A Note on Supersymmetric Gödel Black Holes, Strings and Rings of Minimal $d = 5$ Supergravity

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Abstract

We show how any asymptotically flat supersymmetric solution of minimal $d = 5$ supergravity with flat base space $ds^2(\mathbb{R}^4)$ can be deformed into another supersymmetric asymptotically-Gödel solution and apply this procedure to the recently found supersymmetric black-ring and black-string solutions.
1. Black rings [1] (black holes with regular event horizons of topology \( S^2 \times S^1 \)) are fascinating objects that challenge (in fact, violate), in higher dimensions, the black hole uniqueness theorems that hold in four dimensions, with potentially important consequences for entropy calculations. Several generalizations of the original purely gravitational solution of Ref. [1] have been found to include electric charge [2] and to make them supersymmetric in the context of minimal and non-minimal \( d = 5 \) supergravities [3, 4, 5, 6, 7, 8].

In this note we describe yet another generalization of the supersymmetric black rings of minimal \( d = 5 \) supergravity to include Gödel asymptotics and preserving supersymmetry. We will show that for any asymptotically flat supersymmetric solution of minimal \( d = 5 \) supergravity with flat base space it is possible to construct in a systematic way another supersymmetric solution with Gödel asymptotics, and we will apply this procedure to the supersymmetric BMPV black hole solution and to the supersymmetric black string solution of Ref. [9]. In the three cases (supersymmetric black hole, string and ring) this deformation preserves the existence (something non trivial) and the geometry of the event horizon, but introduces closed timelike curves (CTCs) far from the horizon.

We start by describing minimal \( d = 5 \) supergravity and the supersymmetric black ring solution of Ref. [3].

2. The action of the bosonic fields (metric and 1-form \( V \)) of minimal \( d = 5 \) supergravity [11] is, in our conventions\(^1\)

\[
S = \frac{1}{16\pi G_N^{(5)}} \int d^5x \sqrt{|g|} \left[ R - \frac{1}{4} G^2 + \frac{1}{12\sqrt{3}} \frac{\epsilon}{\sqrt{|g|}} GGV \right],
\]

where \( G = dV \).

All the supersymmetric solutions of this theory can be classified according to the timelike or null nature of the Killing vector \( V^\mu \), which is constructed as the bilinear of Killing spinors \( V^\mu = \bar{\epsilon} \gamma^\mu \epsilon \) [10]. We are interested in the former, since, in principle, they are the ones that may describe stationary black holes and black rings. The metric and 2-form field strength can always be brought to the form

\[
ds^2 = f^2(dt + \omega)^2 - f^{-1} ds^2_{(4)}, \quad (2)
\]

\[
G = -\sqrt{3}d[f(dt + \omega)] + \frac{2}{\sqrt{3}} G^+, \quad (3)
\]

where the function \( f \) and 1-form \( \omega \) are independent of the time coordinate \( t \), \( ds^2_{(4)} \) is a 4-dimensional hyper-Kähler metric and \( G^\pm \) are the self- and anti-selfdual parts of the 2-form \( G = fd\omega \) with respect to \( ds^2_{(4)} \). The scalar function \( f \) and 1-form \( \omega \) satisfy

\[
dG^+ = 0, \quad (4)
\]

\(^1\)Our conventions are those of Refs. [12, 13]. In particular, we use mostly minus signature and our 1-form is \(-2\) times that of Ref. [10].
\[
\n\nabla^2_{(4)} f^{-1} - \frac{2}{5} G^+_{\, mn} G^+_{\, mn} = 0.
\]

Now we make the following observation: given a supersymmetric solution of minimal \(d = 5\) supergravity, characterized by \(f\) and \(\omega\) satisfying Eqs. (4) and (5), we can always add to the 1-form \(\omega\) a piece \(W\) such that \((dW)^+ = 0\). The deformed \(\omega' = \omega + W\) satisfies clearly Eq. (4), and Eq. (5) is satisfied for the same \(f\) since only \((d\omega)^+\) enters it. This freedom is implicit in the wide class of solutions that we are going to describe next.

3. A particularly interesting class of solutions, including those of interest for us (Gödel spacetime and supersymmetric black holes, strings and rings) can be obtained by using as \(ds^2_{(4)}\) the 4-dimensional Euclidean space metric \(ds^2(\mathbb{E}^4)\), written in the form of a Gibbons-Hawking instanton metric. These have the general form

\[

\begin{align*}
\text{(6)} & \quad ds^2 = H d\bar{x}_3^2 + H^{-1}(d\psi + \chi)^2, \\
\text{(6)} & \quad d\chi = \ast_3 dH,
\end{align*}
\]

where \(\ast_3\) is the Hodge star in Euclidean space \(\mathbb{E}^3\) and \(H\) is a harmonic function in \(\mathbb{E}^3\). The metric of \(\mathbb{E}^4\) is recovered with the choices\(^2\) \(H = 1\) or \(H = 1/|\bar{x}_3|\). Further one requires that \(\partial_\psi\) is a Killing vector of the full solution. It was shown in Ref. [10] that, in these conditions, the most general solution is specified by giving three functions \(K, L\) and \(M\) harmonic in the 3-dimensional Euclidean metric, in terms of which

\[

\begin{align*}
\text{(11)} & \quad f^{-1} = L + H^{-1} K^2, \\
\text{(12)} & \quad \omega \equiv \omega_5 (d\psi + \chi) + \hat{\omega}, \\
\text{(13)} & \quad \omega_5 = M + \frac{3}{2} H^{-1} KL + H^{-2} K^3, \\
\text{(14)} & \quad \ast_3 d\hat{\omega} = H dM - M dH + \frac{3}{2} (K dL - L dK).
\end{align*}
\]

\(^2\)If we use 3-dimensional spherical coordinates \(r, \theta, \varphi\) so

\[

\begin{align*}
\text{(7)} & \quad ds^2(\mathbb{E}^3) = d\bar{x}_3^2 = dr^2 + r^2 d\Omega^2, \\
\text{(7)} & \quad d\Omega^2 = d\theta^2 + \sin \theta d\varphi^2.
\end{align*}
\]

then, if \(H = 1/|\bar{x}_3|\)

\[

\begin{align*}
\text{(8)} & \quad \chi = \cos \theta d\varphi.
\end{align*}
\]

The coordinates \(r, \theta, \varphi, \psi\) are related to the standard 4-dimensional spherical coordinates of \(\mathbb{E}^4\) \(\rho, \theta, \varphi, \psi\) in which

\[

\begin{align*}
\text{(9)} & \quad ds^2(\mathbb{E}^4) = d\rho^2 + \rho^2 d\Omega^2_{(3)}, \\
\text{(10)} & \quad d\Omega^2_{(3)} = \frac{1}{4} [d\Omega^2_{(2)} + (d\psi + \chi)^2], \\
\text{(10)} & \quad r = \rho^2/4.
\end{align*}
\]

3
The freedom of modifying any solution by adding a piece \( W \) with \( (dW)^+ \) to \( \omega \) is already contained in this form of the solution, and is associated to the harmonic function \( M \): it is easy to see that the \( M \)-dependent part of \( \omega \) in that solution

\[
W_M = M(d\psi + \chi) + \hat{\omega}_M, \tag{15}
\]

\[
\ast_3 d\hat{\omega}_M = H dM - M dH, \tag{15}
\]

has \( (dW_M)^+ = 0 \), and it is due to this fact that \( f \) does not depend on \( M \) at all. Thus, we can use the linearity of the Laplace equation and the equation for \( \hat{\omega}_M \) to modify \( M \), changing only the piece \( W_M \) of \( \omega \).

Observe that, when \( K \propto H \), then \( L \) enters the equation of \( \omega \) in the same form as \( M \) and the piece of \( \omega \) that depends on it

\[
W_L = \frac{3}{2} H^{-1} KL(d\psi + \chi) + \hat{\omega}_L, \tag{16}
\]

also has \( (dW_L)^+ = 0 \). Although \( L \) appears in \( f^{-1} \), now \( f^{-1} \) does not really depend on it, because we can add to it any other harmonic function. In fact, all the solutions with \( K \propto H \) can be rewritten as solutions with \( K = 0 \) and two independent harmonic functions \( L, M \).

4. In Ref. [4] it was shown that the supersymmetric black ring of Ref. [3] was the case associated to the choice

\[
H = -\frac{1}{|\vec{x}_3|}, \quad K = -\frac{q}{2|\vec{x}_3 - \vec{x}_{31}|}, \quad L = 1 + \frac{Q - q^2}{4|\vec{x}_3 - \vec{x}_{31}|}, \tag{17}
\]

\[
M = \frac{3q}{4} - \frac{3q|\vec{x}_{31}|}{4|\vec{x}_3 - \vec{x}_{31}|}, \quad \vec{x}_{31} = (0, 0, -R^2/4),
\]

which determines \( \omega \) to be given by

\[
\omega_\psi = \frac{3q}{4} \left[ 1 - \frac{|\vec{x}_3|}{|\vec{x}_3 - \vec{x}_{31}|} - \frac{|\vec{x}_{31}|}{|\vec{x}_3 - \vec{x}_{31}|} \right] - \frac{q|\vec{x}_3|}{16|\vec{x}_3 - \vec{x}_{31}|^2} \left[ 3(Q - q^2) + \frac{2q^2|\vec{x}_3|}{|\vec{x}_3 - \vec{x}_{31}|} \right], \tag{18}
\]

\[
\omega_\varphi = \frac{3q}{4} \left[ 1 - \frac{|\vec{x}_3|}{|\vec{x}_3 - \vec{x}_{31}|} - \frac{|\vec{x}_{31}|}{|\vec{x}_3 - \vec{x}_{31}|} \right] - \frac{q|\vec{x}_3|}{16|\vec{x}_3 - \vec{x}_{31}|^2} \cos \theta \left[ 3(Q - q^2) + \frac{2q^2|\vec{x}_3|}{|\vec{x}_3 - \vec{x}_{31}|} \right]. \tag{18}
\]

This asymptotically flat solution is characterized by four physical parameters: mass \( M \), angular momenta \( J_1, J_2 \) and electric charge \( Q \) given in terms of the three independent parameters \( Q, q, R \), by

\[
M = \frac{3\pi}{4G_N^{(5)}} Q, \quad J_1 = \frac{\pi}{8G_N^{(5)}} q(3Q - q^2), \quad J_2 = \frac{\pi}{8G_N^{(5)}} q(6R^2 + 3Q - q^2), \quad Q = \frac{\sqrt{3}\pi}{G_N^{(5)}} Q. \tag{19}
\]
The BPS bound $M \geq \sqrt{3}|Q|$ is saturated. These parameters are constrained by the condition $Q > q^2 + 2qR$ to avoid CTCs. A 10-dimensional type IIB supertube configuration that reduces to the above solution of minimal $d = 5$ SUGRA has been given in Ref. [5] (see also Ref. [6]). There is an event horizon at $\vec{x}_3 = \vec{x}_3 1$ that has topology $S^2 \times S^1$ and metric

$$ds^2_{(3)} = -\frac{Q^2}{4}d\Omega_{(2)}^2 - l^2d\psi^2.$$  

$$l = \sqrt{3\left[\frac{(Q - q^2)^2}{4q^2} - R^2\right]}.$$  

This solution has several different limits:

(i) $q = R = 0$ corresponds to the extreme 5-dimensional Reissner-Nordström black hole $M - \sqrt{3}|Q| = J_1 = J_2 = 0$, which also belongs to the class of solutions given in Eqs. (6)-(14) with $H = 1/|\vec{x}_3|$ and is given by a single non-vanishing harmonic function ($K = M = \omega = 0$)

$$f^{-1} = L = 1 + \frac{Q}{4|\vec{x}_3|} = 1 + \frac{Q}{\rho^2}.$$  

The event horizon is placed at $\rho = 0$ in these isotropic coordinates, and its constant time sections have induced metric

$$ds^2_{(3)} = -Qd\Omega_{(3)}^2,$$  

and so they are round 3-spheres of radius $Q^{1/2}$, with $A = 2\pi^2Q^{3/2}$.

(ii) $R = 0$ corresponds to the embedding in minimal $d = 5$ supergravity of the charged, rotating, supersymmetric BMPV black hole [14, 15] ($M = \sqrt{3}|Q| = \frac{3\pi}{4G_N}Q$, $J_1 = J_2 = \frac{\pi}{8G_N^5}q(3Q - q^2)$) [14]. This solution also belongs to the same class, with $H = 1/|\vec{x}_3|$, $K \propto H$ and $\hat{\omega} = 0$, and thus it can be written as a solution with $K = 0$ and

$$f^{-1} = L = 1 + \frac{Q}{4|\vec{x}_3|},$$  

$$\omega = W_M = M(d\psi + \chi) = \frac{J}{|\vec{x}_3|}(d\psi + \chi).$$  

Then one can see the supersymmetric BMPV black hole as a supersymmetric deformation of the extreme Reissner-Nordström black hole with $W = W_M$ [16].

This solution has a regular horizon at $\rho = 0$ whenever $J < \frac{1}{2}Q^{3/2}$. Its constant time sections have induced metric

$$ds^2_{(3)} = -\frac{Q}{4}(d\theta^2 + \sin^2\theta d\psi^2) - \left[\frac{Q}{4} - \frac{4J^2}{Q^2}\right](d\varphi + \cos\theta d\psi)^2,$$
and so they are squashed 3-spheres with 

\[ A = 2\pi^2 \sqrt{Q^3 - 4J^2} \] [15].

(iii) The \( R \rightarrow \infty \) limit gives the black string solution of Ref. [9], which belongs to the same class with

\[ H = 1, \quad K = -\frac{q}{2|x_3|}, \quad L = 1 + \frac{Q}{2|x_3|}, \quad M = -\frac{3q}{4|x_3|}, \] (26)

and, thus, it is given by

\[ f^{-1} = 1 + \frac{Q}{|x_3|} + \frac{q^2}{4|x_3|^2}, \]

\[ \omega = -\left( \frac{3q}{2|x_3|} + \frac{3qQ}{4|x_3|^2} + \frac{q^3}{8|x_3|^3} \right) d\psi, \] (27)

Actually, this solution is not quite a string\(^3\): when \( q = 0 \), \( \omega \) vanishes and the solution is an electrically charged extreme Reissner-Nordström black hole smeared in the direction \( \psi \), in which it is not asymptotically flat. There is a horizon at \( x_3 = 0 \) with constant time slices of topology \( S^2 \times \mathbb{R} \) and metric

\[ ds^2_{(3)} = -\frac{q^2}{4} d\Omega^2_{(2)} - 3\frac{Q^2 + q^2}{q^2} d\psi^2. \] (28)

The \( q = 0, R \neq 0 \) limit gives a solution with naked singularities.

5. Now we want to introduce, via \( W \)s with \( (dW)^+ = 0 \) further deformations of the above solutions that preserve the regularity of the horizons and, if possible, do not introduce other pathologies like Dirac strings and CTCs. There seems to be very few, or none at all, deformations with all these properties, and we are going to settle deformations that do introduce CTCs but no Dirac strings or naked singularities.

In Ref. [16] two particularly interesting \( W \)s were considered\(^4\)

\[ W_B \equiv \frac{j}{\rho^2} (d\psi + \cos \theta d\varphi) \equiv \frac{j}{\rho^2} \sigma^3_R, \] (29)

\[ W_G \equiv \lambda \rho^2 (d\varphi + \cos \theta d\psi) \equiv \lambda \rho^2 \sigma^3_L. \] (30)

\( W_B \) is precisely the deformation that introduces rotation in the extreme Reissner-Nordström black hole and is associated to a harmonic function \( M \sim 1/|x_3| \). \( W_G \) changes the asymptotic behavior of the Reissner-Nordström black hole, which still has a regular horizon basically because \( W_G \sim \rho^2 \rightarrow 0 \) in the near-horizon limit, but the solution now asymptotes to a space of the same general form:

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\(^3\)The supersymmetric string solution of minimal \( d = 5 \) supergravity does not belong to the class of solutions we are studying, for which \( V^\mu \) is null [10].

\(^4\)Our orientation is \( e^{\theta \varphi r} = +1 \), so as to coincide with those of Refs. [3, 4], and seems to coincide with of Ref. [16], which is not explicitly stated.
\[ ds^2 = (dt + W_G)^2 - d\rho^2 - \rho^2 d\Omega^2_{(3)}, \]  
\[ G' = -\sqrt{3}dW_G, \]

which is the celebrated maximally supersymmetric 5-dimensional analog of Gödel’s solution [10]. Thus, the deformed solution is an asymptotically-Gödel extreme Reissner-Nordström black hole with an event horizon again placed at \( \rho = 0 \) and with the same geometry and area as in the undeformed black hole.

It is clear that we can use both deformations simultaneously, taking \( f \) as above and \( \omega = W_B + W_G \), and obtain an asymptotically-Gödel BMPV black hole\(^5\). To see this, we have to prove that the surface \( \rho = 0 \) has finite area and is null, but these results follow immediately from the fact that \( W_G \) vanishes fast enough near the surface \( \rho = 0 \). The area of the horizon will be exactly that of the BMPV black hole. On the other hand, \( W_G \) will dominate the asymptotic behavior and, for large enough values of \( \rho \) there will be CTCs even in the regions in parameter space in which the asymptotically flat BMPV black hole is free of them. Finally, the physical parameters originally referred to asymptotic flat spacetime have to be redefined in accordance with the new Gödel asymptotics, something delicate that requires a detailed study that falls out of the scope of this paper.

At this point it should be clear that any asymptotically flat supersymmetric solution of minimal \( d = 5 \) supergravity with flat base space \( ds^2(\mathbb{E}^4) \) can be deformed into another supersymmetric asymptotically-Gödel solution using \( W_G \). In principle, though, we will have to check case by case if the deformation changes the properties of the original solution, in particular the values of charges and the existence and area of the event horizon, although, as we have seen, this is highly unlikely. We expect the presence of CTCs in all the asymptotically-Gödel solutions.

It is natural to perform a similar deformation of the supersymmetric black ring solution with \( W_G \). Now we have to check explicitly that the deformed solution is in fact a black ring with a regular horizon. We can do this by writing the metric in \( x\phi_1, y, \phi_2 \) coordinates\(^6\), in which

\[ W_G = \frac{2\lambda R^2}{(x - y)^2} \left[ (1 - x^2)d\phi_1 - (y^2 - 1)d\phi_2 \right], \]  
\[ (34) \]

\(^5\)A similar solution of (non-minimal) 5-dimensional supergravity has been given in Refs. [17, 18]. The metric is almost identical, the only difference being that \( \sigma_3^R \) in \( W_G \) is replaced by \( \sigma_3^L \).

\(^6\)These are related to the spherical coordinates \( \theta, \varphi, \psi, \rho \) by

\[ \rho \sin \theta/2 = \frac{R\sqrt{y^2 - 1}}{x - y}, \quad \rho \cos \theta/2 = \frac{R\sqrt{1 - x^2}}{x - y}, \quad \varphi = \phi_1 - \phi_2, \quad \psi = \phi_1 + \phi_2. \]  
\[ (32) \]

and, in this coordinate system

\[ ds^2(\mathbb{E}^4) = \frac{R^2}{(x - y)^2} \left[ \frac{dy^2}{(y^2 - 1)} + (y^2 - 1)d\phi_2^2 + \frac{dx^2}{(1 - x^2)} + (1 - x^2)d\phi_1^2 \right]. \]  
\[ (33) \]
redefining \( y = -R/\bar{r} \) and taking the \( \bar{r} \to 0 \) limit as in Ref. [3]. As the explicit calculation shows, in the asymptotically flat case for constant \( t \) and \( \bar{r} = 0 \) we obtain a regular metric for the horizon and in this calculation only the terms singular in \( \bar{r} \) contribute to this metric. Then, it is clear that \( W_G \) will not modify the geometry and area of the supersymmetric black ring event horizon. The regularity of the full 5-dimensional metric on the horizon is proven by analytical extension of the solution, and this is not affected by the presence of \( W_G \) either. The same deformation can be used in the supersymmetric concentric black ring solutions of Ref. [4] without changing their near-horizon properties.

The \( R = 0 \) limit of the new solution gives the Gödel-BMPV black hole. The \( R \to \infty \) limit, though, gives again the same black string solution as before. This is not surprising, since this limit is also a sort of near-horizon limit and \( W_G \) is irrelevant in that limit. However, being supersymmetric, we are free to add directly to the black string \( W_G \) expressed in the new coordinates

\[
W_G = \lambda (|\vec{x}_3|^2 \sin \theta d\varphi - 2|\vec{x}_3|d\psi).
\]  

(35)

This term leads to Gödel asymptotics in the directions orthogonal to \( \psi \) and does not change the geometry of the horizon since it vanishes on it.

6. If the very existence of the maximally supersymmetric Gödel vacuum of minimal \( d = 5 \) supergravity is already quite remarkable, the possibility to “excite” this vacuum placing a large variety of supersymmetric (and also non-supersymmetric [19]) objects such as black holes [16, 17, 18] and black strings [20] in it is also very surprising. All these solutions can be uplifted to solutions of 10-dimensional superstring effective theories in which supersymmetric black rings will appear as black superstubes in Gödel-type or (T-dualizing) in \( pp \)-wave vacua. Non-supersymmetric extensions of most of these solutions should also exist. Work in this direction is in progress.

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References


