, in 2002, Gauntlett, Gutowski, Hull, Pakis and Reall proposed to translate the Killing spinor equation to tensor language and they solved the problem in minimal \( N = 1, d = 5 \) SUGRA [11]. This opened the gates to new results: in 2002 the problem was solved in gauged minimal \( N = 1, d = 5 \) SUGRA [12], in 2003 in minimal \( N = (1, 0), d = 6 \) SUGRA [13, 14] and gauged \( N = 2, d = 4 \) SUGRA [15], and in 2004 and 2005 in gauged minimal \( N = 1, d = 5 \) SUGRA coupled to Abelian vector multiplets [16, 17] and in \( N = 4, d = 4 \) SUGRA [6], completing the work started by Tod on this theory.

There is by now a well-defined recipe to attack this problem (at least in low dimensions) starting with only one assumption: the existence of one Killing spinor \( \epsilon \). The recipe encompasses the following steps:

I Translate the Killing spinor equations and KSIs into tensorial equations.

With the Killing spinor \( \epsilon \) one can construct scalar, vector, and \( p \)-form bilinears \( M \sim \bar{\epsilon} \epsilon, \quad V_\mu \sim \bar{\epsilon} \gamma_\mu \epsilon, \ldots \) that are related by Fierz identities. These bilinears satisfy certain equations because they are made out of Killing spinors, for instance, if the KSE is of the general form
\[ \delta_{\epsilon} \psi_{\mu} = \tilde{D}_\mu \epsilon = [\nabla_\mu + \Omega_\mu] \epsilon = 0, \quad \Rightarrow \quad \nabla_\mu M + 2 \Omega_\mu M = 0, \] (4.1)

The set of all such equations for the bilinears should be equivalent to the original spinorial equation or at least it should contain most of the information contained in it (but, certainly, not all of it).

II One of the vector bilinears (say \( V_\mu \)) is always a Killing vector which can be timelike or null. These two cases are treated separately.

III One can get an expression of all the gauge field strengths of the theory using the Killing equation for those scalar bilinears: \( \Omega_\mu \) is usually of the form \( F_{\mu\nu} V^\nu \) and, then Eq. (4.1) tells us that \( F_{\mu\nu} V^\nu \sim \nabla_\mu \log M \). When \( V \) is timelike this determines completely \( F \) and, when it is null, it determines the general form of \( F \). Of course, Eq. (4.1) is an oversimplified KSE and in real-life situations there are additional scalar factors, \( SU(N) \) indices etc.
IV The Maxwell and Einstein equations and Bianchi identities are imposed on those field strengths $F$, getting second order equations for the scalar bilinears $M$.

V The KSIAs guarantee that these three different sets of equations (plus the equations of the scalar fields, if any) are complicated combinations a a reduced number of simple equations involving a reduced number of scalar unknowns. Solving these equations for the scalar unknowns gives full solutions of the theory. The tricky part is, usually, identifying the right variables that satisfy simple equations and finding these equations as combinations of the Maxwell, Einstein etc. equations.

VI Finally, with the results obtained, the KSEs have to be solved, which may lead to additional conditions on the fields.

Let us see how this recipe works in the examples considered before.

4.1 Example: $N = 1, d = 4$ Supergravity

With one (Majorana) Killing spinor $\epsilon$ the only bilinear that one can construct is a real vector bilinear $V_\mu$ which is always null. $V_\mu$ is also covariantly constant (i.e. it is a Killing vector and $V_\mu dx^\mu$ is an exact 1-form, which allows us to write $V_\mu dx^\mu = du$): $\delta_\epsilon \psi_\mu = \nabla_\mu \epsilon = 0, \Rightarrow \nabla_\mu V_\nu = 0, \ R^\mu_\nu V_\nu = 0, \ (\bar{\epsilon} R^a_\mu \gamma^a \epsilon = 0).$ \hspace{1cm} (4.2)

All the metrics with covariantly constant null vectors are Brinkmann pp-waves and have the form $ds^2 = 2du (dv + Kdu + A_idx^i) + \tilde{g}_{ij} dx^i dx^j,$ \hspace{1cm} (4.3)

where all the components are independent of $v$, where $v$ is defined by $V_\mu \partial_\mu \equiv \partial / \partial v$.

It can be checked that for all these metrics the KSE has solutions. These, then, are all the supersymmetric field configurations of $N = 1, d = 4$ SUGRA, but only those with $R_{\mu \nu} = 0$ are supersymmetric solutions.

4.2 Example: $N = 2, d = 4$ Supergravity

With two Weyl spinors $^4 \epsilon^I$ one can construct the following independent bilinears

- A complex scalar $\bar{\epsilon}^I \epsilon^J \equiv M^{I,J}$
- A Hermitean matrix of null vectors $V^I J_\mu \equiv i \bar{\epsilon}^I \gamma_\mu \epsilon^J$

The KSEs imply the following equations for the bilinears:

$\nabla_\mu M \sim F^+_{\mu \nu} V^I J_\nu,$ \hspace{1cm} (4.4)

$\nabla_\mu V^I J_\nu \sim \delta^I J [M F^+_{\mu \nu} + M^* F^-_{\mu \nu}] - \Phi^{KJ}(\mu) ^{\rho} \bar{\epsilon}^{KJ} F^-_{\nu} \rho - \bar{\Phi}^{I K}(\mu) ^{\rho} \bar{\epsilon}^{KJ} F^+_{\nu} \rho; \hspace{1cm} (4.5)$

so the vector $V_\mu \equiv V^I J_\mu$ is Killing and the other three are exact forms. The Fierz identities tell us that $V_\mu V^\mu \sim |M|^2 \geq 0$ can be timelike or null. When it is timelike, $V_\mu \partial_\mu \equiv \sqrt{2} \partial / \partial t$ and the metric can be put in the conformastationary form $ds^2 = |M|^2 (dt + \omega)^2 - |M|^{-2} d\vec{x}^2,$ \hspace{1cm} (4.6)

\footnote{\text{In this theory one can use pairs of Majorana or Weyl spinors or single Dirac spinors. We now use, for convenience, pairs of Weyl spinors.}}
where, for consistency, the 1-form $\omega$ has to be related to $M$ by

$$d\omega = i |M|^{-2} i^* [M dM^* - c.c.] .$$

(4.7)

On the other hand, Eq. (4.4) gives

$$F^+ \sim |M|^{-2} \{ V \wedge dM + i^* [V \wedge dM] \} .$$

(4.8)

The KSIs are satisfied if Eq. (4.7) is satisfied. It can be seen that, then, any metric and 2-form field strength of the above form admit Killing spinors. On the other hand, all the equations of motion are combinations of the simple equation in 3-dimensional Euclidean space

$$\nabla^2 M^{-1} = 0 .$$

(4.9)

Thus, solving this equation for some $M$ gives us a supersymmetric solution of all the equations of motion (all the fields are determined by $M$). These solutions of the Einstein-Maxwell theory are the Israel-Wilson-Perjés family [18, 19].

The case in which $V$ is null is very similar to the $N = 1$ case and we will not study it here in detail for lack of space.

## 5 Tod’s problem in $N = 4, d = 4$ supergravity

This theory can be obtained by toroidal compactification on $T^6$ of $N = 1, d = 10$ SUGRA [20] (the effective field theory of the Heterotic String) and subsequent (consistent) truncation of the matter vector fields. $N = 4, d = 4$ supergravity has, therefore, contains the fields $\{ e^a_\mu, A^{IJ}_\mu, \tau, \psi_1 \mu, \chi_I \}$, (the Vierbein, vector fields, complex scalar, gravitinos and dilatinos, respectively). The $I, J = 1, \cdots, 4$ indices are $SU(4)$ indices. the theory is invariant under global $SU(4)$ rotations. This symmetry is a remnant of the symmetry in the 6 dimensions which have been compactified in $T^6$, since $SO(6)$ is isomorphic to $SU(4)$. The $SU(4)$ indices of the vector fields are antisymmetric and, further, obey a reality constraint $A_{IJ} = (A^{IJ})^* = \frac{1}{2} \varepsilon_{IKL} A^{KL}$ and, thus, there are only 6 independent vector fields.

A special role is played by the complex scalar field $\tau = a + i e^{-\phi}$, which is called the axidilaton. The field $a$ (axion) is dual to the 4-dimensional Kalb-Ramond 2-form and plays the role of local $\theta$ parameter and $\phi$ is the 4-dimensional dilaton, which plays its usual role of local coupling constant.

It is convenient to start by studying the pure supergravity theory (without the vector supermultiplets) [21], for simplicity. The theory has global $SU(4)$ symmetry (duality) and, furthermore, only at the level of the equations of motion, an $SL(2, \mathbb{R})$ invariance ($S$ duality) that rotates Maxwell equations into Bianchi identities and acts on the axidilaton according to

$$\tau' = \frac{\alpha \tau + \beta}{\gamma \tau + \delta} , \quad \alpha \delta - \gamma \beta = 1 .$$

(5.1)

Observe that the $N = 2$ and $N = 1$ are included as truncations.

The bosonic action of the theory is

$$S = \int d^4x \sqrt{|g|} \left[ R + \frac{1}{2} \frac{\partial \mu \tau \partial ^\mu \tau^*}{(3 \Im \tau)^2} - \frac{1}{16} \Im \tau F^{IJ}_\mu^\nu F_{IJ}^\mu^\nu - \frac{1}{16} \Re \tau F^{IJ}_\mu^\nu \star F_{IJ}^\mu^\nu \right] .$$

(5.2)

It is convenient to denote the equations of motion by

$$\mathcal{E}^a_\mu = - \frac{1}{2} \frac{\delta S}{\delta e^a_\mu} , \quad \mathcal{E} = \frac{1}{\sqrt{|g|}} \frac{\delta S}{\delta \tau} , \quad \mathcal{E}^{IJ}_\mu = \frac{8}{\sqrt{|g|}} \frac{\delta S}{\delta A^{IJ}_\mu} .$$

(5.3)
The Maxwell equation $\mathcal{E}^{I,J}_\mu$ transforms as an $SL(2, \mathbb{R})$ doublet together with the Bianchi identity
\begin{equation}
B^{I,J}_\mu \equiv \nabla_\nu F^{I,J}_{\nu\mu}.
\end{equation} 

For vanishing fermions, the supersymmetry transformation rules of the gravitini and dilatini, generated by 4 spinors $\epsilon_l$ of negative chirality, are
\begin{align}
\delta \epsilon_I \psi_I &\mu = \mathcal{D}_\mu \epsilon_l - \frac{i}{2\sqrt{2}} (\Im \tau)^{1/2} F_{IJ}^+ \mu \gamma^\nu \epsilon^J, \\
\delta \epsilon_I \chi_I & = \frac{1}{2\sqrt{2}} \frac{\partial}{\partial \tau} \epsilon_l - \frac{1}{3} (\Im \tau)^{1/2} F_{IJ} \epsilon^J,
\end{align}

where $\mathcal{D}$ is the Lorentz plus $U(1)$ covariant derivative and where the $U(1)$ connection is given by
\begin{equation}
Q_\mu = \frac{1}{4} \frac{\partial}{\partial \tau} \Im \tau.
\end{equation}

The supersymmetry transformation rules of the bosonic fields take the form
\begin{align}
\delta \epsilon_I e^a &\mu = - \frac{i}{8} (\bar{\epsilon}^I \gamma^a \psi_I + \epsilon_I \gamma^a \psi^I), \\
\delta \epsilon_I \tau & = - \frac{i}{\sqrt{2}} \Im \tau \bar{\epsilon}^I \chi_I, \\
\delta \epsilon_I A_{IJ} &\mu = \frac{\sqrt{2}}{(\Im \tau)^{1/2}} \left[ \bar{\epsilon}^I \psi_{(J} \nu \epsilon^{(I} + \frac{i}{\sqrt{2}} \bar{\epsilon}^I \gamma_{(J} \chi_{I)} + \frac{1}{8} \epsilon_{IJKL} \left( \bar{\epsilon}^K \psi_{(J} \nu \epsilon^{(I} + \frac{i}{\sqrt{2}} \bar{\epsilon}^K \gamma_{(J} \chi_{I)} \right) \right].
\end{align}

Given $N$ chiral commuting spinors $\epsilon_I$ and their complex conjugates $\epsilon^I$ we can construct the following independent bilinears:

1. A complex, antisymmetric, matrix of scalars
\begin{equation}
M_{IJ} \equiv \bar{\epsilon}_I \epsilon_J, \quad M^{IJ} \equiv \epsilon^I \epsilon^J = (M_{IJ})^*,
\end{equation}

2. A complex matrix of vectors
\begin{equation}
V^I_{\ J} = i \bar{\epsilon}^I \gamma_a \epsilon_J, \quad V^I_{\ J} = i \epsilon_I \gamma_a \epsilon^J = (V^I_{\ J})^*,
\end{equation}

which is Hermitean:
\begin{equation}
(V^I_{\ J})^* = V^I_{\ J} = V^J_{\ I} = (V^I_{\ J})^T.
\end{equation}

Using the supersymmetry transformation rules of the bosonic fields, one can find the KSIs of this theory, associated to the gravitini and dilatini, respectively. However, just as in the $N = 2, d = 4$ example, since the Bianchi identities do not appear in these equations, they break S-duality covariance. This covariance can be restored by hand or re-deriving the KSIs from the KSEs integrability conditions. The result is
\begin{align}
\mathcal{E}^a \gamma^\alpha \epsilon_I - \frac{i}{\sqrt{2}} (\mathcal{E}_I)_\mu + \tau^* \mathcal{B}_I \nu \epsilon^J & = 0, \\
\mathcal{E}^* \epsilon_I - \frac{1}{\sqrt{2}} (\mathcal{E}_I)_\mu + \tau \mathcal{B}_I \nu \epsilon^J & = 0.
\end{align}
It is useful to derive tensorial equations from these KSIs. Combining them we arrive to the following, which are chosen among the many possible tensorial KSIs by their interest. For timelike $V^a = V'^a_I$, we get

\begin{equation}
\mathcal{E}^{ab} - \frac{1}{2} \Im \mathcal{E} V^a V^b - \frac{1}{\sqrt{2}} (3m \tau)^{1/2} \Im (M^{IJ} B_{IJ}^a) V^b = 0, \tag{5.16}
\end{equation}

\begin{equation}
\mathcal{E}^* V^a - \frac{i}{\sqrt{2} (3m \tau)^{1/2}} M^{IJ} (E_{IJ}^a - \tau B_{IJ}^a) = 0, \tag{5.17}
\end{equation}

\begin{equation}
\Im m [M^{IJ} (E_{IJ}^a - \tau^* B_{IJ}^a)] = 0. \tag{5.18}
\end{equation}

Observe that the first equation implies the off-shell vanishing of all the Einstein equations with one or two spacelike components. Further, the Einstein equation is automatically satisfied when the Maxwell, Bianchi and complex scalar equations are satisfied and the scalar equation is automatically satisfied when the Maxwell and Bianchi are.

When $V^a$ is null (we denote it by $l^a$), all the spinors $\epsilon_J$ are proportional and we can parametrize all of them by $\epsilon_1 = \phi_1 \epsilon$, where $\phi^I \phi_I = 1$. In order to construct tensor bilinears we define an auxiliary spinor $\eta$ normalized by $\bar{\eta} \eta = \frac{1}{2}$. With these two spinors we can construct a standard complex null tetrad

\begin{equation}
l_\mu = i \bar{\epsilon} \gamma_\mu \epsilon, \quad n_\mu = i \bar{\eta} \gamma_\mu \eta, \quad m_\mu = i \bar{\epsilon} \gamma_\mu \eta^*, \quad m^*_\mu = i \bar{\eta} \gamma_\mu \epsilon^*. \tag{5.19}
\end{equation}

Then, in the null case, the KSIs take the form

\begin{equation}
(E^a_{\mu} - \frac{1}{2} \epsilon^a_{\mu} \mathcal{E}^a_{\rho}) l^a = (E^a_{\mu} - \frac{1}{2} \epsilon^a_{\mu} \mathcal{E}^a_{\rho}) m^a = 0, \tag{5.20}
\end{equation}

\begin{equation}\mathcal{E} = 0, \tag{5.21}\end{equation}

\begin{equation}
(E_{IJ}^\mu - \tau^* B_{IJ}^\mu) \phi^I = 0. \tag{5.22}
\end{equation}

In this case supersymmetry implies that the scalar equations of motion are automatically satisfied.

We are now ready to follow the recipe to find all the supersymmetric configurations of this theory. The first step consists in finding (Killing) equations for the spinor bilinears. From the vanishing of the gravitini supersymmetry transformation rule we find

\begin{equation}
\mathcal{D}_{\mu} M_{IJ} = \frac{1}{\sqrt{2}} (3m \tau)^{1/2} F_{K[I}^+ [IJ] \nu^\nu, \tag{5.23}
\end{equation}

\begin{equation}
\mathcal{D}_{\mu} V^I_{ J \nu} = - \frac{1}{2} (3m \tau)^{1/2} \left[ M_{KJ} F^I - [\mu^\nu + M^{KJ} F_{KJ} + [\mu^\nu \right. \end{equation}

\begin{equation}
\left. - \Phi_{KJ} \phi^I_{\rho} F_{KJ}^+ [\nu | \rho] + \Phi_{KJ} [\mu^\rho F_{KJ}^+ [\nu | \rho] \right] , \tag{5.24}
\end{equation}

and from that of the dilatini, we find

\begin{equation}
V^K_{ I } \cdot \partial \tau - \frac{1}{2} (3m \tau)^{3/2} F_{IJ}^+ \cdot \Phi_{KJ} = 0, \tag{5.25}
\end{equation}

\begin{equation}
F_{IJ - \rho \sigma} V^J_{ K } \sigma + \frac{1}{\sqrt{2}} (3m \tau)^{-3/2} (M_{IK} \partial_{\rho \tau} - \Phi_{IK} \rho^\mu \partial_{\mu} \tau) = 0. \tag{5.26}
\end{equation}
It is immediate to see that $V \equiv V_I$ is a Killing vector and that
\[ V^\mu \partial_\mu \tau = 0. \tag{5.27} \]

Further, using Eq. (5.23) and the antisymmetric part of Eq. (5.26) we find
\[ F_{SR} - \mu \nu V^\nu = -\sqrt{2i} (\mathbb{I} m \tau)^{3/2} M_{SR} \partial_\mu \tau - \sqrt{2} (\mathbb{I} m \tau)^{1/2} \epsilon_{SRIJ} D_\mu M^{IJ}, \tag{5.28} \]
which determines completely the vector field strengths in terms of the scalar bilinears, $\tau$ and the Killing vector $V^\mu$ when this is timelike. In the null case, this equation gives us important constraints on the form of the field strengths, but does not completely determine them. From now on we will focus on the timelike case since it illustrates our procedure best. In this case we can write the metric in the conformastationary form Eq. (4.6), but, while in the $N = 2$, $d = 4$ case one could show that three of the vector bilinears were exact 1-forms and then the metric on the constant-time slices could be chosen to be Euclidean, in the $N = 4$, $d - 4$ case this is not possible and we have to live with a non-trivial 3-dimensional metric $\gamma_{ij}$. Thus
\[ ds^2 = |M|^2 (dt + \omega)^2 - |M|^{-2} \gamma_{ij} dx^i dx^j, \quad i, j = 1, 2, 3, \tag{5.29} \]
where $\omega$ has to satisfy the equation
\[ d\omega = \frac{1}{\sqrt{2}} \Omega = \frac{i}{2\sqrt{2}} |M|^{-4} \left[ (M^{IJ} D M_{IJ} - M_{IJ} D M^{IJ}) \wedge \hat{V} \right]. \tag{5.30} \]

Having the field strengths expressed in terms of the scalars $M^{IJ}$, $\tau$, we move on to the next step and impose the Maxwell equations and Bianchi identities on them, to obtain equations that only involve those scalars. We also substitute the field strengths into the $\tau$ equation, obtaining another equation that only involves $M^{IJ}$ and $\tau$. Now comes the magic of supersymmetry: these three sets of equations are combinations of just two sets of much simpler equations in the 3-dimensional metric $\gamma_{ij}$:

\[ n^{IJ}_{(3)} \equiv (\nabla_\perp + 4i \xi) \left( \frac{\partial \tau M^{IJ}}{|N|^2} \right), \tag{5.31} \]
\[ e^{ij}_{(3)} \equiv (\nabla_\perp + 4i \xi) \left( \frac{\partial_\tau}{|N|^2} \right), \tag{5.32} \]

where $N^{IJ} \equiv (3m \tau)^{1/2} M^{IJ}$ and $\xi$ is defined by
\[ \xi \equiv \frac{1}{4} |M|^{-2} (M_{IJ} d M^{IJ} - M^{IJ} d M_{IJ}), \tag{5.33} \]
and acts as a $U(1)$ connection.

In fact, we can write all the components of the equations of motion define above in terms of these two
\[ \mathcal{E}_{00} = |M|^2 \left[ |M|^2 \Im e_i^3 - 2 \Re (N_{KL} n_{KL}^3) + \frac{1}{2} e_k^k \right], \quad (5.34) \]
\[ \mathcal{E}_{0i} = 0, \quad (5.35) \]
\[ \mathcal{E}_{ij} = |M|^2 (e_{ij} - \frac{1}{2} \delta_{ij} e_k^k), \quad (5.36) \]
\[ B^{IJ} = -2|M|^2 V^a \left\{ \frac{N^{IJ} - \hat{N}^{IJ}}{3m} \Re \epsilon_{(3)} - i(n^{IJ}_{(3)} - \tilde{n}^{IJ}_{(3)}) \right\}, \quad (5.37) \]
\[ \mathcal{E}^{IJ} = -\sqrt{2}|M|^2 \left[ |M|^2 \epsilon_{(3)} + 2in_{KL} \tilde{n}_{KL}^3 \right], \quad (5.39) \]

and a set of equations \( e_{ij} \) defined by
\[ e_{ij} \equiv R_{ij}(\gamma) - 2\partial_i \left( \frac{N^{IJ}}{|N|} \right) \partial_j \left( \frac{N_{KL}}{|N|} \right) \left( \delta_{IJ}^{KL} - J_I^I J_J^J \right), \quad (5.40) \]

and which have to vanish in order to satisfy the KSIs and have supersymmetry\(^5\). These equations are conditions for the 3-dimensional metric \( \gamma_{ij} \), but are not easy to solve directly. We have to substitute our results into the original KSEs or into their integrability conditions. The solution one finds is that, in order to solve the \( e_{ij} = 0 \) equations have supersymmetry, the 3-dimensional metric has to take the form
\[ \gamma_{ij} dx^i dx^j = dx^2 + 2e^{2U(z,z^*)} dz dz^*, \quad (5.41) \]
and the connection \( \xi \) has to take the form
\[ \xi = \pm \frac{i}{2} (\partial \bar{z} U dz - \partial z \bar{U} dz^*) + \frac{1}{2} d\lambda(x,z,z^*). \quad (5.42) \]

Since \( \xi \) is defined in terms of the \( M^{IJ} \) scalars, this is a condition that these scalars have to fulfill, on top of Eqs. (5.31,5.32).

Further, to have supersymmetry, the integrability condition for the equation defining \( \omega \) has to be satisfied as well. It takes the form
\[ \nabla_{\bar{z}} \left( \frac{Q_{\bar{z}} - \xi_{\bar{z}}}{|M|^2} \right) = 0. \quad (5.43) \]

The timelike case now has been completely solved. Let us put together the results: any supersymmetric configuration of \( N = 4, d = 4 \) supergravity in this class is given by a set of 7 complex functions \( M^{IJ}, \tau \) which have to satisfy the following conditions:

1. \( M^{IJ} M^{KL} = 0 \). This is a condition that the scalar bilinears satisfy due to the Fierz identities.
2. \( |M|^2 \neq 0 \). We have assumed this, as definition of the timelike case (\( V^2 \sim |M|^2 > 0 \)).
3. Eq. (5.43) has to be satisfied.

\(^5\) The integrability condition of the equation for \( \omega \) has to be satisfied as well in order to have supersymmetry. We are going to discuss it later.
Given 7 complex functions satisfying these conditions, then, a supersymmetric field configuration of $N = 4, d = 4$ is given by the metric Eqs. (5.29,5.41) and the field strengths Eq. (5.28). These field configurations will be supersymmetric solutions if the expressions Eqs. (5.31,5.32) vanish.

This is the main result in the timelike case.

Now comes the problem of finding sets of 7 complex functions satisfying the above conditions, which is not an easy. We have been able to find two families of supersymmetric solutions based on the Ansatz for the $M_{IJ}$s

$$M_{IJ} = e^{i\lambda(x,z,z^*)}M(x, z, z^*)k_{IJ}(z), \quad M = M^*, \quad \lambda = \lambda^*, \quad k_{[IJ}k_{K]L} = 0. \quad (5.44)$$

which give a connection $\xi$ of the form Eq. (5.42) with

$$U = + \ln |k|, \quad |k|^2 \equiv k^{IJ}(z^*)k_{IJ}(z). \quad (5.45)$$

This Ansatz satisfies all the conditions except for Eq. (5.43). In the following two cases, at least, this last condition is also satisfied:

1. If the $k_{IJ}$ are constants, then, normalizing $|k|^2 = 1$ for simplicity, $\xi = \frac{1}{2}d\lambda$ and $U = 0$. This case was considered by Tod in Ref. [10] and studied in detail in Ref. [22]. Defining $H_1 \equiv \left[(3\Im \, \tau)^{1/2}e^{-i\lambda}M\right]^{-1}$, and $\tau = H_1/H_2$ we get solutions if $\partial_{z_2}H_1 = \partial_{z_2}H_2 = 0$.

2. With $e^{i\lambda} = M = 1$ and constant $\tau$ we solve all constraints and all equations using the holomorphicity of the $k_{IJ}s$. The metric takes the form

$$ds^2 = |k|^2(dt + \omega)^2 - |k|^{-2}dx^2 - 2dzdz^*. \quad (5.46)$$

The metric and the supersymmetry projectors correspond to stationary strings lying along the coordinate $x$, in spite of the trivial axion field. These solutions clearly deserve more study. Observe that this family is precisely the one that cannot be embedded in $N = 2, d = 4$ supergravity plus matter fields [23] and it is genuinely $N = 4$.

6 Conclusions

The landscape approach offers an interesting, even if controversial, point of view over the vacuum selection problem. It also gives additional reasons to work on the problem of classification of supersymmetric solutions, whose 4-dimensional structure we have reviewed in this talk, emphasizing the difference between general supersymmetric configurations and solutions and showing how the KSI can be used in this problem. We have applied the recipes to an interesting case: pure $N = 4, d = 4$ supergravity, but is should be clear that the same procedure could be used in more general contexts ($N = 4, d = 4$ coupled to matter, gauged etc. and other 4-dimensional theories [36]). We also expect some of the techniques could also be of use in solving the much more complicated 11- and 10-dimensional problems [24, 35].

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References